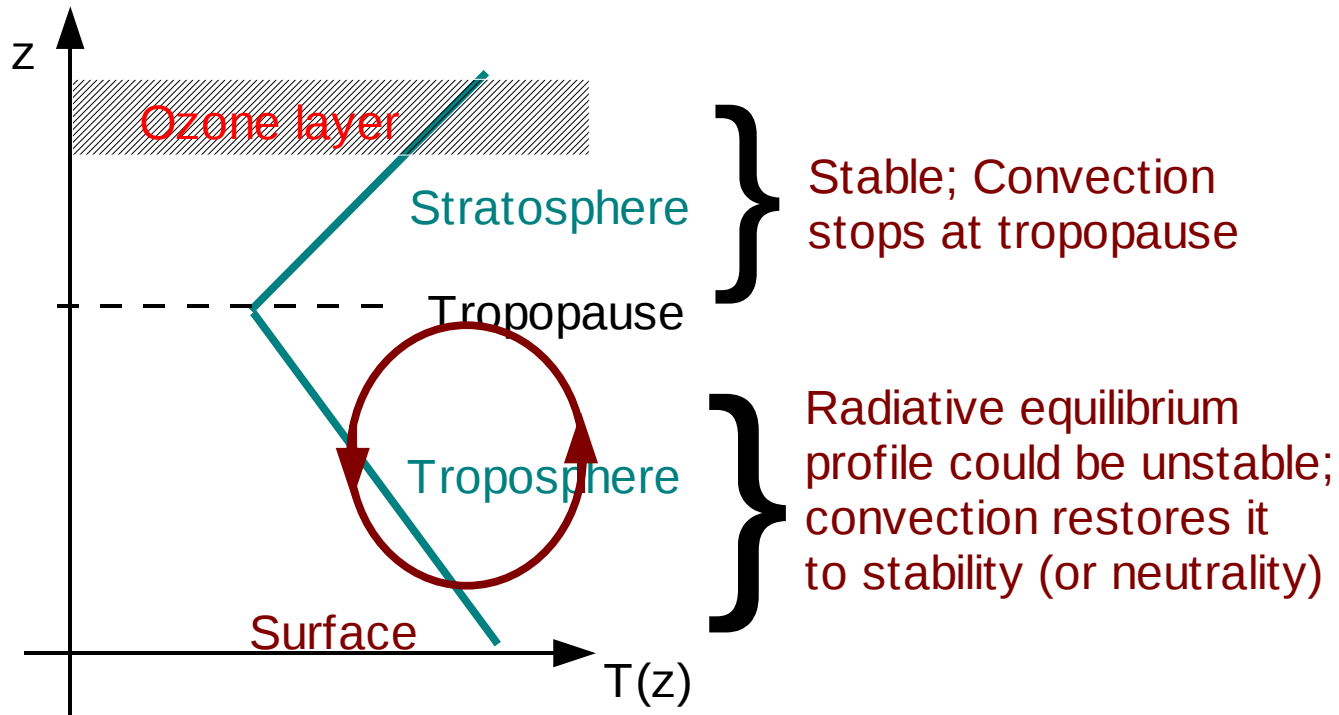# Vertical structure of the atmosphere - Hydrostatic balance

Recap:


Radiative equilibrium:

Stratosphere: $T(z)$ increases with z while $p(z)$ decreases with z

$\Rightarrow$ density $\rho(z) = p/RT$ always decreases with z

Troposphere: $T(z)$ decreases with z while $p(z)$ also decreases with z

$\Rightarrow \rho(z)$ may decrease or increase with height, depending on the detail of $T(z)$ and $p(z)$

Recap:

For a system of liquid fluid, "static stability" (that indicates whether convection can happen spontaneously) can be determined by density stratification.
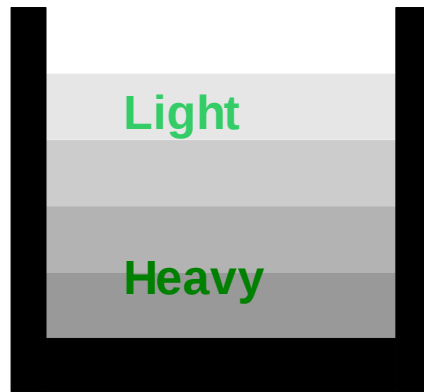

