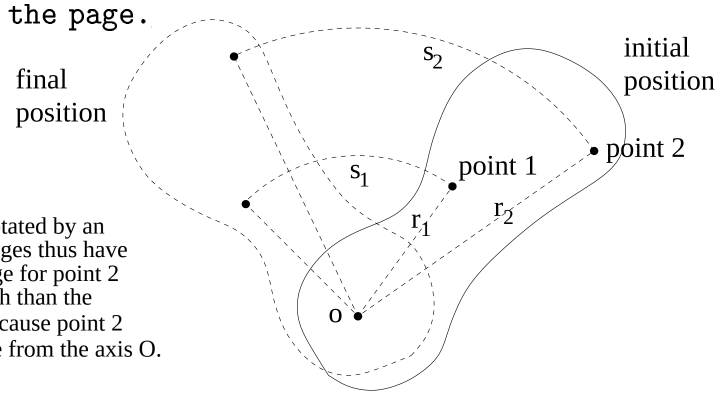
So far this semester, we have had an introduction to KINEMATICS, the description of motion, and to DYNAMICS, the causes of motion -- or, more accurately, the causes of changes in motion. We have learned to look at dynamics from three perspectives, the FORCE perspective, the WORK AND ENERGY perspective, and the MOMENTUM perspective; we can either say, "A net force causes an acceleration", or that "net work causes a change in kinetic energy", or that "an impulse causes a change in momentum". Now we will apply the things we have learned thus far to a problem which is very important for any technological society, ROTATION OF RIGID BODIES, including ROLLING WITHOUT SLIPPING.

A RIGID BODY is just a system of particles that are rigidly connected to one another. We can imagine a set of point masses connected by massless rigid sticks, but more often we just want to think of any solid object; the particles are the atoms that make up that solid object -- these atoms cannot change places, so they are "rigidly connected". We will begin our study by considering a rigid body that is constrained to rotate about a fixed axis. The axis can either pass through the body at some point, or it can even be outside the rigid body, but in this case it would have to be connected to the rigid body by "massless" rods -- like the spokes that connect the axle of a bicycle wheel to the hoop and tire (where almost all the mass lies).

We start at the beginning, i.e. with the kinematics of the rotation of such an object. What are the variables that we will use to describe the rotation of such an object, or the motion of any particle within the rotating object? Since each particle in the object will be going in a circle around the fixed axis, it should come as no surprise that we will use the variables of circular motion. Consider the crazily-shaped rotating object that I have drawn below. It is rotating about a fixed axis $\perp$ to the page and passing through point O, so the rotation is in the plane of the page.



The rigid object is rotated by an angle Δθ. Both wedges thus have angle Δθ. The wedge for point 2 has a longer arclength than the wedge for point 1 because point 2 is at a larger distance from the axis O.

Every particle is going in a circle about point O. Every particle is possibly at a different radius, so we will use r_i for the radius of particle i. We will use $\Delta\theta$ for the angular displacement of any particle; IN THIS CASE WE DON'T NEED AN i SUBSCRIPT BECAUSE THE ANGULAR DISPLACEMENT OF EVERY PARTICLE MUST BE THE SAME -- our object is a rigid body. As opposed to our work in Chapter 5, now we want $\Delta\theta$ to be a 1D vector; traditionally CCW is positive and CW is negative. The units of $\Delta\theta$ are still radians. And we will use s_i for the arclength turned through by particle i. Since by the DEF of $\Delta\theta$, $s_i = r_i\Delta\theta$ (see required reading 5 if you have any uncertainty), s_i is now also a 1D vector, with the same sign convention as for $\Delta\theta$.

We will finish our introduction to the variables of rotational kinematics with formal definitions. The definitions are very similar to our definitions for "translational" kinematics.

DEF The AVERAGE ANGULAR VELOCITY ($\bar{\omega}$) of any point in a rigid rotating object which turns through angular displacement $\Delta\theta$ in time Δt is

$$\bar{\omega} \equiv \frac{\Delta\theta}{\Delta t} \qquad \text{units are rad/s}$$

$\bar{\omega}$ is a 1D vector with the same sign convention as for $\Delta\theta$.

DEF The INSTANTANEOUS ANGULAR VELOCITY is the angular velocity at some instant in time.

DEF The AVERAGE ANGULAR ACCLERATION ($\bar{\alpha}$) of any point in a rigid rotating object which undergoes a change in angular velocity $\Delta\omega$ in time Δt is

$$\bar{\alpha} \equiv \frac{\Delta\omega}{\Delta t} \qquad \text{units are rad/s}^2$$

$\bar{\alpha}$ is a 1D vector with the same sign convention as for $\Delta\theta$.

DEF The INSTANTANEOUS ANGULAR ACCELERATION is the angular acceleration at some instant in time.

