

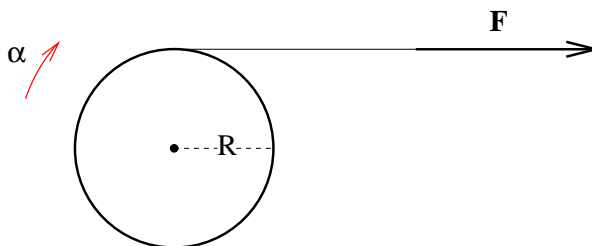
In lecture 14, we discussed the TORQUE perspective on the DYNAMICS of rotation. We will begin lecture 15 with a final example using the 2nd Law for rotation; that example will include a couple of PRS questions. After that example, we will move on to the ENERGY and MOMENTUM perspectives for rotation. We will find that those perspectives are at least as useful for the rotation of rigid bodies as they were for problems involving translating particles.

A discussion of the ENERGY perspective must begin with work. What is the work done by a constant torque acting over some angular displacement of our rigid rotating object? We do not have to make a new DEF. Instead, we can figure it out from the DEF of the work done by a constant force; all we have to do is to apply that DEF to the case of an external force applied to our rigid system of particles rotating about a fixed axis. Instead of doing the proof, I will give you the result, and then do a simple example that hopefully will convince you that the result must be correct. The result we get for the work done by a constant torque (τ) during an angular displacement ($\Delta\theta$) is

$$W_\tau = \tau\Delta\theta \quad (1)$$

Both τ and $\Delta\theta$ are 1D vectors. The work W_τ is of course a signed scalar; if W_τ is positive, then there will be more KE , and if W_τ is negative, there will be less KE . τ is in N·m and $\Delta\theta$ is in radians, and N·m·radians is just N·m, which we now call Joules, since we are now dealing with energy. Eq. (1) will appear on your equation sheet. You should note its similarity to the DEF of work done by a constant force; in both cases the work done is essentially the cause of acceleration times the displacement.

Now for the example. Consider a spool (on a fixed horizontal axis) with a string wrapped around it. The spool is being rotationally accelerated by pulling on the string with a constant force \mathbf{F} . The situation is drawn below:



If we pull $|\Delta x|$ meters of string off the spool, then the magnitude of the displacement of the end of the string is $|\Delta x|$ meters, and the work done by \mathbf{F} on the string is thus $F|\Delta x|$ Joules. But if we are using our typical "massless" string, then the string cannot accumulate any KE , so all of the KE created must be transferred to the spool. So the work done ON the spool BY THE TORQUE from the tension T in the string must equal the work done by the force \mathbf{F} on the string. By calculating the work done by the torque, and algebraically showing that it equals $F|\Delta x|$, we will thus confirm the correctness of Eq. (1):

$$\text{from Eq. (1)} \qquad W_\tau = \tau\Delta\theta$$

$$\text{torque on spool by string is } \tau_T = -TR \text{ (neg. for CW)} \qquad = -TR\Delta\theta$$

$$\text{use DEF of } \Delta\theta \equiv \frac{s}{r} \text{ (let } s \text{ be at the rim of the spool)} \qquad = -TR\frac{s}{R}$$

$$\text{and since } T = F \text{ and } s = -|\Delta x| \text{ (neg. for CW)} \Rightarrow \qquad = F|\Delta x|$$

I hope this example convinces you that $W_\tau = \tau\Delta\theta$ must be the correct expression for the work done by a constant torque.

In required reading 14, we have already shown that the KE of our system of particles, rigidly rotating about a fixed axis O , is given by

$$KE_O = \frac{1}{2}I_O\omega^2 \qquad (2)$$

This equation will appear on your equation sheet. You should note its similarity to the DEF of KE for a particle; in both cases KE is $\frac{1}{2}$ the inertia times the speed squared. But you should always remember that this is NOT a new

DEF; rather, we worked out Eq. (2) by applying our original DEF of KE to our system of rigidly-connected particles.

