Summary of Chapter 5
(Why we have orthogonal eigenfunctions for so many physical problems)

Key: A **Sturm-Liouville problem** has orthogonal eigenfunctions

Sturm-Liouville (eigenvalue) problem:

\[
\frac{d}{dx} \left[ P(x) \frac{du}{dx} \right] + Q(x) u - \lambda R(x) u = 0 , \tag{1}
\]

for \( u(x) \) defined on \( x \in [a, b] \), plus homogeneous b.c.'s

\[
A \ u(a) + B \ u'(a) = 0 \quad (u' \equiv du/dx) \quad (I)
\]
\[
C \ u(b) + D \ u'(b) = 0 . \quad (II)
\]

Remarks:

(1) The forms of the ODE and b.c.'s above are general enough that many physical problems can be converted to a standard Sturm-Liouville problem \( \Rightarrow \) Orthogonality of eigenfunctions

(2) It is crucial that the b.c.'s are homogeneous. If they are not, there may not be orthogonal eigenfunctions for the system.
Proof of orthogonality...

**Step 1:** Define the operator $L$ as

$$L\{u\} \equiv \frac{d}{dx} \left[P(x) \frac{du}{dx}\right] + Q(x)u,$$

such that the original Eq. (1) in the Sturm-Liouville system can be written as

$$L\{u\} = \lambda \ R(x) \ u \quad (2)$$

Let $u$ and $v$ be two solutions (need not be eigenfunctions at this point) to the Sturm-Liouville problem, then

$$\int_a^b v L\{u\} - u L\{v\} \ dx = \int_a^b v \left[ d \left(\frac{d}{dx} [P \frac{du}{dx}] + Q u \right) - u \left[ d \left(\frac{d}{dx} [P \frac{dv}{dx}] + Q v \right) \right] \right] \ dx$$

$$= \int_a^b v \frac{d}{dx} \left[P \frac{du}{dx}\right] - u \left[P \frac{dv}{dx}\right] \ dx$$

$$= \int_a^b \frac{d}{dx} \left[P (v \frac{du}{dx} - u \frac{dv}{dx}) \right] \ dx$$

$$= \left[ P (v \frac{du}{dx} - u \frac{dv}{dx}) \right]_a^b$$

$$= P(b)[v(b)u'(b) - u(b)v'(b)] - P(a)[v(a)u'(a) - u(a)v'(a)]$$

$$= 0 \quad (3)$$

See next page for an explanation why the green-colored expression is identically zero.

[ Note: There was a typo in my blackboard derivation (April 8 lecture) for the above equation. I accidentally put $Qdu/dx$ in the place of $Qu$. Please correct your notes. -- HPH ]
Since $u$ and $v$ are two solutions to the Sturm-Liouville system, they both satisfy the b.c.'s (I) and (II),

\begin{align*}
A u(a) + B u'(a) &= 0 \quad \text{(I-u)} \\
C u(b) + D u'(b) &= 0 \quad \text{(II-u)} \\
A v(a) + B v'(a) &= 0 \quad \text{(I-v)} \\
C v(b) + D v'(b) &= 0 \quad \text{(II-v)}
\end{align*}

From (I-u) and (I-v), we have

\[ u(a) = \left(-\frac{B}{A}\right) u'(a) \quad \text{and} \quad v'(a) = \left(-\frac{A}{B}\right) v(a) \Rightarrow v(a)u'(a) - u(a)v'(a) = 0 \]

Similarly, from (II-u) and (II-v) we can establish that

\[ v(b)u'(b) - u(b)v'(b) = 0 \]

**Step 2:** Now, consider that $u$ and $v$ are two eigenfunctions, $u = \phi_m$, $v = \phi_n$, of the Sturm-Liouville problem corresponding to eigenvalues $\lambda_m$ and $\lambda_n$. Then,

\[ L\{\phi_m\} = \lambda_m R(x) \phi_m \quad \text{and} \quad L\{\phi_n\} = \lambda_n R(x) \phi_n .\]

Using Eq. (3), we have

\[
0 = \int_a^b \phi_n L\{\phi_m\} - \phi_m L\{\phi_n\} \, dx = \int_a^b (\lambda_m - \lambda_n) \phi_m \phi_n R(x) \, dx .
\]

Therefore, as long as $\lambda_m \neq \lambda_n$, we have the orthogonal relationship

\[
\int_a^b \phi_m \phi_n R(x) \, dx = 0 .
\]
Remarks:

In addition to orthogonality of eigenfunctions, it can be shown that

- The eigenvalues of the Sturm-Liouville system are real and have a lower bound (but no upper bound); The eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, with $\lambda_1$ the smallest eigenvalue

- There is a one-to-one correspondence between an eigenvalue and an eigenfunction

- The eigenfunctions form a complete basis for piece-wise continuous functions defined on $[a, b]$, meaning that any function $f(x)$ that is piece-wise continuous can be represented by the eigenfunction expansion,

$$ f(x) \approx \sum_n a_n \phi_n. $$

See p. 163 in textbook for further detail.