You may find the following formula useful:

\[
\int_0^\infty e^{-x^2} \cos(2b x) \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.
\]

In Prob 2, it can be used to evaluate the forward Fourier transform. (The inverse transform for that problem would rely on numerical integration.) For Prob 3 and 4, it can be applied twice in both forward and inverse Fourier transform to produce a close-form solution.

**Prob 1** (3 points)
For \(u(x,t)\) defined on the domain of \(0 \leq x \leq 2\pi\) and \(t \geq 0\), solve the PDE

\[
\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - 0.1 \frac{\partial^4 u}{\partial x^4},
\]

with the boundary conditions (the first 4 conditions simply indicate that the system is periodic in the \(x\)-direction):

(i) \(u(0, t) = u(2\pi, t)\), (ii) \(u_x(0, t) = u_x(2\pi, t)\), (iii) \(u_{xx}(0, t) = u_{xx}(2\pi, t)\), (iv) \(u_{xxx}(0, t) = u_{xxx}(2\pi, t)\)

(v) \(u(x, 0) = \cos(x) + \sin(2x)\).

We expect a closed-form solution without any unevaluated integral(s) or summation of infinite series.

**Prob 2** (4 points)
For \(u(x,t)\) defined on the infinite domain of \(-\infty < x < \infty\) and \(t \geq 0\), solve the PDE

\[
\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + \exp[-(x^2 + t)]
\]

with the boundary conditions:

(i) \(u(x, t)\) and its partial derivatives in \(x\) vanish as \(x \to \pm \infty\)

(ii) \(u(x, 0) = \exp[-(x-3)^2]\).

It is acceptable to express the solution as an integral. Plot the solution as a function of \(x\) at \(t = 0, 0.3,\) and 2. Please collect all 3 curves in one plot.

*Note: For this problem, numerical integration (e.g., by the trapezoidal method) might be needed to evaluate \(u(x, t)\) for the plot. Since numerical integration cannot go all the way to \(\infty\), one has to "truncate" the integral at a finite value of \(\omega\). This is analogous to truncating a Fourier series at a finite \(n\). A useful way to determine where to truncate the integral is to plot, for a give \(t\), \(U(\omega, t)\) (the Fourier transform of \(u(x, t)\)) as a function of \(\omega\) and observe how \(U(\omega, t)\) decays with \(\omega\).*
**Prob 3** (1.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, use the method of Fourier transform to solve the PDE

$$\frac{\partial u}{\partial t} = (1 + 2t) \frac{\partial^2 u}{\partial x^2},$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives in $x$ vanish as $x \to \pm \infty$
(ii) $u(x,0) = \exp(-x^2)$.

To receive full credit, the final solution should have a closed-form expression of a real function which contains no unevaluated integral(s).

**Prob 4** (1.5 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, use the method of Fourier transform to solve the PDE

$$\frac{\partial u}{\partial t} = 3 \frac{\partial u}{\partial x} - \frac{u}{1+t},$$

with the boundary conditions:

(i) $u(x, t)$ and its partial derivatives in $x$ vanish as $x \to \pm \infty$
(ii) $u(x,0) = \exp(-x^2)$.

To receive full credit, the final solution should have a closed-form expression of a real function which contains no unevaluated integral(s). Plot the solution, $u(x,t)$, as a function of $x$ at $t = 0, 0.5,$ and $1$. Please collect all 3 curves in a single plot.