Proof of a Theorem for System Kinetic Energy

Consider a rigid system of particles of mass $M = \Sigma_i m_i$, translating with center-of-mass (CM) velocity $\vec{\mathbf{V}}_{CM}$ and at the same time rotating about an axis through the CM with angular speed ω. $\vec{\mathbf{V}}_{CM}$ is as measured by a stationary observer O. We wish to decompose the total kinetic energy of the system as measured by the stationary observer, $K = \Sigma_i \frac{1}{2} m_i v_{iO}^2$, into the translational kinetic energy of the CM plus the rotational kinetic energy of the system about an axis through the CM. We will need to use what we know about relative velocity to write the velocity of each particle as measured by O, $\vec{\mathbf{v}}_{iO}$, in terms of $\vec{\mathbf{V}}_{CM}$ plus the velocity of each particle with respect to the CM, $\vec{\mathbf{v}}_{iC}$.

$$\vec{\mathbf{v}}_{iO} = \vec{\mathbf{v}}_{iC} + \vec{\mathbf{V}}_{CM}.$$

We are going to need $v_{iO}^2 = \vec{\mathbf{v}}_{iO} \cdot \vec{\mathbf{v}}_{iO}$.

$$v_{iO}^2 = v_{iC}^2 + V_{CM}^2 + 2(\vec{\mathbf{v}}_{iC} \cdot \vec{\mathbf{V}}_{CM})$$

According to the CM observer (traveling along with the CM), the motion of the system of objects is pure rotation; therefore the speed of each particle with respect to the CM, v_{iC}, can be gotten from

$$v_{iC} = r_{iC}|\omega|.$$

Now we are ready to put it all together

$$K = \sum_i \frac{1}{2} m_i v_{iO}^2$$

sub for v_{iO}^2
$$= \sum_i \frac{1}{2} m_i v_{iC}^2 + \sum_i \frac{1}{2} m_i V_{CM}^2 + (\sum_i m_i \vec{\mathbf{v}}_{iC}) \cdot \vec{\mathbf{V}}_{CM}$$

sub for v_{iC} and factor
$$= \frac{1}{2}(\sum_i m_i r_{iC}^2)\omega^2 + \frac{1}{2}(\sum_i m_i)V_{CM}^2 + (\sum_i m_i \vec{\mathbf{v}}_{iC}) \cdot \vec{\mathbf{V}}_{CM}$$

The third term $(\Sigma_i m_i \vec{\mathbf{v}}_{iC}) \cdot \vec{\mathbf{V}}_{CM}$ is zero because the part in parentheses is just numerator for the velocity of the CM as measured in the CM frame of reference, and in the CM frame of reference the CM is not moving. The second term $\frac{1}{2}(\Sigma_i m_i)V_{CM}^2$ is just the translational kinetic energy of the CM. And finally, the first term $\frac{1}{2}(\Sigma_i m_i r_{iC}^2)\omega^2$ is just the rotational kinetic energy of the system about an axis through the CM, *i.e.* $\frac{1}{2} I_{CM}\omega^2$. We have thus proved

$$K_{SYS} = \frac{1}{2} M V_{CM}^2 + \frac{1}{2} I_{CM}\omega^2.$$