CSE 591: Theoretical aspects of CPS

ODEs, Physical system modeling and numerical solutions of ODEs

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Slides slightly adapted from lecture 1 of UPenn
ESE 601: Hybrid Systems by A. Julius

25 Jan. 2010

References

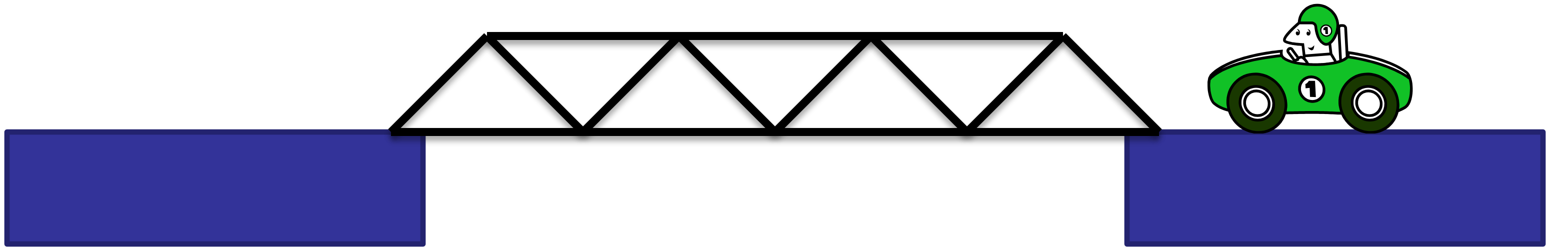
- Textbooks or lecture notes on linear systems or systems theory.
- T. Burton, Introduction to dynamic systems analysis, McGraw-Hill
- M. R. Spiegel: Advanced mathematics, Schaum
- J. Lygeros, Lecture notes on hybrid systems, 2006

Contents

- Modeling with differential equations
- Taxonomy of systems
- Solution to linear ODEs
- General solution concept
- Simulation and numerical methods
- State space representation
- Stability
- Reachability

Dynamic systems

- Algebra can describe static systems adequately



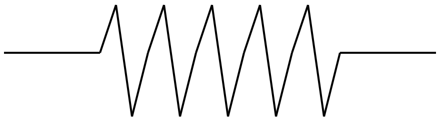
- Dynamic phenomena: We must relate rates of change

- Differential equation: An equation that relates an unknown function with its derivatives of various order

