

In required reading 13, you learned that the DEF of the TORQUE (τ) about axis O due to force \mathbf{F} acting at a distance r from O is $\tau_F \equiv (F_t)r$ with units of N·m. For the official DEF, see the end of required reading 13. Then, in lecture 13, you saw that the magnitude of this torque can be written in three equivalent ways:

$$|\tau_F| \equiv |(F_t)|r \equiv F(\sin \theta)r \equiv F\ell$$

where θ is the angle between \mathbf{F} and the radial direction, and ℓ is the lever arm for force \mathbf{F} with respect to axis O . Questions 3-5 in Self-Assessment Test 9.1 will check your understanding of the DEF of torque. We will begin lecture 14 with two PRS questions involving the DEF of torque.

You also learned in reading 13 that torque is the cause of angular acceleration. This concept was demonstrated in lecture by hanging weights on either end of a metal bar that was free to rotate about an axis through its center. When we began our study of Newton's Laws we learned that force (specifically a nonzero net force) is the cause of translational acceleration; the connection between force and acceleration is given by Newton's 1st and 2nd Laws. Similarly, the connection between torque and angular acceleration is given by Newton's 1st and 2nd Laws for rotation; your task for this lecture is to understand these two rules.

Newton's 1st Law for rotation is quite simple -- if there is no net torque, then there is no angular acceleration, and vice versa. The DEF of NET TORQUE is just what you would expect:

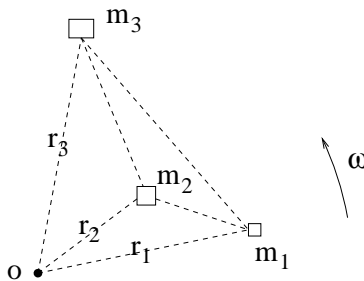
DEF For a rigidly connected system of particles, the NET TORQUE ($\Sigma\tau$) about axis O is simply the 1D vector sum of the torques about axis O due to all the forces acting on that system of particles. Synonyms are the RESULTANT TORQUE, the TOTAL TORQUE, and the SUM OF ALL TORQUES.

In symbols, Newton's 1st Law for rotation may be written in the following way:

$$\Sigma\tau_O = \text{zero} \quad \Leftrightarrow \quad \omega_O \text{ is constant}$$

where the O subscript indicates the axis of rotation. During the first half of lecture 14, I will do several examples for which $\Sigma\tau = 0$.

Now on to the 2nd Law for rotation. Newton's 2nd Law is $\Sigma\mathbf{F} = m\mathbf{a}$. The net force is the cause of the acceleration, and mass is the resistance to acceleration, also known as (AKA) inertia. We already know that torque (specifically a nonzero net torque) is the cause of angular acceleration, but what is the right definition for the resistance to angular acceleration, i.e. ROTATIONAL INERTIA. The easiest way to figure this out is to think about kinetic energy, specifically the KE of a rigid system of particles rotating about a fixed axis O . I have drawn a very simple rigidly-connected system below. It consists of three particles which are connected by rigid massless sticks to one another and to fixed axis O , which is \perp to the page. The massless sticks are drawn with dotted lines; their only function is to make the system rigid, so that it can only rotate about the fixed axis.



Given that the system is rotating about O with angular velocity ω , our job is to calculate the KE of the rotating system. Since the KE for a translating particle is defined by $KE \equiv \frac{1}{2}mv^2$, or $\frac{1}{2}$ the inertia times the speed squared, we will assume that the "right" DEF of rotational inertia will result in a similar rule for the KE of our rigid rotating object, namely $\frac{1}{2}$ the rotational inertia

times the angular speed squared. Our strategy is to use the DEF of $KE \equiv \frac{1}{2}mv^2$ to calculate the KE of this system and then write our result as $\frac{1}{2}$ times "something" times the angular speed squared; the "something" must then be the rotational inertia. Here we go:

$$\begin{array}{ll}
 \text{DEF of } KE \text{ applied to this system} & KE_{sys} = \sum_i \left(\frac{1}{2} m_i v_i^2 \right) \\
 \text{use } v_i = r_i \omega & = \sum_i \left(\frac{1}{2} m_i r_i^2 \omega^2 \right) \\
 \text{factor out the } \frac{1}{2} \text{ and the } \omega^2 \text{ found in every term} & = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2
 \end{array}$$

So the "something" turns out to be $\sum_i m_i r_i^2$; this must be rotational inertia:

DEF The ROTATIONAL INERTIA (I) about axis O for a rigidly-connected system of particles is

$$I_O \equiv \sum_i m_i r_{iO}^2 \quad \text{units are kg} \cdot \text{m}^2$$

where r_{iO} is the distance from axis O to m_i . I is a positive-only scalar.

This DEF tells us that if a particle is twice as far from the axis as another particle of equal mass, then its contribution to the rotational inertia of the system is FOUR times as large. Check your understanding of this DEF by doing Questions 1-2 from Self-Assessment Test 9.2. We will have PRS questions for the DEF of rotational inertia during lecture.