Once again, we will primarily restrict our study to cases in which the acceleration is constant. For those cases, it should be no surprise that we get the same five equations that we used in cases of constant "translational" acceleration. Here are those five equations for rotation (compare with Eqs. (1)-(5) in required reading 3 with $\Delta x \to \Delta\theta$, $v \to \omega$, and $a \to \alpha$):

DEF of average velocity $$\bar{\omega} \equiv \frac{\Delta\theta}{\Delta t} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} \tag{1}$$

DEF of (average) acceleration $$\alpha \equiv \frac{\Delta\omega}{\Delta t} \equiv \frac{\omega_f - \omega_i}{\Delta t} \tag{2}$$

simple fact for constant a $$\bar{\omega} = \frac{\omega_i + \omega_f}{2} \tag{3}$$

derived equation $$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha(\Delta t)^2 \tag{4}$$

derived equation $$\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta \tag{5}$$

Eqs. (4) and (5) will appear on your equation sheet. We use these equations so frequently in 111 that students often have trouble remembering that they are only valid when the acceleration (in this case the angular acceleration α) is constant.

Now we have to remember that our rotating object is a system of particles, with particle i going around in a circle of radius r_i. So each particle is in circular motion, but unless ω is constant (α = 0) it is NOT UCM. Instead, the particles may be speeding up or slowing down as they go around in their circles. We can get the equations for the motions of the particles from the DEF of $\Delta\theta$, namely $\Delta\theta \equiv s_i/r_i$. In required reading 5, we have already used this DEF to get the speed of the particle in circular motion, $v = r\omega$. Since ω is now a 1D vector, we now write this equation as $v_{t,i} = r_i\omega$, where $v_{t,i}$ stands for the tangential velocity of particle i, and has the same sign convention as for $\Delta\theta$ (i.e. $r_i\omega$ still gives us the speed of particle i, but now there is also a sign indicating whether the particle is going around CCW or CW). The radial velocity of each particle is of course zero. The radial acceleration of each particle is still given by $a_{r,i} = -(v_{t,i}^2/r_i)$; just keep in mind that this value can now change with time (it is often useful to substitute $v_{t,i} = r_i\omega$ into this expression, yielding $a_{r,i} = -(\omega^2 r_i)$). The tangential acceleration of the particle is zero only if ω is constant; if ω is not constant, we get the average tangential acceleration $\bar{a}_{t,i}$ from the following short derivation:

$$\bar{a}_{t,i} \equiv \frac{\Delta v_{t,i}}{\Delta t} = \frac{\Delta(\omega r_i)}{\Delta t} = r_i \frac{\Delta\omega}{\Delta t} = r_i \bar{\alpha}$$

A version of this equation will appear on your equation sheet; because we usually deal with constant acceleration, the average bars are often omitted. (I hope that you have noticed that any particle's tangential quantity is gotten by multiplying the corresponding angular quantity by the radius at which the particle is located, i.e. $s_i = r_i\Delta\theta$, $v_{t,i} = r_i\omega$, and $a_{t,i} = r_i\alpha$.) Remember that to get the total acceleration of particle i from the radial and tangential components of its acceleration, you must use your knowledge of vectors from Chapter 1, i.e. $a_i = \sqrt{a_{t,i}^2 + a_{r,i}^2}$, etc.

All of the above relationships are for a RIGID BODY rotating about a FIXED AXIS. In lecture, we will extend some of these ideas to the problem of ROLLING WITHOUT SLIPPING, in which the axis of rotation is translating at the same time that the object is rotating. To check your grasp of the concepts involved in ROTATIONAL KINEMATICS, I suggest that you try Self-Assessment Test (SAT) 8.1. Before starting on the HW for lecture 13, I suggest you also try SAT 8.2.

We begin the study of ROTATIONAL DYNAMICS in the same way that we began translational dynamics -- we ask the question, "What is the cause of acceleration?" (angular acceleration in this case). The answer is TORQUE.

DEF The TORQUE (τ) about axis O due to force $\mathbf{F}$ acting at a distance r from O is

$$\tau_F \equiv (F_t)r \qquad \text{units are N} \cdot \text{m}$$

where F_t is the tangential component of $\mathbf{F}$. τ is thus a 1D vector with CCW usually taken as positive.

The symbol "τ" is the Greek letter "tau". While the units of torque, N·m, have the same dimensions as the units of energy, J, it is traditional to reserve the J unit for energy only. This duplication of units arises (as you will see in lecture 15) because radians is a dimensionless unit. Sometimes multiple subscripts are used for torque, i.e. τ_{OF} would mean the torque about axis O due to force $\mathbf{F}$, and sometimes only the axis subscript is used; our choice may depend on the particular problem we are doing.

In lecture 13, I will do a DEMO which will hopefully convince you that this is the right DEF for the cause of angular acceleration, and we will do some examples to help you understand the DEF. As was the case for the DEF of work, we will find that the DEF of torque can be written in three equivalent ways. I will give you these three ways in lecture.