- Ordinary differential equation (ODE): 1 unknown function

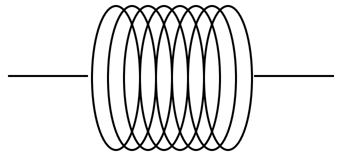
Physical systems

Resistor



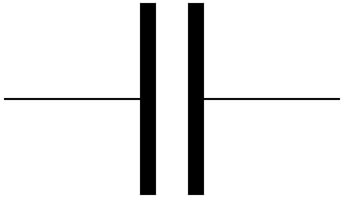
$$V(t) = R \cdot I(t)$$

Inductor



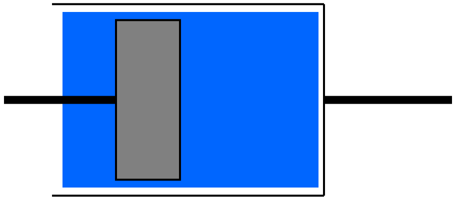
$$V(t) = L\frac{dI}{dt}$$

Capacitor



$$I(t) = C\frac{dV}{dt}$$

Damper

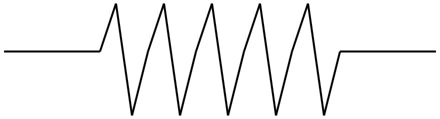


$$F(t) = b \cdot v(t)$$

Mass

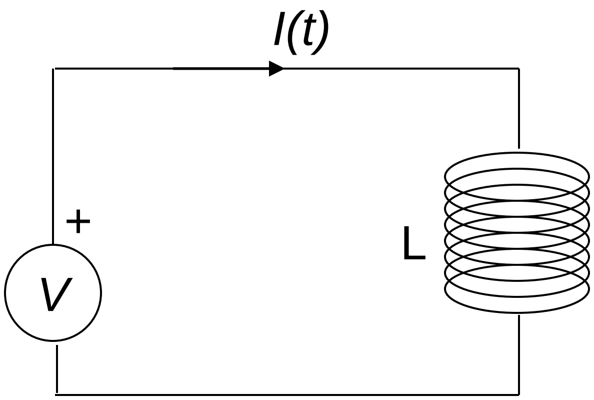
$$F(t) = M\frac{dv}{dt}$$

Spring

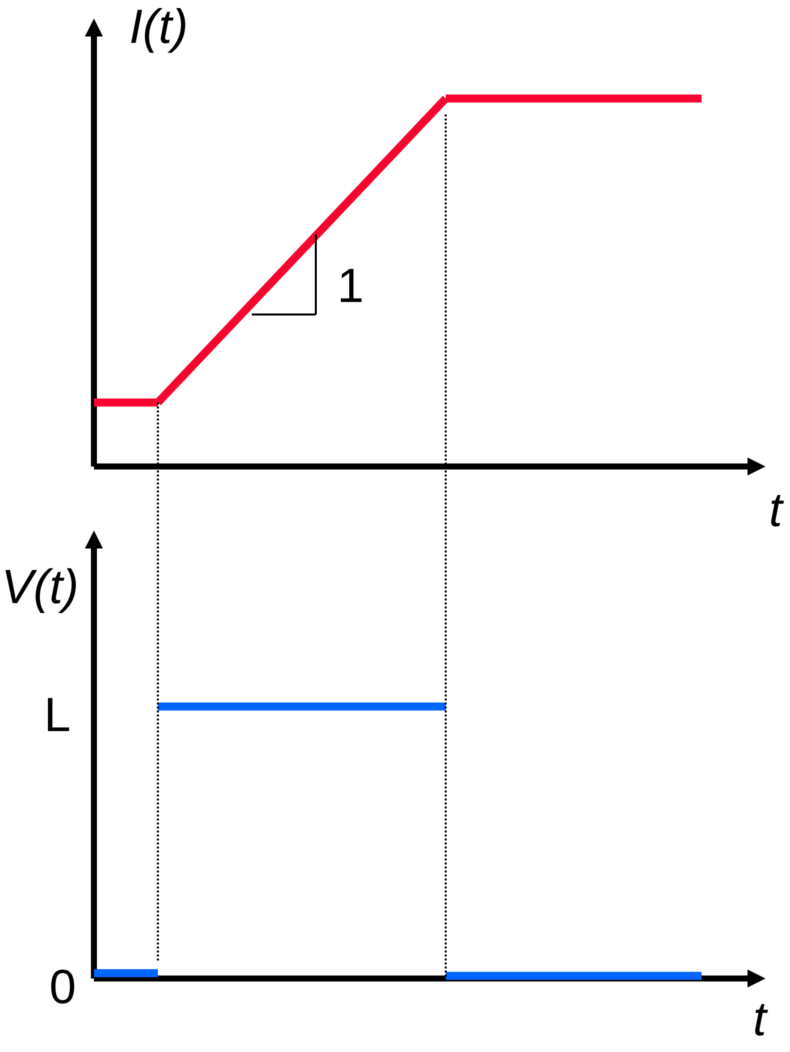


$$v(t) = \frac{1}{k}\frac{dF}{dt}$$

Electric circuit

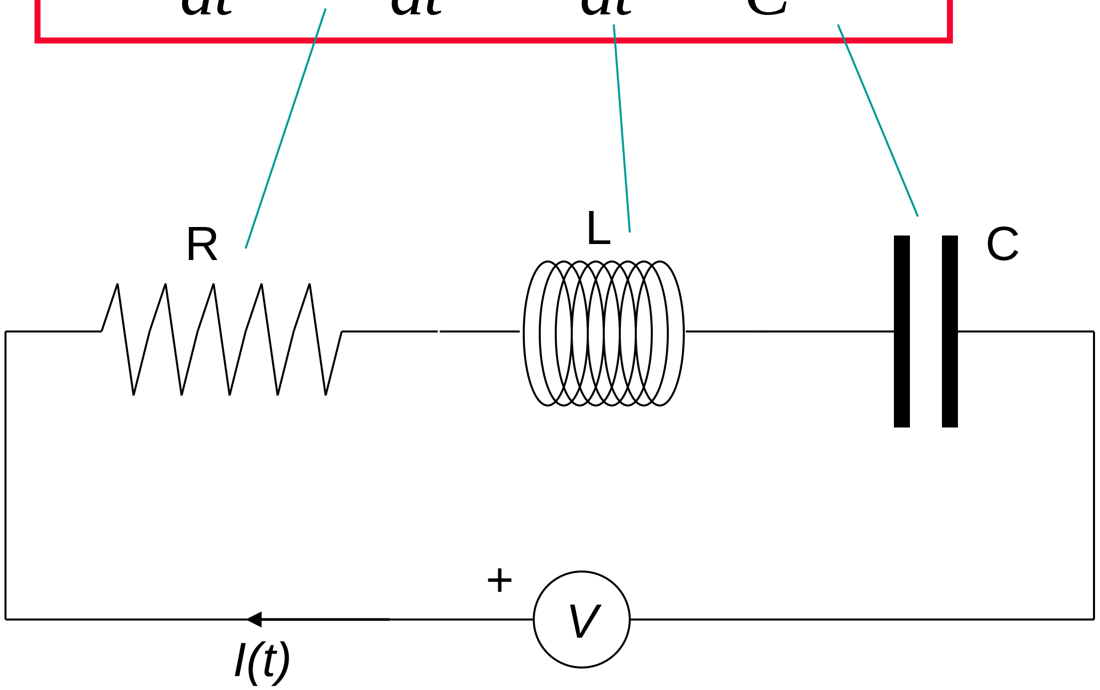


