MAE/MSE 502 Fall 2014, Homework #4

Prob. 1 (4 points)
(a) For a function defined on the interval of $0 \leq x \leq 1$ given as

\[
 f(x) = \begin{cases} 
 2x, & 0 \leq x \leq 1/2 \\
 1-x, & 1/2 < x \leq 1 
\end{cases}
\]

work out its Fourier Cosine series expansion,

\[
 F(x) \approx F_C(x) = \sum_{n=0}^{\infty} a_n \cos(n \pi x),
\]

where $F(x)$ is the even extension of $f(x)$ and $F_C(x)$ denotes the Fourier Cosine series representation of $F(x)$. A sketch of $f(x)$ is shown at right; Notice a discontinuity at $x = 1/2$.

(b) Plot the original $f(x)$ and its Fourier Cosine series representation, $F_C(x)$, truncated (inclusively) at $n = 5, 10, \text{ and } 30$. Please collect all four curves in a single plot. What are the values of $F_C(x)$ at $x = 0.35$ for the three cases truncated at $n = 5, 10, \text{ and } 30$? Compare them to the exact value, $f(0.35)$, to determine the percentage error (using the exact value as denominator) for the three cases. Repeat the exercise for $x = 0.49$ (a point close to the discontinuity). Discuss the results.

(c) Define $S(N)$ as the value of $F_C(1/2)$ calculated from the Fourier Cosine series truncated (inclusively) at $n = N$, plot $S(N)$ as a function of $N$ over the range of $1 \leq N \leq 30$. What value does $S(N)$ converge to at large $N$?

Prob. 2 (3 points)
For $u(x,t)$ defined on the domain of $0 \leq x \leq 2\pi$ and $t \geq 0$, solve the PDE

\[
 \frac{\partial u}{\partial t} = \frac{\partial^5 u}{\partial x^5} + \frac{\partial^2 u}{\partial x^2},
\]

with the boundary conditions (the first five simply indicate that the system is periodic in $x$),

(i) $u(0, t) = u(2\pi, t)$
(ii) $u_t(0, t) = u_t(2\pi, t)$
(iii) $u_{xx}(0, t) = u_{xx}(2\pi, t)$
(iv) $u_{xxxx}(0, t) = u_{xxxx}(2\pi, t)$
(v) $u_{xxxxx}(0, t) = u_{xxxxx}(2\pi, t)$
(vi) $u(x, 0) = 1 + \sin(2x) + \cos(3x)$.

We expect a closed-form solution without any unevaluated integral or summation of infinite series.
**Prob. 3** (3 points)

(a) For \( u(x,y,t) \) defined on the domain of \( 0 \leq x \leq 1, 0 \leq y \leq 1, \) and \( t \geq 0, \) solve the modified two-dimensional heat equation (be aware of a factor of 4 in the last term),

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial y^2},
\]

with the boundary conditions

(i) \( u(0, y, t) = 0 \)
(ii) \( u(1, y, t) = 0 \)
(iii) \( u(x, 0, t) = 0 \)
(iv) \( u(x, 1, t) = 0 \)
(v) \( u(x, y, 0) = \sin(2\pi x) \sin(3\pi y) . \)

We expect a closed-form solution without any unevaluated integral or summation of infinite series.