Next, does the Work-Kinetic Energy (WK) Theorem hold for our rigid object rotating about a fixed axis? The answer is yes. It is worth doing the little proof; we must calculate the work done by a net torque $\Sigma\tau$ during an angular displacement $\Delta\theta$ and show that it equals the change in KE of our system:

$$\begin{aligned}
 \text{DEF of work done by a constant torque for net work} & \quad \Sigma W \equiv \Sigma\tau_O \Delta\theta \\
 \text{2nd law for rotation } \Sigma\tau_O = I_O\alpha & \quad = I_O\alpha\Delta\theta \\
 \text{use } \omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta & \quad = I_O\left(\frac{1}{2}\right)(\omega_f^2 - \omega_i^2) \\
 \text{algebra and Eq. (2)} & \quad = KE_f - KE_i
 \end{aligned}$$

So the WK Theorem, $\Sigma W = \Delta KE$, still holds. Also, since our Energy Conservation Laws (for mechanical energy and for total energy) were all derived from the WK Theorem, all of our Energy Conservation Laws hold as well. During lecture, we will do multiple examples in which we apply Energy Conservation to problems involving rotation.

Even though we will not do any problems involving POWER and rotation, I want to mention it here. This idea is not discussed at all in our book, and yet most engines are rotary in the sense that they involve torques, and all engines are limited by their maximum power output. The DEF of average power is still the same, $\bar{P}_\tau \equiv \frac{W_\tau}{\Delta t}$; i.e. average power is the average rate of doing work. In required reading 11, we showed that the instantaneous power due to a force \mathbf{F} can be gotten from $P_F(t) = (F_v)v(t)$ where $v(t)$ is the speed of the object to which the force is being applied. It should not be a surprise, then, that the instantaneous power due to a torque τ can be gotten from $P_\tau(t) = \tau\omega(t)$ where τ and $\omega(t)$ are now both 1D vectors (note that in both cases the instantaneous power is roughly gotten by cause of acceleration times speed). So if the maximum power output of your engine is P_{max} Watts (you can always convert to

horsepower), then for a given amount of torque $|\tau|$, the maximum angular speed of your engine is $(P_{max}/|\tau|)$ rad/s (you can always convert to rpm). This concludes our discussion of POWER due to torque, for which you are not responsible in any way in this class.

We do want to consider the possibility that our rigid system of particles is not attached to a fixed axis; in other words, what if our system of particles is translating at the same time that it is rotating? We want to consider this possibility because we want to do at least a few problems with rolling objects, which are obviously translating and rotating at the same time. We can work out the KE of such a rigid system of particles by using the DEF of KE for each particle in the system, i.e. $KE_i \equiv \frac{1}{2}m_i v_i^2$, and using $KE_{sys} = \sum_i KE_i$. The calculation would be similar to that we did in reading 14, when we worked out the KE of a rigid system rotating about a fixed axis. But the algebra here would be a bit more complex, and the result is not really surprising, so once again I will just give you the result and then do an example that, hopefully, will convince you that the result must be correct. In general, for any system of rigidly-connected particles, the total KE is given by

$$KE_{sys} = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 \quad (3)$$

where M is the total mass of the system,

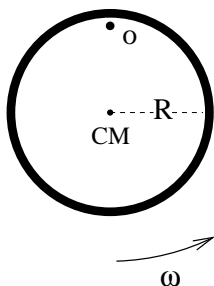
V_{CM} is the speed of the CM of the system,

I_{CM} is the rotational inertia about an axis through the CM, and

ω is the angular speed about that same axis.

This equation will appear on your equation sheet. I will now do an example in which the rigid system is rotating about a fixed axis, but that axis does not pass through the CM; so, we should be able to use either Eq. (2) or Eq. (3) to calculate the KE . By seeing that we get the same result either way, I hope

you will be convinced that Eq. (3) must be correct for a system that is both translating and rotating. The example is of a hoop of mass M which is being rotated about a fixed axis next to the rim and \perp to the plane of the hoop:



I_{CM} , for an axis \perp to the plane of the hoop, is easy for a hoop, it is just MR^2 . We get I_O from the parallel-axis theorem (we did this in lecture 14); it is $2MR^2$. So Eq. (2) yields

$$KE_O = \frac{1}{2}I_O\omega^2 = \frac{1}{2}(2MR^2)\omega^2 = MR^2\omega^2$$

If we want to use Eq. (3) instead, we need V_{CM} . Since the CM is going in a circle with angular velocity ω , and since it is a distance R from the axis, $V_{CM} = R|\omega|$. So Eq. (3) yields

$$KE_{sys} = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}(MR^2)\omega^2 = MR^2\omega^2$$

which is the same as we got by using Eq. (2). So, for this case we could use either Eq. (2) or Eq. (3) to get the KE of this system of particles; but, usually we use the more complex equation, Eq. (3), only when there is no fixed axis, as in the important case of rolling without slipping. I will do a demonstration and example of rolling without slipping in lecture in which we use both Eq. (3) and the Conservation of Mechanical Energy.