$$V(t) = L\frac{dI}{dt}$$

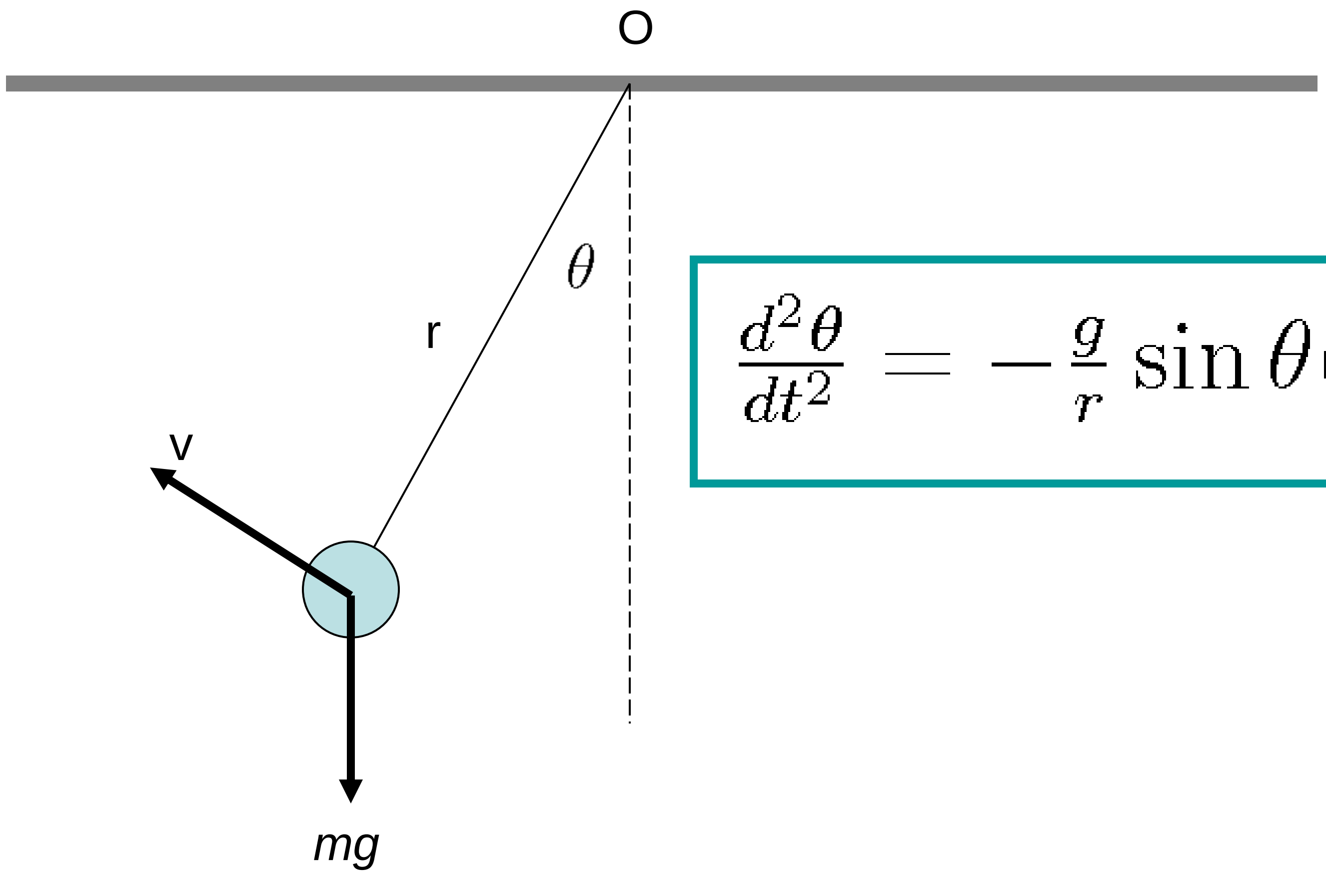


More electric circuit

$$\frac{dV}{dt} = R\frac{dI}{dt} + L\frac{d^2 I}{dt^2} + \frac{1}{C} I$$



A pendulum



$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta(t)$$

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Linear vs nonlinear

- Linear systems: if the set of solutions is closed under linear operation, i.e. scaling and addition.

$$\left\{ \begin{array}{c} V_1(t) = L\frac{dI_1}{dt} \\ V_2(t) = L\frac{dI_2}{dt} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \alpha V_1(t) = L\frac{d(\alpha I_1)}{dt} \\ V_1(t) + V_2(t) = L\frac{d(I_1+I_2)}{dt} \end{array} \right\}$$

- All the examples are linear systems, except for the pendulum.

$$\left\{ \frac{d^2\theta_1}{dt^2} = -\frac{g}{r}\sin\theta_1(t) \right\} \not\Rightarrow \left\{ \frac{d^2\alpha\theta_1}{dt^2} = -\frac{g}{r}\sin\alpha\theta_1(t) \right\}$$

Time invariant vs time varying

- Time invariant: the set of solutions is closed under time shifting.

$$\left\{\frac{d^2\theta_1}{dt^2} = -\frac{g}{r}\sin\theta_1(t)\right\} \Rightarrow \left\{\frac{d^2\theta_1(t-\Delta)}{dt^2} = -\frac{g}{r}\sin\theta_1(t-\Delta)\right\}$$

- Time varying: the set of solutions is not closed under time shifting.

