

In reading 16, you learned the DEF of the SPRING CONSTANT or, equivalently, the DEF of an IDEAL SPRING. In lecture 17, we will begin by studying the motion of a mass on a spring, which is called SIMPLE HARMONIC MOTION (SHM). Then we will do examples with the ENERGY OF SHM, i.e. the energy of a moving mass-spring system. The ENERGY OF SHM is the main topic of this required reading.

Because a spring can be used to store energy for later use, we expect to be able to define a potential energy for a spring. We need to start our study of the energy of SHM by figuring out what is meant by "the POTENTIAL ENERGY of a spring". First you need to make sure that you understand the potential energy due to gravity (reread that part of required reading 10 if necessary). Recall that the DEF of  $\Delta PE_G$  is  $\Delta PE_G \equiv -W_G$ , and that potential energy is always a property of a system of two (or more) objects. Our DEF for the CHANGE IN POTENTIAL ENERGY OF A SPRING ( $S$ ) will have the same form, i.e.  $\Delta PE_S \equiv -W_S$ , and the system of objects in this case will be the spring and the two objects to which the ends of the spring are connected (usually we think of one end as being fixed to a wall or a ceiling, but both ends may be connected to moveable objects). Here is the formal DEF:

DEF The CHANGE IN SPRING POTENTIAL ENERGY ( $\Delta PE_S$ ) for a system of two objects connected by a spring, as the spring moves between two configurations denoted by  $\mathbf{r}_i$  and  $\mathbf{r}_f$ , is (the  $\mathbf{r}_i \rightarrow \mathbf{r}_f$  subscript is usually omitted)

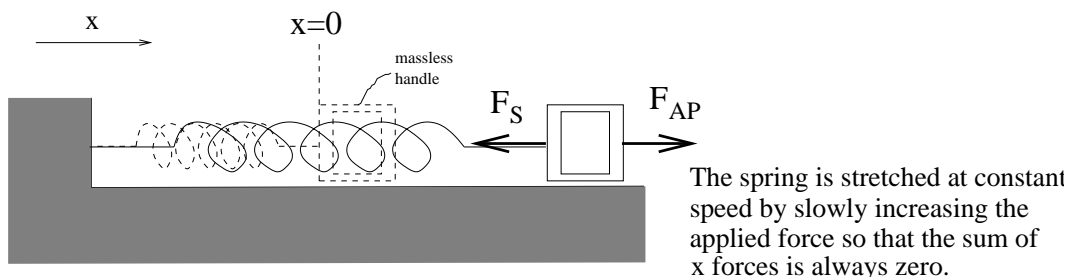
$$\Delta PE_{S,\mathbf{r}_i \rightarrow \mathbf{r}_f} \equiv -W_S \quad (\text{units are Joules})$$

where  $W_S$  is the work done by the spring on the objects as the configuration changes from  $\mathbf{r}_i$  to  $\mathbf{r}_f$ .  $\Delta PE_S$  is a signed scalar.

This DEF may seem a bit vague; the vagueness arises because we want to include the possibility that both ends of the spring are connected to moveable objects. Usually, one end of the spring is fixed; in that case,  $\mathbf{r}_i$  and  $\mathbf{r}_f$  simply

indicate the initial and final position of the moveable object -- that is the case we will deal with here.

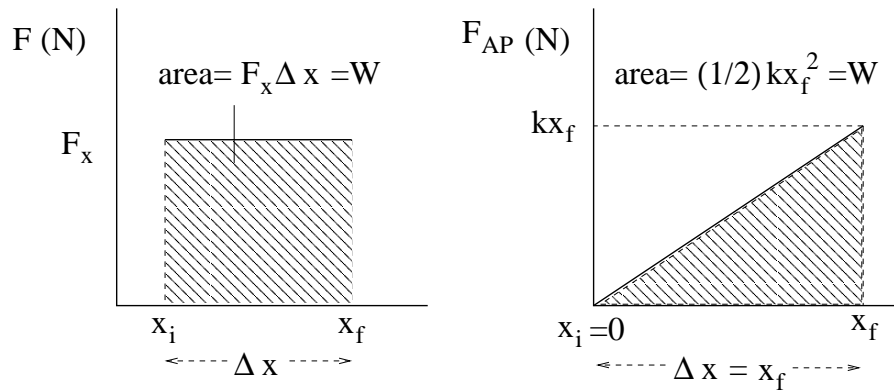
Since only the change in potential energy is defined, we must select some position for which the potential energy will be set to zero. Let's consider first a horizontal spring, then at the end of our discussion, we will relate our result to a vertical spring.