Density decreases
with height - stable
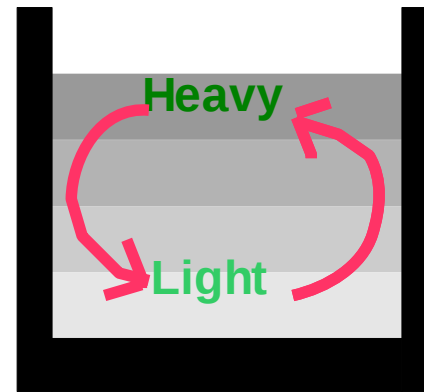
Density increases
with height - unstable

For the atmosphere (close to an ideal gas), density depends strongly on both temperature and pressure, $\rho = p/RT$. When an air parcel is moved up and down adiabatically, its density can change. This needs to be taken into account to determine whether an air parcel has positive or negative buoyancy.

Energetics of convection
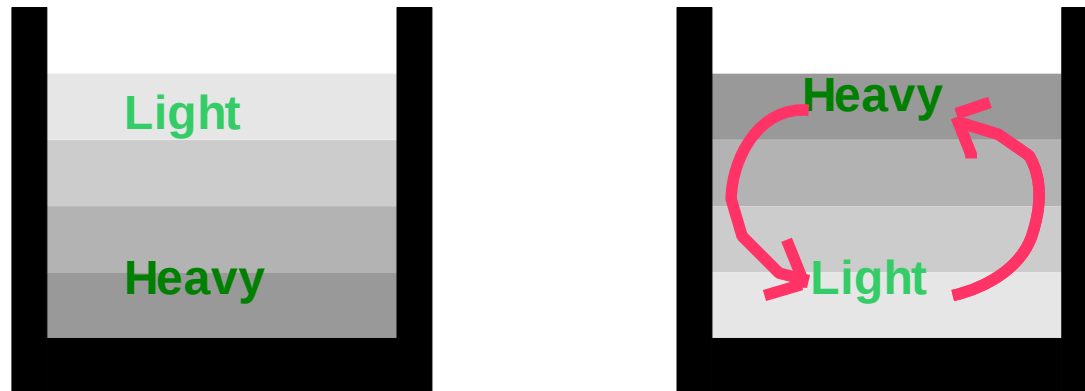
The configuration at right has a higher gravitational potential energy than the one at left. When convection happens in the system at right, the overturning circulation gains its kinetic energy at the expense of the loss of potential energy of the system as the density field evolves toward the (stable) configuration at left.

This "conversion from potential to kinetic energy" is akin to the process of releasing (say) a metal ball in mid-air and let it fall freely; The ball loses its potential energy while accelerating toward the ground.

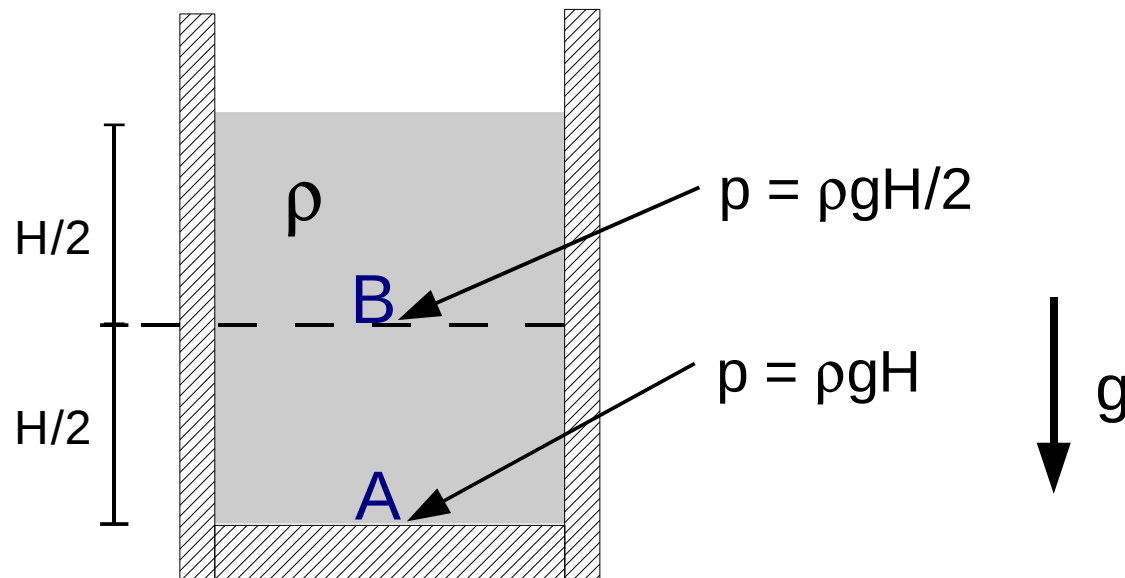To determine the static stability of the atmosphere, we need the knowledge of both $T(z)$ and $p(z)$