$$\frac{dy}{dt} = tx(t)$$

Autonomous vs non-autonomous

- Autonomous systems: given the past of the signals, the future is already fixed.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta(t)$$

- Non-autonomous systems: there is possibility for input, non-determinism.

$$\frac{dV}{dt} = R\frac{dI}{dt} + L\frac{d^2 I}{dt^2} + \frac{1}{C}I$$

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Techniques for autonomous systems

First order linear ODE:

$$\frac{dx}{dt} = \gamma x,$$
$$x(t) = k \cdot e^{\gamma t}.$$

Higher order linear ODEs, denote the differential operator by s,

$$\{\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0\} \Rightarrow \{s^2 + 3s + 2 = 0\}$$

Take the roots of the characteristic polynomial.

$$x(t) = k_1 \cdot e^{-2t} + k_2 \cdot e^{-t}$$

Techniques for non-autonomous systems

Use Laplace transform,

$$\mathcal{L}(x(t)) = X(s) = \int_0^{\infty} x(t)e^{-st}dt.$$

$$\mathcal{L}\left(\frac{dx}{dt}\right) = sX(s) - x(0)$$

Obtain the solution in the frequency domain $X(s)$, and use inverse transform to time domain.

$$\mathcal{L}^{-1}(X(s)) = x(t) = \int_{-\infty}^{+\infty} X(s)e^{st}ds$$

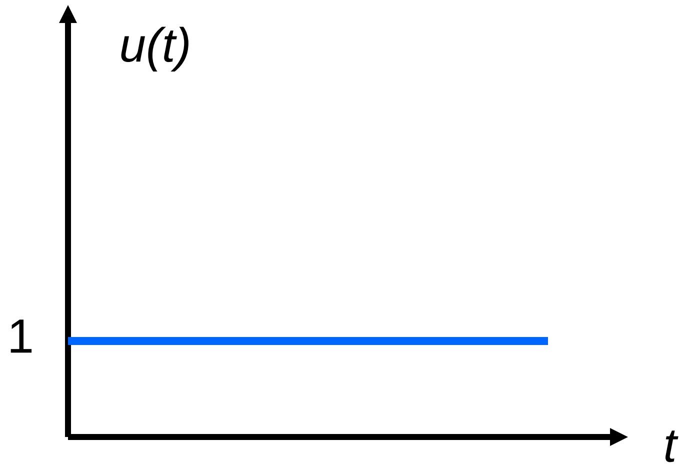
Techniques for non-autonomous systems

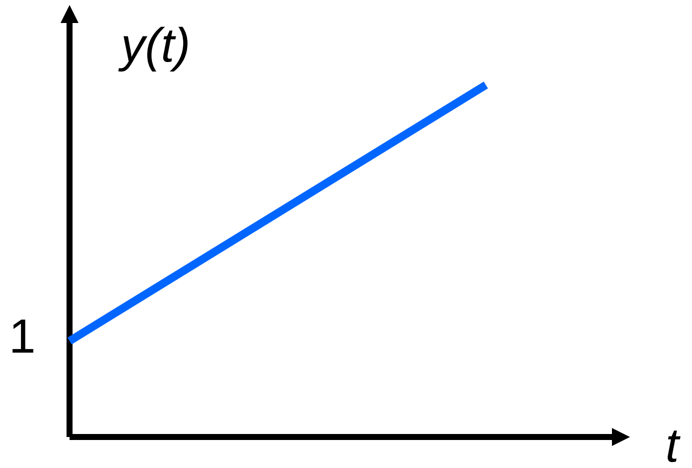
- Example:

$$\frac{dy}{dt} = u(t)$$

$$u(t) = \mathbb{1}(t), y(0) = 1.$$

$$sY(s) - 1 = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s^2} + \frac{1}{s}, y(t) \quad = t\mathbb{1}(t) + \mathbb{1}(t).$$