I have again drawn the horizontal spring with a massless handle. The two objects to which the spring is connected are the handle and the wall. It makes most sense (and our work will be easiest) if we choose  $PE_S = 0$  when the spring is in its relaxed position, neither compressed nor stretched. In my drawing, the relaxed spring is drawn with a dotted line, and the right-hand end of the relaxed spring is located at  $x = 0$ . To calculate  $PE_S$  for any other configuration of the spring, we need to use the DEF of  $\Delta PE_S$ , i.e. calculate  $W_S$  for a displacement giving that configuration, and take the negative of our result. My strategy will be to stretch the spring by applying a VARIABLE rightward pulling force  $F_{AP}$  which is always equal to the leftward pulling force of the stretched spring  $F_S$ , so that the handle moves to the right at a constant speed. Then, since by Newton's 1st Law (or the 2nd Law with  $a = 0$ ) the two forces will be equal and opposite, and since the displacement will be the same for both forces,  $W_{F_{AP}} = -W_S$ . Finally, since  $\Delta PE_S \equiv -W_S$ , therefore  $\Delta PE_S = W_{F_{AP}}$ . Before going any further, make sure you have understood this argument (and my drawing) by deciding which of the three changes in energy,

i.e. which of  $W_{F_{AP}}$ ,  $W_S$ , and  $\Delta PE_S$ , are positive and which are negative. You will find the answers at the bottom of the last page of this reading.

The only problem with my strategy is that  $F_{AP}$  is NOT a constant force, so I can't use the DEF of work that we know and love, i.e.  $W_F \equiv (F_{\Delta r})\Delta r$ . We will solve this problem with the two graphs drawn below:

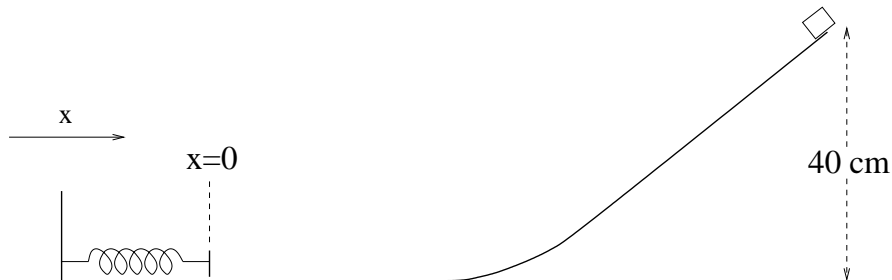


On the left, I have graphed force in Newtons vs. position in meters for a constant rightward  $x$  force  $F_x$  during a rightward displacement  $\Delta x$  from  $x_i$  to  $x_f$  (for example, imagine  $F_x$  being used to displace a box from  $x_i$  to  $x_f$ ). The "area" under this line will have units of N·m and can be gotten by  $F_x \Delta x$ ; in other words, that "area" is equal to the work done by  $F_x$  in units of Joules. THIS strategy for calculating work is always valid, whether the force is constant or not. On the right, my graph of force vs. position is for the variable  $x$  force  $F_{AP,(x)}$  in my drawing. The maximum value of  $F_{AP}$  occurs at the maximum stretch of  $x_f$ , namely  $kx_f$ . The work done by  $F_{AP}$  is the "area" of the triangle under the line. The area of a triangle is one-half the base times the height, so that  $W_{F_{AP}} = \frac{1}{2} kx_f^2$ . Therefore, at displacement  $x_f$ ,  $PE_S = \frac{1}{2} kx_f^2$ ; similarly, at a general displacement  $x$ ,  $PE_S = \frac{1}{2} kx^2$ . This equation will appear on your equation sheet. While we found this equation by considering a stretch, it is equally valid for a compression (since the  $x$  is squared, the sign of  $x$  makes no difference). Note that  $PE_S$  can never be negative; this is

a consequence of choosing  $PE_S = 0$  when the spring is relaxed.

So now we have two kinds of potential energy,  $PE_G$  and  $PE_S$ . The total mechanical energy of a system is now  $E_{sys} \equiv KE_{sys} + PE_G + PE_S$ . Since our "ideal springs" don't get hotter when used, they conserve mechanical energy just as well as does gravity. Therefore, mechanical energy is conserved whenever only gravity and/or spring forces are doing any work.

Before discussing the energy of a vertical spring, here is an example of mechanical energy conservation using an ideal horizontal spring. The drawing below shows a frictionless ramp by means of which the 8.0 kg block can be "dropped" onto a horizontal spring. The spring constant is 784 N/m, and the block is dropped (from rest) from a height of 40 cm. This "frictionless" system forms a perpetual motion machine; the block will compress the spring and then be shot back up the ramp to a height of 40 cm, after which the action will repeat forever. Our task is to find the maximum compression distance.