Our final topic in ROTATIONAL DYNAMICS is the MOMENTUM perspective.

Today, we will just define what we mean by ANGULAR MOMENTUM, and we will work out the angular momentum for our rigid system of particles rotating about a

fixed axis. During the next lecture, we will do lots of examples using the CONSERVATION OF ANGULAR MOMENTUM.

So far, during our lessons on rotational dynamics, we have had two new DEFs, the DEF of torque and the DEF of rotational inertia. We now need to define angular momentum. To make the right DEF, we need to think about the momentum perspective on the 2nd Law. We learned in required reading 12 that, when an impulsive net force is not constant, it is useful to rewrite the 2nd Law in the following form: $\Sigma \bar{\mathbf{F}} = \Delta \mathbf{p} / \Delta t$. We want to do the same thing for an impulsive net torque and angular momentum, in other words, we want $\Sigma \bar{\tau} = \Delta L / \Delta t$ where L stands for angular momentum. So we see that we can define angular momentum L by analogy; L must have the same relationship to \mathbf{p} that τ has to \mathbf{F} . All we have to do is to copy down the DEF of torque from required reading 13, and change F in that DEF to p .

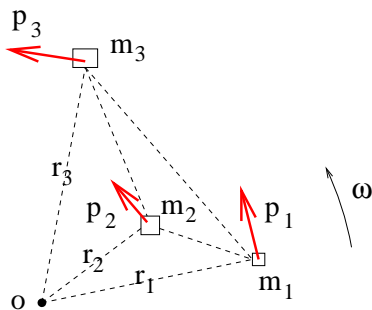
DEF The ANGULAR MOMENTUM (L) about axis O due to momentum \mathbf{p} occurring at a distance r from O is

$$L_p \equiv (p_t)r \quad \text{units are kg} \cdot \text{m}^2/\text{s}$$

where p_t is the tangential component of \mathbf{p} . L is thus a 1D vector with CCW usually taken as positive.

So just like every force with a nonzero tangential component can create a torque about an axis, every momentum with a nonzero tangential component creates an angular momentum about an axis. Calculating angular momentum is JUST EXACTLY like calculating torque; you can use the same three different ways ($|L| \equiv |p_t|r \equiv p(\sin\theta)r \equiv p\ell$). We will primarily be concerned with the angular momentum of our rigid system of particles rotating about a fixed axis, so as an example lets calculate its angular momentum. I will use the same

system and picture that we used to calculate the rotational KE in reading 14, but this time I have also drawn in the momentum vectors for each particle; they are drawn in red. Note that, for this system of three masses connected by massless sticks and rotating about fixed axis O (\perp to the page), the momentum vectors are necessarily purely tangential; this will make our work easy.



We want to calculate $L_{sys} = \sum_i L_i$. To get each L_i we use our DEF of L :

$$L_i \equiv (p_{t,i})r_i = m_i(v_{t,i})r_i = m_i(r_i\omega)r_i = (m_i r_i^2)\omega$$

You should check each step above to make sure that you understand the reason for that step. We are now ready to calculate L_{sys} about fixed axis O :

$$L_{sys} = \sum_i L_i = \sum_i ((m_i r_i^2)\omega) = (\sum_i (m_i r_i^2))\omega = I_O \omega. \quad (4)$$

Once again, you should make sure that you understand the reason for each step. This result, which is usually written as $L_O = I_O \omega$ (to emphasize the axis on both sides of the equation), will appear on your equation sheet. Once again our result for the rigid rotating object turns out to be very similar to the analogous DEF for translating particles, the momentum is just the product of inertia and velocity. This should not be surprising by now; we have been able to make these analogies at every step in our study of rigid-body rotation. But you should always remember that this is NOT a definition (as is implied in our textbook); instead we worked this out using our DEFs and results for translating particles. At the end of lecture, you can expect a PRS question or two about the DEF of angular momentum and Eq. (4).