Nonhomogeneous PDE - Heat equation with a forcing term

Example 1 Solve the PDE + boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t) ,$$  \hspace{1cm} \text{Eq. (1)}

(I) \quad u(0,t) = 0
(II) \quad u(1,t) = 0
(III) \quad u(x,0) = P(x)

Strategy:
Step 1. Obtain the eigenfunctions in x, $G_n(x)$, that satisfy the PDE and boundary conditions (I) and (II)
Step 2. Expand $u(x,t)$, $Q(x,t)$, and $P(x)$ in series of $G_n(x)$. This will convert the nonhomogeneous PDE to a set of simple nonhomogeneous ODEs.
Step 3. Solve the nonhomogeneous ODEs, use their solutions to reassemble the complete solution for the PDE

For the current example, our eigenfunctions are $G_n(x) = \sin(n\pi x)$, so we should try

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x) ,$$  \hspace{1cm} \text{Eq. (2)}

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(n\pi x) \quad \Rightarrow \quad q_n(t) = 2 \int_{0}^{1} Q(x,t) \sin(n\pi x) \, dx ,$$  \hspace{1cm} \text{Eq. (3)}

$$P(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sin(n\pi x) \quad \Rightarrow \quad u_n(0) = 2 \int_{0}^{1} P(x) \sin(n\pi x) \, dx ,$$  \hspace{1cm} \text{Eq. (4)}
From Eqs. (3) and (4), \( q_n(t) \) and \( u_n(0) \) have already been determined. Our task is to solve \( u_n(t) \) and express it in \( u_n(0) \) (the initial condition of \( u_n(t) \)) and \( q_n(t) \) (the forcing that acts on \( u_n(t) \)).

Plugging Eq. (2) into the original PDE, we have

\[
\frac{\partial}{\partial t} \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x) = \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x) + \sum_{n=1}^{\infty} q_n(t) \sin(n\pi x)
\]

\[
\Rightarrow \sum_{n=1}^{\infty} \frac{d}{dt} u_n(t) \sin(n\pi x) = \sum_{n=1}^{\infty} -n^2 \pi^2 u_n(t) \sin(n\pi x) + \sum_{n=1}^{\infty} q_n(t) \sin(n\pi x)
\]

\[
\Rightarrow \sum_{n=1}^{\infty} \left( \frac{d}{dt} u_n(t) + n^2 \pi^2 u_n(t) - q_n(t) \right) \sin(n\pi x) = 0 \quad \Rightarrow \quad \frac{d}{dt} u_n(t) + n^2 \pi^2 u_n(t) - q_n(t) = 0
\]

or,

\[
\frac{d}{dt} u_n(t) = -n^2 \pi^2 u_n(t) + q_n(t) \quad n = 1, 2, 3, \ldots
\]

Eq. (5)

Equation (5) has the standard solution,

\[
u_n(t) = u_n(0) e^{-n^2 \pi^2 t} + e^{-n^2 \pi^2 t} \int_0^t q_n(t') e^{n^2 \pi^2 t'} \ dt'.
\]

Eq. (6)

Since \( u_n(0) \) and \( q_n(t) \) are known from Eqs. (3) and (4), we have the complete solution once the integral in Eq. (6) is evaluated to obtain \( u_n(t) \); \( u(x,t) \) can be evaluated by Eq. (2) once \( u_n(t) \) is known.
Example 2: In example 1, find the solution for the case with

\[ Q(x,t) = \sin(3\pi x) \, S(t), \]

where

\[ S(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq T \\ 0, & \text{if } t > T \end{cases}, \]

and

\[ P(x) = 5 \sin(2\pi x) + 2 \sin(3\pi x). \]

From Eqs. (3) and (4), we immediately obtain

\[ u_2(0) = 5, \quad u_3(0) = 2, \quad \text{and} \quad u_n(0) = 0 \quad \text{for all other } n \]

\[ q_3(t) = S(t), \quad \text{and} \quad q_n(t) = 0 \quad \text{for all other } n \, . \]