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Solution concepts

- Given a differential equation, $\frac{dx}{dt} = f(x, u)$, and a function $\tilde{x}(t)$. When can we say that $(\tilde{x}(t), \tilde{u}(t))$ is a solution of the differential equation?
- When $\tilde{x}(t)$ is differentiable, then it is straightforward. This is called a strong solution to the equation.
- When $\tilde{x}(t)$ is not differentiable, then $(\tilde{x}(t), \tilde{u}(t)$ is a solution if there exists an x_0 such that

$$\tilde{x}(t) = x_0 + \int_0^t f(\tilde{x}(\tau), \tilde{u}(\tau))\, d\tau$$

This is called a weak solution to the equation.

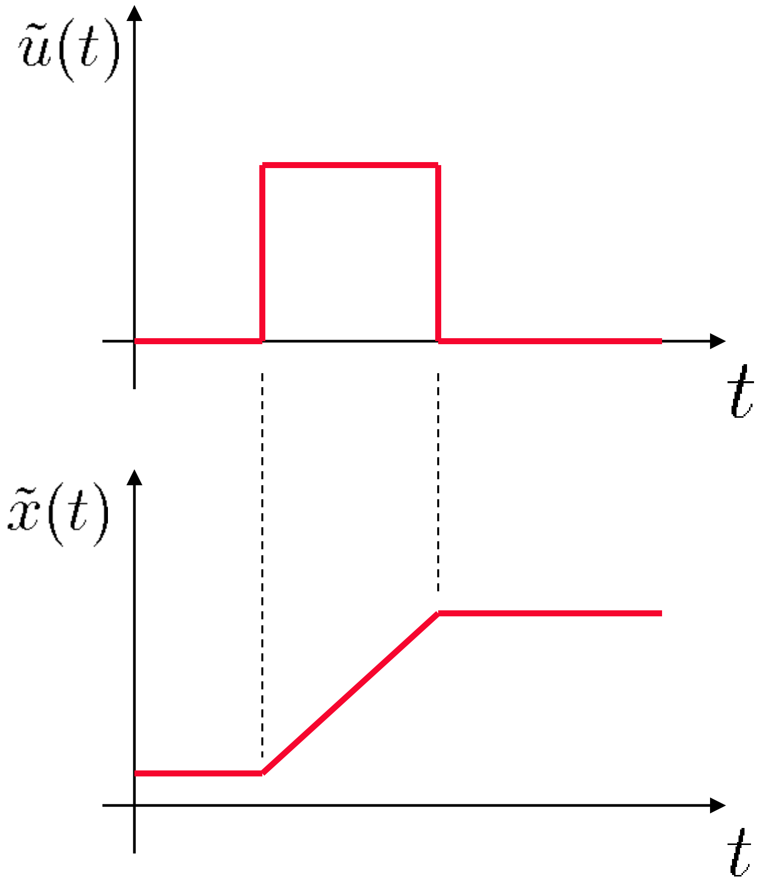
Example of weak solution

Suppose that $\frac{dx}{dt} = u(t)$.

$$\tilde{x}(t) = \begin{cases} 1/4, & t \le 1 \\ t - 3/4, & 1 < t \le 2 \\ 5/4, & t > 2 \end{cases},$$

$$\tilde{u}(t) = \begin{cases} 0, & t \le 1 \\ 1, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}.$$

is a weak solution since
$\tilde{x}(t) = \frac{1}{4} + \int_0^t \tilde{u}(\tau)\, d\tau.$



Existence

- Consider the system

$$\frac{dx}{dt} = -\operatorname{sgn}(x) = \begin{cases} -1 & if \quad x \geq 0 \\ 1 & if \quad x < 0 \end{cases}$$

with initial condition x(0) = 0. A solution to this differential equation does not exist for any t≥0.

