In lecture 15, you learned the DEF of ANGULAR MOMENTUM ($L_p \equiv (p_t)r$), and we worked out the angular momentum for our system of rigidly-connected particles which is rotating about a fixed axis ($L_O = I_O\omega$). Our DEF was chosen so that, just as $\Sigma\overline{\mathbf{F}} = \Delta\mathbf{p}/\Delta t$ for translation, $\Sigma\overline{\tau}_O = \Delta L_O/\Delta t$ for motion with respect to a fixed axis O (i.e. L_p is to $\mathbf{p}$ as τ_F is to $\mathbf{F}$).

When we first studied systems of particles, we learned that

$$\sum \mathbf{F}_{ext} = 0 \quad \Longleftrightarrow \quad \mathbf{p}_{sys} \text{ is constant},$$

This rule is called the CONSERVATION OF MOMENTUM, (Cof$\mathbf{p}$) and it still holds true for our rotating system of rigidly-connected particles. But it is usually not useful -- if the axis O doesn't go through the CM (as in my example using the hoop at the end of reading 15), then $\mathbf{F}_{ext} \neq 0$, i.e. there is some force at the axis which enables the CM to move in a circular path -- and if the axis does go through the CM, then the CM never moves so $\mathbf{p}_{sys}$ is always zero; either way, Cof$\mathbf{p}$ is not likely to help us in the analysis of such a system.

A rule more likely to be useful for rotational problems comes from the equation $\Sigma\overline{\tau}_O = \Delta L_O/\Delta t$. If $\Sigma\tau_{O,ext} = 0$ (the ext subscript makes it clear now that we are talking about torques due to EXTERNAL forces), then $L_{O,sys}$ cannot change (the sys subscript makes it clear now that we are talking about the angular momentum of a system of particles), where the O subscript on both sides indicates the fixed axis O. This rule is called

THE PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM.

For any system of N particles, and with respect to fixed axis O, the $\sum \tau_{O,ext}$ acting on the system is zero if and only if the total angular momentum of the system, $\sum_{i=1}^{N} L_{i,O}$, is constant. In particular, if the system is closed, so that there can be no external torques acting on the system, then the total system angular momentum is constant. Therefore, if the Universe is a closed system, then the total angular momentum of the Universe is constant.

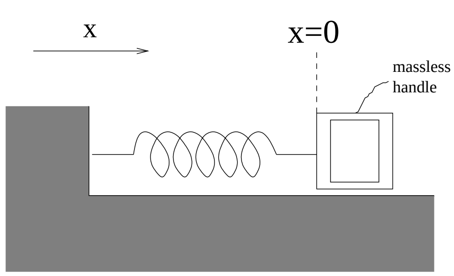
Conservation of Angular Momentum (CofL) is the third great conservation principle of classical physics, after Conservation of Energy (CofE) and Conservation of (Translational) Momentum (Cof$\mathbf{p}$). Here it is in symbols:

$$\sum \tau_{O,ext} = 0 \quad \Longleftrightarrow \quad L_{O,sys} \text{ is constant},$$

CofL is useful in a remarkable variety of situations. We will do between three and five examples during lecture.

The topic of CofL ends our study of ROTATIONAL DYNAMICS, in which we applied the principles of in Chapters 1-7 to problems involving rigid rotating objects and rolling without slipping. Now we will apply those same principles to the problem of motion in the presence of an ELASTIC or SPRINGLIKE force.

We begin by defining what is meant by an IDEAL SPRING. First, an ideal spring is frictionless, meaning it doesn't get hotter as it is being used (all real springs get a little hotter as you repeatedly stretch and compress them). Second, an ideal spring is massless, so it has no inertia and never accumulates any kinetic energy; also, it can be perfectly horizontal, as drawn below:



This spring is drawn in the relaxed position, which we will call x=0.

I have drawn the spring with a massless handle, that we can use to stretch or compress the spring (usually we will imagine a mass of some kind attached to the spring instead). But the most important quality of an ideal spring is that it is "linear" -- if you double the pushing (or pulling) force on the spring then you double the amount the spring is compressed (or stretched). (The word "linear" is chosen only because the graph of the stretching force versus displacement is a straight line.) The amount of stretch (or compression) of the spring away from its relaxed position is the DISPLACEMENT Δx (or Δy for a vertical spring) of the spring. To make our spring equations simpler, we usually consider the relaxed position of the spring to be at $x = 0$ (or $y = 0$); this allows us to write the displacement of the spring with a simple x (or y), instead of having to use Δx (or Δy) every time. Our book does this, and I will adopt this notation as well. Using the idea of the "linearity" of an ideal spring, we can define the spring constant of an ideal spring:

DEF The SPRING CONSTANT (k) of an ideal spring is given by

$$k \equiv \frac{F_{AP}}{x} \qquad \text{units are N/m or kg/s}^2$$

where F_{AP} is the force necessary to hold the spring (AT REST) at a displacement of x. In this DEF, both F_{AP} and x are 1D vectors. Since F_{AP} and x must always have the same direction (and thus the same sign) k is a positive-only scalar.