There is no friction and the normal force on the block by the ramp does no work (since it is always  $\perp$  to the displacement), so only gravity and the spring force do any work. Therefore,  $E_i = E_f$ , where  $i$  stands for the instant the block is released at the top of the ramp, and  $f$  stands for the instant of maximum compression. At both instants,  $i$  and  $f$ , nothing is moving, so both  $KE_i$  and  $KE_f$  are zero. We will define  $PE_G = 0$  at the bottom of the ramp. Finally, at

instant  $i$ , the spring is in its relaxed position so that  $PE_S = 0$ . Therefore,

$$E_i = E_f$$

$$\cancel{KE}_i + PE_{G,i} + \cancel{PE}_S = \cancel{KE}_f + \cancel{PE}_G + PE_{S,f}$$

$$(8.0 \text{ kg})(9.8 \text{ m/s}^2)(0.40 \text{ m}) = \frac{1}{2}(784 \text{ N/m})x_f^2$$

$$\Rightarrow |x_f| = 0.283 \text{ m}$$

So the maximum compression distance is 28.3 cm.

Now let's consider the energy of a mass bobbing up and down at the end of a vertical spring. Conservation of mechanical energy is applicable, but it is a little complicated, since not only is the  $KE$  of the mass changing as it bobs up and down, but also both  $PE_S$  and  $PE_G$  are changing. We can simplify our work considerably by using what we learned about vertical springs in required reading 16; if we measure displacements from the equilibrium position of the hanging mass (i.e.  $y' = 0$  in the figure on page 5 of reading 16), then  $-ky'$  is the TOTAL FORCE acting on the hanging mass (or more accurately the sum of the spring force and the force of gravity). If  $-ky'$  is the total force, then  $\frac{1}{2}k(y')^2$  must be the TOTAL POTENTIAL ENERGY (the spring potential energy alone would be  $\frac{1}{2}ky^2$ , where  $y$  was measured from the relaxed position of the spring). So you can calculate the total mechanical energy from  $E = KE + PE_{total} = KE + \frac{1}{2}k(y')^2$ , as long as you remember to measure displacements from  $y' = 0$ . We will do an example using conservation of energy for a vertical spring during lecture; you will get a chance to practice with HW problem Ch. 10, #33. I must issue one warning. Don't try to use the shortcut if the mass becomes separated from the spring during the action of the problem; in that case your work must consider  $PE_S$  and  $PE_G$  separately as in my example above (and in HW problem Ch. 10, #28).

We will need a few new DEFs to talk about SHM. You will note that some of these DEFs are very similar to earlier DEFs for Uniform Circular Motion (UCM). That is not an accident, as you will see during lecture. Learn these DEFs.

DEFs The FREQUENCY ( $f$ ) of SHM is simply the number of vibrations (or cycles) per second; the units of  $f$  are cycles/s, which is given the name Hertz (Hz).

The PERIOD ( $T$ ) of SHM is the time required for one complete vibration (or cycle); the units of  $T$  are seconds. The maximum distance between the vibrating object and its equilibrium position (i.e. the position where  $\Sigma F$  on that object is zero) during the SHM is the AMPLITUDE ( $A$ ) of the motion.

SHM is an important topic because almost 100% of naturally occurring vibrations, if the amplitude of the vibrations is not too large, are SHM. For example, I will show during lecture that the vibratory motion of a pendulum, for small amplitude oscillations, is SHM. Of course, most naturally occurring vibrations involve some kind of resistive (i.e. frictional) force. Also, we are often concerned with an unwanted natural vibration in an engine (or in a frame connected to an engine) which is being driven by the motion of the engine. We will discuss these cases at the end of lecture. Learn these DEFs.

DEFs SHM with friction is called DAMPED HARMONIC MOTION; in this type of motion all of the mechanical energy is gradually turned into thermal energy and the motion ceases. In DRIVEN HARMONIC MOTION, energy is added to an vibrating system by an engine, animal, or human (an example is pushing a child on a swing); if the energy added per vibration is as great as the energy removed by resistive forces, then the motion can continue indefinitely. The NATURAL FREQUENCY of any oscillating system is the frequency with which it vibrates when there are no significant damping forces nor any driving forces. The addition of energy by a driving force is most efficient when the frequency with which the driving force is applied matches the natural frequency of the system; this situation is called RESONANCE (I will give demonstrations of resonance in lecture).

ANSWERS:  $W_{F_{AP}}$  and  $\Delta PE_S$  are positive,  $W_{F_S}$  is negative.