Thus, the expansion in Eq. (2) is reduced to just two terms,

\[ u(x,t) = u_2(t) \sin(2\pi x) + u_3(t) \sin(3\pi x), \quad \text{Eq. (7)} \]

where

\[ u_2(t) = u_2(0) e^{-2^2\pi^2 t} = 5 e^{-4\pi^2 t}, \quad \text{Eq. (8)} \]

\[ u_3(t) = u_3(0) e^{-3^2\pi^2 t} + e^{-3^2\pi^2 t} \int_0^t q_3(t') e^{3^2\pi^2 t'} \, dt'. \quad \text{Eq. (9)} \]
Case 1: Solution for $t > T$

For $t > T$, Eq. (9) will become

$$u_3(t) = u_3(0) e^{-3\pi^2 t} + e^{-3\pi^2 t} \int_0^T e^{3\pi^2 t'} dt'$$

$$= 2 e^{-9\pi^2 t} + e^{-9\pi^2 t} \left( \frac{e^{9\pi^2 T} - 1}{9\pi^2} \right),$$

and the complete solution is

$$u(x, t) = 5 e^{-4\pi^2 t} \sin(2\pi x) + \left[ 2 e^{-9\pi^2 t} + e^{-9\pi^2 t} \left( \frac{e^{9\pi^2 T} - 1}{9\pi^2} \right) \right] \sin(3\pi x) . \quad \text{Eq. (10)}$$

Note #1: In this case, the solution decays to zero as $t \to \infty$

Note #2: In the absence of the forcing (setting $Q(x,t)$ to zero), the solution is reduced to the familiar solution for the homogeneous heat equation,

$$u(x, t) = 5 e^{-4\pi^2 t} \sin(2\pi x) + 2 e^{-9\pi^2 t} \sin(3\pi x) .$$

Note #3: If the initial state is $P(x) = 0$, the solution is contributed entirely by the forcing:

$$u(x, t) = e^{-9\pi^2 t} \left( \frac{e^{9\pi^2 T} - 1}{9\pi^2} \right) \sin(3\pi x) . \quad \text{Eq. (11)}$$
Note #4: For the case with a very small $T$ (i.e., "impulsive forcing"; the forcing $Q$ is turned on at $t = 0$ then turned off after a very short amount of time), $\exp(\alpha T) \approx 1 + \alpha T$, so Eq. (11) can be approximated by

$$u(x, t) \approx e^{-9\pi^2 t} T \sin(3\pi x).$$

In this case, at the time when the forcing is switched off, i.e., at $t = T$, the system reaches an amplitude $T$. Afterward, it decays exponentially just like the solution for the unforced heat equation.

**Case 2: Solution for $t < T$**

This is the case when the forcing is kept on for a long time (compared to the time, $t$, of our interest). If it is kept on forever, the equation might admit a nontrivial steady state solution depending on the forcing. In general, for $t < T$, Eq. (9) becomes

$$u_3(t) = u_3(0) e^{-3\pi^2 t} + e^{-3\pi^2 t} \int_0^t e^{3\pi^2 t'} d t'$$

$$= 2 e^{-9\pi^2 t} + \left( \frac{1 - e^{-9\pi^2 t}}{9\pi^2} \right),$$

and the complete solution is

$$u(x, t) = 5 e^{-4\pi^2 t} \sin(2\pi x) + \left[ 2 e^{-9\pi^2 t} + \left( \frac{1 - e^{-9\pi^2 t}}{9\pi^2} \right) \right] \sin(3\pi x). \quad \text{Eq. (12)}$$

Notably, in this case a nontrivial steady state exists: $u(x, t) \to \frac{1}{9\pi^2} \sin(3\pi x)$ as $t \to \infty$ (and $T \to \infty$).
Exercise: In Example 2, what would be the behavior of the solution if the forcing is periodic in time. For example, if $S(t)$ is replaced by $\sin(t)$?

Exercise: Note that the steady state solution in the preceding page can be readily obtained by setting $\frac{\partial u}{\partial t}$ to zero in the original PDE. Work out the detail and show that the result agrees with our conclusion at the bottom of the preceding page.