Summary of Chapter 5  
(When do we have orthogonal eigenfunctions for our boundary value problem?)

Key: A **Sturm-Liouville problem** has orthogonal eigenfunctions

Sturm-Liouville (eigenvalue) problem:

\[
\frac{d}{dx} \left[ P(x) \frac{du}{dx} \right] + Q(x)u - \lambda R(x)u = 0 , \quad (1)
\]

for \( u(x) \) defined on \( x \in [a, b] \), plus homogeneous b.c.'s

\[
A \ u(a) + B \ u'(a) = 0 \quad (u' \equiv du/dx) \quad (I)
\]
\[
C \ u(b) + D \ u'(b) = 0 . \quad (II)
\]

Remarks:

(1) The forms of the ODE and b.c.'s above are general enough that many physical problems can be converted to a standard Sturm-Liouville problem => Orthogonality of eigenfunctions

(2) It is crucial that the b.c.'s are homogeneous. If they are not, there may not be orthogonal eigenfunctions for the system.
Proof of orthogonality...

**Step 1:** Define the operator $L$ as

$$L\{u\} \equiv \frac{d}{dx} \left[ P(x) \frac{du}{dx} \right] + Q(x) u,$$

such that the original Eq. (1) in the Sturm-Liouville system can be written as

$$L\{u\} = \lambda R(x) u \quad (2)$$

Let $u$ and $v$ be two solutions (need not be eigenfunctions at this point) to the Sturm-Liouville problem, then

$$\int_a^b v L\{u\} - u L\{v\} \, dx = \int_a^b v \left[ \frac{d}{dx} \left( P \frac{du}{dx} \right) + Q u \right] - u \left[ \frac{d}{dx} \left( P \frac{dv}{dx} \right) + Q v \right] \, dx$$

$$= \int_a^b \frac{d}{dx} \left[ P \frac{du}{dx} \right] - u \frac{d}{dx} \left[ P \frac{dv}{dx} \right] \, dx$$

$$= \int_a^b \frac{d}{dx} \left[ P \left( v \frac{du}{dx} - u \frac{dv}{dx} \right) \right] \, dx$$

$$= \left[ P \left( v \frac{du}{dx} - u \frac{dv}{dx} \right) \right]_a^b$$

$$= P(b)[v(b)u'(b) - u(b)v'(b)] - P(a)[v(a)u'(a) - u(a)v'(a)]$$

$$= 0 \quad (3)$$

See next page for an explanation why the green-colored expression is identically zero.
(Addendum to the derivation in previous page)

Since \( u \) and \( v \) are two solutions to the Sturm-Liouville system, they both satisfy the b.c.'s (I) and (II),

\[
\begin{align*}
A \ u(a) + B \ u'(a) &= 0 \quad \text{(I-u)} & \quad A \ v(a) + B \ v'(a) &= 0 \quad \text{(I-v)} \\
C \ u(b) + D \ u'(b) &= 0 \quad \text{(II-u)} & \quad C \ v(b) + D \ v'(b) &= 0 \quad \text{(II-v)} 
\end{align*}
\]

From (I-u) and (I-v), we have

\[
\begin{align*}
u(a) &= (-B/A) \ u'(a) , \quad \text{and} \quad v'(a) = (-A/B) \ v(a) \quad \Rightarrow \quad v(a)u'(a) - u(a)v'(a) = 0
\end{align*}
\]

similarly, from (II-u) and (II-v) we can establish that \( v(b)u'(b) - u(b)v'(b) = 0 \).

**Step 2:** Now, consider that \( u \) and \( v \) are two eigenfunctions, \( u = \phi_m, \ v = \phi_n \), of the Sturm-Liouville problem corresponding to eigenvalues \( \lambda_m \) and \( \lambda_n \). Then,

\[
\begin{align*}
L\{\phi_m\} = \lambda_m \ R(x) \ \phi_m , \quad L\{\phi_n\} = \lambda_n \ R(x) \ \phi_n 
\end{align*}
\]

Using Eq. (3), we have

\[
0 = \int_a^b \phi_n L\{\phi_m\} - \phi_m L\{\phi_n\} \ dx = \int_a^b (\lambda_m - \lambda_n) \ \phi_m \phi_n \ R(x) \ dx 
\]

Therefore, as long as \( \lambda_m \neq \lambda_n \), we have the orthogonality relation

\[
\int_a^b \phi_m \phi_n \ R(x) \ dx = 0 
\]
**Remarks:**

In addition to orthogonality of eigenfunctions, it can be shown that

- The eigenvalues of the Sturm-Liouville system are **discrete** and **real**, and they have a lower bound (but no upper bound); The eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, with $\lambda_1$ the smallest eigenvalue.

- There is a one-to-one correspondence between an eigenvalue and an eigenfunction.

- The eigenfunctions form a **complete basis** for piece-wise continuous functions defined on $[a, b]$, meaning that any function $f(x)$ that is piece-wise continuous can be represented by the eigenfunction expansion,

$$f(x) \approx \sum_n a_n \phi_n.$$ 

See p. 163 in textbook for further detail.