Saying that k is constant for any amount of stretch is equivalent to saying that the spring is "linear". This DEF is often written in the form $F_{AP,x} \equiv kx$, where the x subscript on $F_{AP,x}$ is there to remind you that F_{AP} is a 1D vector and not simply a magnitude. I will sometimes add the x subscript, but I will usually leave it off.

We will sometimes be interested in F_{AP}, but just as often, we will want to know the force exerted BY the stretched (or compressed) spring; we will denote this force by $F_{S,(x)}$ (the x subscript will usually be understood). In the situation described above, F_{AP} must be acting on the right side of the handle (either pushing or pulling) and F_S must be acting on the left side of the handle. Since the handle is at rest, Newton's 1st Law applies; and since there are no more x forces, we must have $F_S = -F_{AP}$ (remember that both symbols are 1D vectors). Therefore,

$$F_S \equiv -kx$$

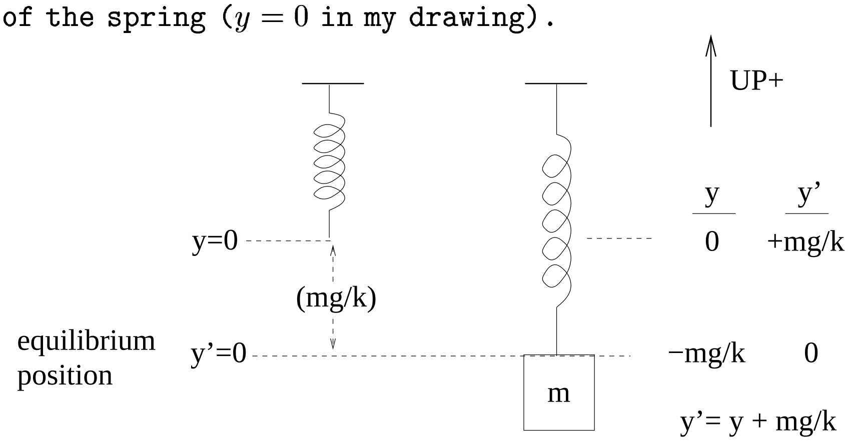
This equation can also be taken as the DEF of the spring constant, or either as the DEF of the ideal spring force.

Horizontal springs are convenient for problems, but in experiments (with all but very light and stiff springs) it is usually more convenient to use vertical springs. Let's do an example with a vertical spring. Imagine that I attach a 100 g mass to a relaxed vertical spring and then lower the mass slowly, until the magnitude of the spring force (which is upward) exactly matches the weight of the 100 g mass. I can then remove my hand and the mass will hang at rest at the end of the spring. When I do this experiment with a selected spring, that spring ends up being stretched by $|y|$ = 10 cm. Let's use this fact to calculate the spring constant k of that spring. Draw the FBD of the hanging mass (after my hand is removed) in the space I have provided below. To the right of the space I have applied the 2nd Law to the problem and solved for k.

$$\begin{aligned}
\Sigma F_y &= m a_y \\
+|F_S| - F_G &= 0 \\
+k|y| - mg &= 0 \\
\Rightarrow \quad k &= \frac{mg}{|y|} \\
&= \frac{(0.100\ \texttt{kg})(9.80\ \texttt{m/s}^2)}{(0.100\texttt{m})} \\
&= 9.8\ \texttt{N/m}
\end{aligned}$$

Make sure that you understand each step in my work above. If there is a step you do not understand, I will give you a chance to ask questions during lecture. Note that I have had to write $|F_S|$ for the magnitude of the spring force, since F_S is defined to be a 1D vector.

There is an important point to make about vertical springs before we leave this example behind. If we know the spring constant of the spring, then by rearranging our result from the bottom of previous page (k = $mg/|y|$), we can always get the amount of stretch $|y|$ in this experiment from the expression mg/k. So, as drawn below, the EQUILIBRIUM POSITION of the spring with the mass attached (i.e. the position at which $\Sigma F_y = 0$) is a distance mg/k below the relaxed position of the spring. I will now show you why, once we attach a mass to a vertical spring, we always choose to measure displacements from the equilibrium position ($y' = 0$ in my drawing) rather than from the relaxed position of the spring ($y = 0$ in my drawing).



How will a displacement measured from $y' = 0$ compare to a displacement measured from $y = 0$? To answer this question, I will choose UP as positive. From the picture, you can then see that when $y' = 0$ the value of y is $-mg/k$. Therefore, y = y' - mg/k, or y' = y + mg/k. Now multiply this last equation by $-k$:

$$-ky' = -ky - mg$$

$$= F_{S,y} + F_{G,y}$$

$$= \sum F_y$$

where I have written the forces with y subscripts to make sure that you realize that I mean 1D vectors and not magnitudes of forces. The last equation, $-ky' = \sum F_y$, is the key to understanding the vertical spring. The spring force is $-ky$; but $-ky'$ is the TOTAL FORCE acting on the hanging mass (or more accurately the sum of the spring force and the force of gravity -- it is the total force only if there are no additional forces acting on the hanging mass, like the force of contact by a hand, etc.). This is a wonderful result that makes working with vertical springs much easier that it might have been; you just have to remember to measure displacements from the equilibrium position of the hanging mass, and not from the relaxed position of the spring.