Vertical structure of **pressure** ; **Hydrostatic balance**

Simple example: Liquid fluid with constant density

Pressure = force (or weight) per unit area
Weight = mass $\times$ gravitational acceleration = (density $\times$ volume) $\times$ g

Pressure at point A = $\rho \times$ (depth of fluid above point A) $\times$ g
$= \rho\, g\, H$

As long as the fluid remains static (or at least does not accelerate in the vertical direction), pressure always increases with depth (or decreases with height)

$\Rightarrow$ One-to-one correspondence between p and z
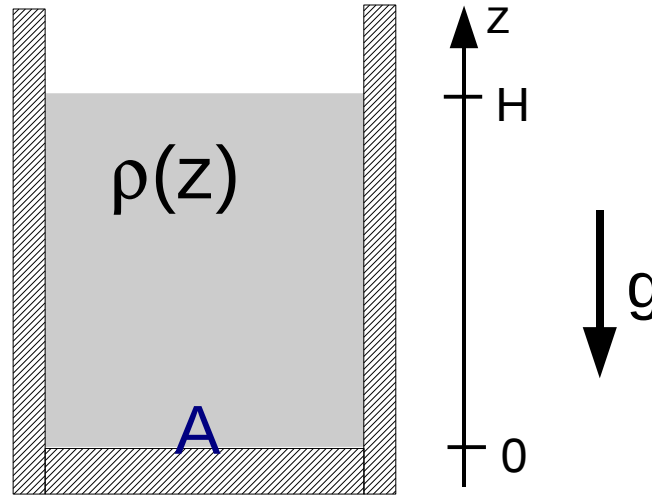
If density depends on depth (or height, z), the total weight per unit area (i.e., pressure) at bottom is simply $p = g \int_0^H \rho(z)\,dz$. Or, the pressure at any height is

$$p(z) = g \int_z^H \rho(z')\,dz' \ .$$

This is the integral form of the "hydrostatic balance" relation that connects pressure to density. Differentiating this equation w.r.t. z, we may obtain the differential form of the **hydrostatic equation**,

$$\frac{dp}{dz} = -\rho g$$

Or, we can derive the hydrostatic equation by considering the vertical pressure difference across a thin slice of fluid

