Prob. 1 (2.5 points)
Consider the function (see sketch below) defined on the interval of \(0 \leq x \leq 1\),

\[
f(x) = \begin{cases} 
1, & 0 \leq x \leq 0.5 \\
1 - x, & 0.5 < x \leq 1
\end{cases}
\]

(a) Work out the Fourier Sine series expansion,

\[
F_S(x) \approx \sum_{n=1}^{\infty} a_n \sin(n\pi x)
\]

where \(F_S(x)\) denotes the Fourier Sine series representation of \(f(x)\). Plot the original \(f(x)\) and its Fourier Sine series representation, \(F_S(x)\), truncated (inclusively) at \(n = 5, 10,\) and \(30\). Please collect all four curves in a single plot.

(b) What are the values of \(F_S(x)\) at \(x = 0.75\) for the three cases truncated at \(n = 5, 10,\) and \(30\)? Compare them to the exact value, \(f(0.75)\), to determine the percentage error (using the exact value as denominator) for the three cases. Repeat the exercise for \(x = 0.51\) (a point close to the discontinuity). Discuss the results.

(c) Define \(S(N)\) as the value of \(F_S(0.5)\) calculated from the Fourier Sine series truncated at \(n = N\), plot \(S(N)\) as a function of \(N\) for the range of \(1 \leq N \leq 30\). What value does \(S(N)\) converge to at large \(N\)?

(d) Repeat (a) but now work out the Fourier Cosine series expansion,

\[
F_C(x) \approx \sum_{n=0}^{\infty} a_n \cos(n\pi x)
\]

where \(F_C(x)\) denotes the Fourier Cosine series representation of \(f(x)\). (Beware that the summation starts at \(n = 0\).) Plot the \(F_C(x)\) truncated (inclusively) at \(n = 5, 10,\) and \(30\), along with the original \(f(x)\).
**Prob. 2 (3.5 points)**

For \( u(x,t) \) defined on the domain of \( 0 \leq x \leq 2\pi \) and \( t \geq 0 \), solve the PDE

\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \frac{\partial^4 u}{\partial x^4},
\]

with the boundary conditions (the first four simply indicate that the system is periodic in \( x \)),

(i) \( u(0, t) = u(2\pi, t) \)
(ii) \( u_x(0, t) = u_x(2\pi, t) \)
(iii) \( u_{xx}(0, t) = u_{xx}(2\pi, t) \)
(iv) \( u_{xxx}(0, t) = u_{xxx}(2\pi, t) \)
(v) \( u(x, 0) = 5 + 2 \cos(3x) \).

Evaluate \( u(x, t) \) at \( x = 1, t = 0.01 \).

**Prob. 3 (1 point)**

(a) Given the following function defined on the semi-infinite interval, \( 0 \leq x < \infty \),

\[
f(x) = 1 , \text{ if } 0 \leq x \leq 1, \quad \text{Eq. (1)}
= 0 , \text{ if } x > 1 ,
\]
determine the Fourier Sine transform of \( f(x) \), \( F(\omega) \), that satisfies

\[
f(x) = \int_0^\infty F(\omega) \sin(\omega x) \, d\omega .
\]

Plot \( F(\omega) \) as a function of \( \omega \) for the range \( 0 \leq \omega \leq 30 \).

(b) If the \( f(x) \) in Eq. (1) is instead defined on a finite interval, \( 0 \leq x \leq L \) (but otherwise retains its definition in Eq. (1), i.e., \( f(x) = 0 \) if \( 1 < x \leq L \)), find the coefficients, \( a_n \), for the Fourier Sine series of \( f(x) \),

\[
f(x) = \sum_{n=1}^\infty a_n \sin\left(\frac{n\pi x}{L}\right).
\]

Plot \( a_n \) as a function of \( n \) for the following cases: (i) For \( L = 2 \), plot \( a_n \) over the range of \( 1 \leq n < 60/\pi \). (ii) For \( L = 5 \), plot \( a_n \) for \( 1 \leq n < 150/\pi \). (iii) For \( L = 100 \), plot \( a_n \) for \( 1 \leq n < 3000/\pi \). Compare these plots with the plot of \( F(\omega) \) in (a). Discuss your results.

(Note: This homework illustrates the correspondence between Fourier series and Fourier integral. When making the plots, beware that the "\( n \)" in Part (b) is an integer while the "\( \omega \)" in Part (a) can be any real number.)