Why is it worthwhile to study springs? The answer is, because any solid material is springlike for small distortions; i.e. real materials act like springs as long as they are not compressed or stretched by large amounts. In fact, you may consider the atoms in a solid to be connected by tiny springs which are very light and very stiff; there is a nice illustration in your textbook on page 282. In order to talk about the springlike behavior of a real material, we need three definitions:

DEF STRESS is the force per unit area that cause the object to change shape; i.e.

$$\text{stress} \equiv \frac{\text{magnitude of the stretching or compressing force}}{\text{area over which that force is applied}} \qquad \text{units are N/m}^2$$

Stress is a positive-only scalar. The stretching or compressing forces must of course be applied equally to opposite sides of the object.

DEF STRAIN is the relative change in shape of the object which results from the applied stress, i.e. where L stands for length and V stands for volume,

$$\text{strain} \equiv \frac{|\Delta L|}{L_0} \;\text{ or }\; \frac{|\Delta V|}{V_0} \qquad \text{no dimensions or units}$$

where L_0 and V_0 are the original values of length or volume before the stress was applied. Strain is a positive-only scalar.

The equivalent of $F_{AP,x} \equiv kx$ for materials is called HOOKE'S LAW, which says that stress is proportional to strain, i.e.

$$\text{stress} \propto \text{strain}$$

$$\text{or} \qquad \text{stress} = (\text{modulus}) \cdot (\text{strain})$$

where "modulus" is just a constant which indicates the size of the effect (i.e. a number that tells you how much stress is required to achieve a given amount of strain). Our third DEF is for the modulus in the case of 1D deformation (either simple compression or elongation):

DEF The YOUNG'S MODULUS (Y) of a given material is given by

$$Y \equiv \frac{\text{stress}}{\text{strain}} \qquad \text{units are N/m}^2$$

where the strain appearing in the denominator is due to a one dimensional deformation produced by the stress appearing in the numerator. The forces producing the stress are collinear with the resulting deformation.

To make sure that we understand these three DEFs, lets do a simple example. Find the Young's modulus of steel (Y_{steel}) from the following experimental data. A steel rod suspended from a clamp in the ceiling was measured to have an original length of 3.0 m and a cross-sectional area of 0.2 cm^2 (i.e. a radius of 2.52 mm). The lower end of the rod had been bent into a hook shape so that masses could be easily attached. When a 500 kg mass was attached to the end of the rod, the rod was found to stretch by 4.0 mm. To find Y_{steel}, we will use the DEF of Y; the stretching force is the force ON the rod by the 500 kg mass. You should draw a picture of the situation, and make a FBD for the hanging mass and for the steel rod. My calculation of Y_{steel} is at the top of the next page:

$$
\begin{aligned}
Y &\equiv \frac{\texttt{stress}}{\texttt{strain}} \\
&= \frac{(|\mathbf{F}_{AP}|/\texttt{area})}{(|\Delta L|/L_0)} \\
&= \frac{((500\ \texttt{kg})(9.8\ \texttt{m/s}^2)/0.00002\ \texttt{m}^2)}{(0.004\ \texttt{m}/3.0\ \texttt{m})} \\
&= 1.84 \times 10^{11}\ \texttt{N/m}^2
\end{aligned}
$$

Notice that Y_{steel} turns out to be a huge number; this means that a huge stress is required to get even a little strain. On page 283, our textbook has a table of Young's Modulus for various materials. Note that the value for Y_{steel} in that table (2.0×10^{11} N/m^2) is slightly different that the value we got in my experiment; Young's modulus for a compound material like steel, brick, or concrete can depend on the manufacturing process of that material.

By rearranging the variables in Hooke's Law for 1D deformation, we can make it take almost exactly the form $F_{AP,x} \equiv kx$:

$$
\texttt{stress} = Y \cdot (\texttt{strain})
$$

$$
(|\mathbf{F}_{AP}|/\texttt{area}) = Y(|\Delta L|/L_0)
$$

$$
\Rightarrow \qquad |\mathbf{F}_{AP}| = (\frac{Y \cdot (\texttt{area})}{L_0})|\Delta L|
$$

This rearrangement allows you to see directly how the spring constant for a solid rod would be related to the Young's modulus of the material from which the rod is made, i.e. $k = (Y \cdot (\texttt{area})/L_0)$. In other words, while the Young's modulus is a property of the MATERIAL, the spring constant depends on the shape of the object made from the material.

This completes our discussion of the stresses and strains in real materials. Our book goes on to discuss both shear and volume deformations, but you will not be responsible for those topics in any way. However, shear deformations can be important in construction and in architecture, and both types of deformations are sometimes covered by the MCAT's, so you may wish to do the reading in the book.