Of course, most of the rigid objects in which we are interested are collections of huge numbers of particles (the number of atoms in a solid object is something like 10^{23} or 10^{24}), too many to even consider calculating the rotational inertia by using the DEF and doing a sum over all the particles. To calculate I for a solid object like a wheel, or a meter stick, requires integral calculus, which is beyond the scope of this course. Since we will want to do problems with objects of this type, the rotational inertias of these

kinds of objects, for one or two possible locations of a fixed axis, have been calculated for you; you will find the results in Table 9.1 on page 244 of your textbook. Please turn to that page before reading the rest of this paragraph. It is VERY IMPORTANT to remember that rotational inertia depends on the choice of axis; it usually makes no sense to speak of "the rotational inertia of an object" without also specifying the orientation of the fixed axis. Use Table 9.1 for HW problems, and you will be given any expressions for I that you need for quizzes or test problems, with one significant exception, that of a hoop of mass M , which is the first entry in the table. The book has been a bit sloppy here, for the axis is drawn in the figure, but it has not been specified in the caption -- it should read "axis \perp to plane of hoop and passing through center" (you have to imagine the axis being connected to the hoop by massless spokes). This one you are expected to know because it is so trivial to get from the DEF of I ; since all of the mass of the hoop is at the same radius R , the DEF $I_O \equiv \sum_i m_i r_{iO}^2$ immediately gives $I = (\sum_i m_i)R^2 = MR^2$.

A table similar to Table 9.1 can be found in any introductory physics book. But often we want to rotate an object about an axis which is not listed in the table. There is a simple theorem that helps us in such cases, as long as we know the rotational inertia of the object about an axis through the center of mass (CM) of the object (as in entries 1-3, 5, 7, and 8 in Table 9.1). This theorem will appear on your equation sheet, and is called the PARALLEL-AXIS THEOREM.

$$\text{Parallel-Axis Theorem} \qquad I_O = I_{CM} + Mh^2$$

where I_{CM} is the rotational inertia of the object about an axis through the CM,

I_O is the rotational inertia about axis O , which must be \parallel to the CM axis,

M is the mass of the object, and

h is the distance from axis O to the axis through the CM.

As an example of using the Parallel-Axis Theorem, I will show you how to get entry 4 in Table 9.1 from entry 3 (of course you would only use the Theorem if the rotational inertia you needed was not listed in the table). We can use the Parallel-Axis Theorem because the axis in entry 4 is parallel to the axis in entry 3, and because the axis in entry 3 passes through the CM of the rod.

Parallel - Axis Theorem I_{CM} is given in entry 3 $h = \frac{L}{2}$ $\frac{1}{12} + \frac{1}{4} = \frac{1}{3}$	$I_O = I_{CM} + Mh^2$ $= \frac{1}{12}ML^2 + Mh^2$ $= \frac{1}{12}ML^2 + M\frac{L^2}{4}$ $= \frac{1}{3}ML^2$
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One of our PRS questions will be to apply the Parallel-Axis Theorem to a similar situation. For interested students, a proof of the Theorem is at

<http://www.public.asu.edu/~gbadams/sum00/parallelaxis.html>

This completes our discussion of rotational inertia, the resistance to angular acceleration. We now have Newton's 2nd Law for rotation, namely $\Sigma\tau_O = I_O\alpha$; τ_O and α are 1D vectors and I_O is a positive-only scalar -- the O subscript emphasizes that the torque and the rotational inertia must be with respect to the same axis. It is important to remember that the 2nd Law for rotation is not a new fundamental rule (therefore it WILL appear on your equation sheet); instead it arises from applying our knowledge of Newton's Laws to a rigid rotating system of particles. In fact, we still need to check that it works, that our DEF of I is, in fact, the correct one. We will start with $I_O\alpha$, plug in the DEF of I , and see what we get:

DEF of I distributive property and $a_{t,i} = r_i\alpha$ 2nd Law for tangential direction, $\Sigma F_{t,i} = m_i a_{t,i}$	$I_O\alpha \equiv \left(\sum_i m_i r_{iO}^2\right)\alpha$ $= \sum_i (m_i r_{iO} a_{t,i})$ $= \sum_i (\Sigma F_{t,i} r_{iO})$
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The next step is not so simple, but we have seen a double sum like this before (when studying the acceleration of the CM). We have the sum over all particles, and for each particle we have the net tangential force on that particle times the distance from axis O to that particle. As we have seen before, in a double sum of this type, all internal forces drop out, leaving only external forces. Particles which have no external forces acting on them will then contribute nothing to the sum. So the r_{iO} 's will become the distances from the axis O to each of the external forces, like so (continuing from above):

$$\begin{aligned}
 I_O \alpha &= \sum_i (\Sigma F_{t,i} r_{iO}) \\
 &= \sum_{\text{ext } \mathbf{F}'s} (F_t r_O) \\
 &= \sum \tau_O
 \end{aligned}$$

by the argument above

and finally, by the DEF of $\tau_F \equiv (F_t)r$

where, in the final line, the sum of torques is understood to include only torques due to external forces. So, the DEF of I that we got by considering KE is the right DEF for rotational inertia, the resistance to angular acceleration. In lecture, we will do a couple of examples using the 2nd Law for rotation. Check your understanding (CYU) of this rule by trying CYU 3 in Chapter 9 on page 248 of your textbook.