Proof that $\Sigma\vec{\tau}_{ext,cm} = d/dt(\vec{\mathbf{L}}_{sys,cm})$ Holds
Even if the Center of Mass is Moving

The basic principle with which we begin is $\Sigma\vec{\tau}_{ext,O} = d/dt(\vec{\mathbf{L}}_{sys,O})$, where O represents a fixed point in space. We first prove that the system angular momentum about point O can be written in terms of the angular momentum OF a moving center of mass plus the angular momentum ABOUT the center of mass.

$$\vec{\mathbf{L}}_{sys,O} \equiv \sum_i (\vec{\mathbf{r}}_{Oi} \times \vec{\mathbf{p}}_i), \quad \text{where } \vec{\mathbf{r}}_{Oi} \text{ is the vector from point } O \text{ to particle } i$$

The vector $\vec{\mathbf{r}}_{Oi}$ may be written as the sum of the vector from O to the center of mass, call it $\vec{\mathbf{r}}_{Oc}$ plus the vector from the center of mass to the point i, call it $\vec{\mathbf{r}}_{ci}$. Make this substitution and simplify.

$$\begin{aligned}
\vec{\mathbf{L}}_{sys,O} &= \sum_i [(\vec{\mathbf{r}}_{Oc} + \vec{\mathbf{r}}_{ci}) \times \vec{\mathbf{p}}_i] \\
&= \sum_i [(\vec{\mathbf{r}}_{Oc} \times \vec{\mathbf{p}}_i) + (\vec{\mathbf{r}}_{ci} \times \vec{\mathbf{p}}_i)] \\
&= \vec{\mathbf{r}}_{Oc} \times \sum_i \vec{\mathbf{p}}_i + \sum_i (\vec{\mathbf{r}}_{ci} \times \vec{\mathbf{p}}_i) \\
&= \vec{\mathbf{r}}_{Oc} \times \vec{\mathbf{p}}_{sys} + \vec{\mathbf{L}}_{sys,cm}
\end{aligned}$$

The first term is just the angular momentum OF the center of mass, so the first step in our proof is complete.

Now we prove that the net external torque about point O may be decomposed into the torque about point O ON the center of mass plus the net external torque ABOUT the center of mass. We make the same substitution for the vector $\vec{\mathbf{r}}_{Oi}$ as we made in the first part of the proof.

$$\begin{aligned}
\Sigma\vec{\tau}_{ext,O} &\equiv \sum_i (\vec{\mathbf{r}}_{Oi} \times \vec{\mathbf{F}}_{ext,i}) \\
&= \sum_i [(\vec{\mathbf{r}}_{Oc} + \vec{\mathbf{r}}_{ci}) \times \vec{\mathbf{F}}_{ext,i}] \\
&= \sum_i [(\vec{\mathbf{r}}_{Oc} \times \vec{\mathbf{F}}_{ext,i}) + (\vec{\mathbf{r}}_{ci} \times \vec{\mathbf{F}}_{ext,i})] \\
&= \vec{\mathbf{r}}_{Oc} \times \sum_i \vec{\mathbf{F}}_{ext,i} + \sum_i (\vec{\mathbf{r}}_{ci} \times \vec{\mathbf{F}}_{ext,i}) \\
&= \vec{\mathbf{r}}_{Oc} \times \sum_i \vec{\mathbf{F}}_{ext,i} + \Sigma\vec{\tau}_{ext,cm}
\end{aligned}$$

The first term is just the torque about point O ON the center of mass due to the net external force acting on the system of particles, so the second step in our proof is complete.

What remains is to prove that

$$\vec{\mathbf{r}}_{Oc} \times \sum_i \vec{\mathbf{F}}_{ext,i} = \frac{d}{dt}(\vec{\mathbf{r}}_{Oc} \times \vec{\mathbf{p}}_{sys}).$$

If this holds true, then the basic principle with which we began will immediately yield our desired result. Here is this third part of our proof:

$$\begin{aligned}\frac{d}{dt}(\vec{\mathbf{r}}_{Oc} \times \vec{\mathbf{p}}_{sys}) &= \frac{d}{dt}(\vec{\mathbf{r}}_{Oc}) \times \vec{\mathbf{p}}_{sys} + \vec{\mathbf{r}}_{Oc} \times \frac{d}{dt}(\vec{\mathbf{p}}_{sys}) \\ &= \vec{\mathbf{v}}_{cm} \times \vec{\mathbf{p}}_{sys} + \vec{\mathbf{r}}_{Oc} \times \sum_i \vec{\mathbf{F}}_{ext,i}\end{aligned}$$

where we have used the definition of the velocity of the center of mass (in the first term), and Newton's Second Law for a system of particles (in the second term). The first term, $\vec{\mathbf{v}}_{cm} \times \vec{\mathbf{p}}_{sys}$ is easily seen to be zero because these two vectors are in the same direction; thus completing our third proof.

Finally, here is a summary of the overall proof, in symbols. We begin with the general principle, apply our first proof on the right-hand side, apply our second proof on the left-hand side, and then use our third proof to simplify to our final result:

$$\begin{aligned}\Sigma\vec{\tau}_{ext,O} &= \frac{d}{dt}(\vec{\mathbf{L}}_{sys,O}) \\ \vec{\mathbf{r}}_{Oc} \times \sum_i \vec{\mathbf{F}}_{ext,i} + \Sigma\vec{\tau}_{ext,cm} &= \frac{d}{dt}(\vec{\mathbf{r}}_{Oc} \times \vec{\mathbf{p}}_{sys} + \vec{\mathbf{L}}_{sys,cm}) \\ \Sigma\vec{\tau}_{ext,cm} &= \frac{d}{dt}(\vec{\mathbf{L}}_{sys,cm}),\end{aligned}$$

which is thus true even if the center of mass is translating. This proof is completely general; the system of particles is not required to be rigid, and the torque and angular momentum are taken about the POINT that marks the center of mass (the result does not refer to any axis).

In order to apply this general result to prove $\Sigma\tau_{ext,cm} = I_{cm}\alpha$ for some AXIS through the center of mass, one must restrict the system to a rigid set of particles and apply appropriate restrictions to the symmetry of the axis (or else run a fixed axle through the center of mass).