p_{top}

$\rho(z)$

$p_{bottom} = p_{top} + \rho(z)g\Delta z$

z

z

z−Δz

g

0
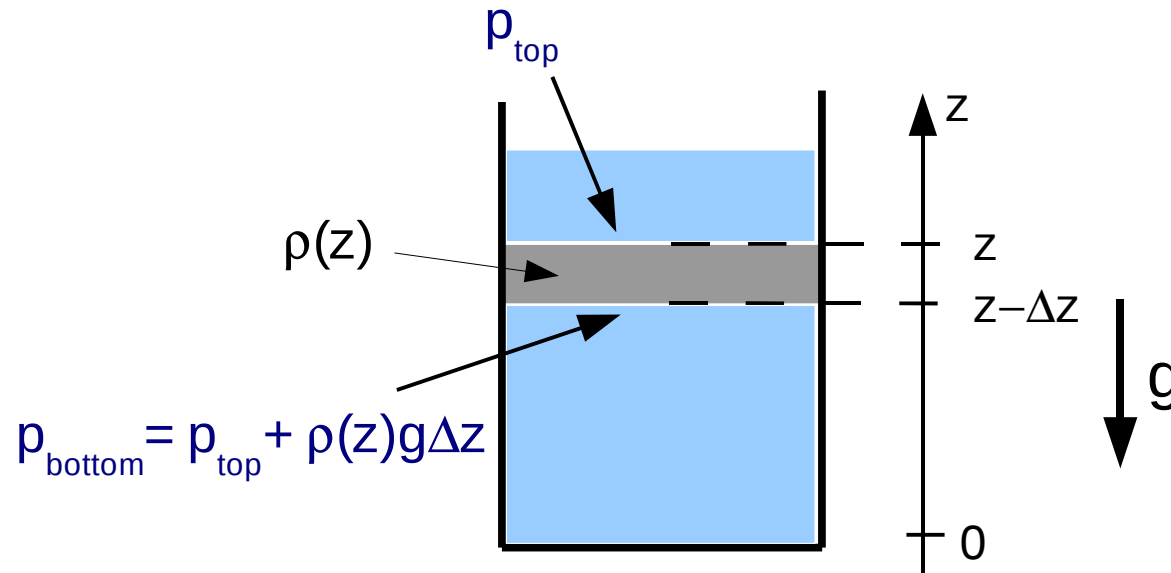
Pressure difference between top and bottom is

$$\Delta p = p_{top} - p_{bottom} = -\rho(z)\, g\, \Delta z$$

$$\Rightarrow \Delta p/\Delta z = -\rho(z)\, g$$

Taking the limit of $\Delta z \to 0$, we have the hydrostatic equation,

$$\frac{dp}{dz} = -\rho\, g$$

Note that the hydrostatic equation depicts the vertical balance of force for a piece of fluid at rest. The balance is between the upward **pressure gradient force** and downward **gravitational force**
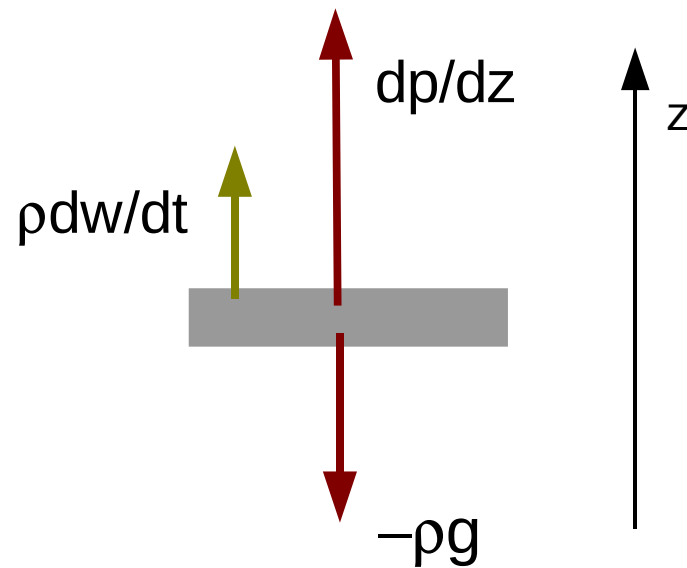
The hydrostatic equation is the vertical component of the momentum equation (Newton's equation of motion) for the fluid parcel when the forces are in
 prefect balance and the net acceleration = 0.
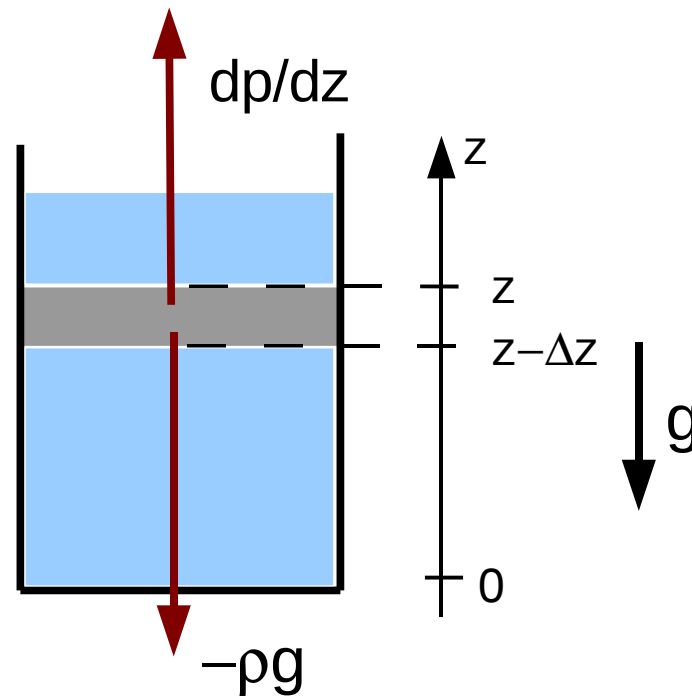
If the two forces are not in perfect balance, there will be vertical acceleration of the fluid parcel according to Newton's 2nd law,

$$\rho \frac{d\,w}{dt} = -\frac{dp}{dz} - \rho g \ , \ \text{ or } \ \frac{d\,w}{dt} = -\frac{1}{\rho}\frac{dp}{dz} - g$$

where w ≡ dz/dt is the vertical velocity of the fluid parcel. **This is the vertical momentum equation for a fluid (in the absence of friction and diabatic forcing) in general.**

dp/dz

z

ρdw/dt

−ρg

Preview of things to come ...

The "non-hydrostatic" equation,

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{dp}{dz} - g \; ,$$

is Newton's equation for a fluid parcel. The vertical velocity of the parcel, w, is a function of the position of the parcel, (x(t), y(t), z(t)), therefore $w \equiv w(x(t),y(t),z(t),t)$. Recall from HW1 Prob 1 that

$$\frac{dw}{dt} \equiv \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\equiv \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \quad ,$$

we have

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \; .$$

This is the vertical component of the Navier-Stokes (momentum) equation, in the absence of friction and diabatic forcing. Note that the w here is now a 3-D field in space and time, not just the velocity for a designated air parcel. This is the prognostic equation for vertical velocity in a *non-hydrostatic model* (for instance for weather prediction).

The atmosphere is very close to hydrostatic balance most of the time, except at isolated locations when the vertical profile becomes statically unstable. In that situation, convection will happen to restore stability. This takes place on a very short time scale (~ a few hours), therefore after some spatial and temporal averaging the atmosphere is generally statically stable; For many applications, it is enough to replace the vertical momentum equation by the hydrostatic equation.

Perhaps the only exception is in the tropics, where the atmosphere could be marginally unstable even in the time mean.

Another way to view the hudrostatic relation is that if we analyze the order of magnitude of the terms in the vertical momentum equation,

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \ ,$$

we will generally find that the two terms in the r.h.s. are much, much bigger than the terms in the l.h.s.

We will revisit this in later chapters when we analyze the full Navier-Stokes equations.

Pressure as a vertical coordinate

Under hydrostatic balance, since pressure decreases monotonically with height, we can use pressure as an alternative coordinate. **This is a very useful alternative for treating the dynamics of large-scale atmosphere.**

Since pressure and height has a one-to-one correspondence, we can swap the two by considering z as a function of p, $z(p)$, as our dynamical variable, and p itself as the coordinate. This works as long as the fluid is in hydrostatic balance.

Pressure as a vertical coordinate

More preview of things to come...

Using p as the vertical coordinate, the horizontal momentum and thermodynamic equations need to be modified as well. For example, in normal (x,y,z,t) coordinate, the "temperature advection" term in thermodynamic equation is

$$\boldsymbol{v}\cdot\nabla T \equiv u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z} \ ,$$

where $\boldsymbol{v} \equiv (u,v,w)$ is the 3-D velocity vector for the fluid. With the new (x,y,p,t) coordinate, it becomes

$$\boldsymbol{v}\cdot\nabla T \equiv u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\omega\frac{\partial T}{\partial p}$$

where $\boldsymbol{v} \equiv (u,v,\omega)$, with $\omega \equiv$ dp/dt is the "vertical velocity" in p-coordinate. Note that $\omega > 0$ implies *downward* motion of the fluid parcel.

The coordinate transformation would rely on the hydrostatic relation. For example,

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial z} = -\rho g\frac{\partial T}{\partial p} \ , \text{ etc.}$$