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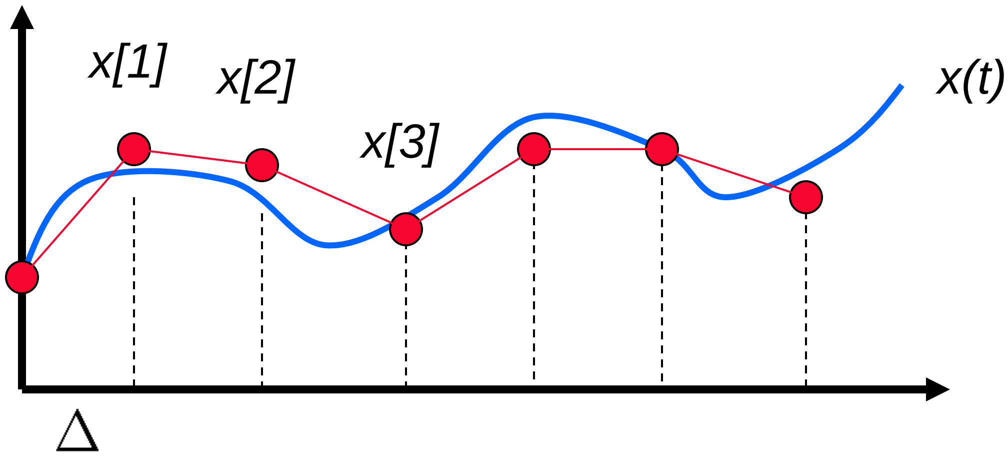
Simulation methods

- Given a differential equation $\frac{dx}{dt} = f(x,t)$.
- To simulate, i.e. numerically compute the solution, we need to discretize.



Forward difference method (Euler) : $\frac{dx}{dt} \approx \frac{x[k+1]-x[k]}{\Delta}$

$$x[k+1] = x[k] + \Delta \cdot f(x[k], k\Delta)$$

Simulation methods

- Backward difference method: $\frac{dx}{dt} \approx \frac{x[k]-x[k-1]}{\Delta}$

$$x[k] = x[k-1] + \Delta \cdot f(x[k], k\Delta)$$

- In each iteration we need to solve an implicit function of $x[k]$. Advantage: the algoritm is more stable.
- Exact discretization is possible for linear time invariant systems.
- There are more sophisticated algoritm, e.g. Runge-Kutta, etc. Most popular algorithms are built in features in most programming/simulation packages, such as MATLAB, MAPLE, etc.

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State space representation

- One of the most important representations of linear time invariant systems.

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

$x(t)$ is called the state of the system, $u(t)$ is the input and $y(t)$ is the output of the system. All variables are vector valued.

A, B, C, D are matrices with appropriate dimensions.

This representation is sometime also called input/state/output representation.

State space representation

- Higher order input/output systems can be cast in state space representation.

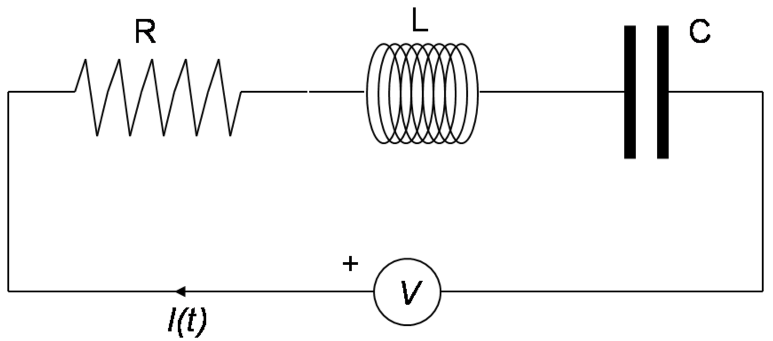
$$\boxed{\ddot{y}(t) + 6\dot{y}(t) + 8y(t) = u(t),}$$
$$x_1(t) = y(t), x_2(t) = \dot{y}(t).$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- Thus, we can transform scalar high order ODE to vector first order ODE.

Back to the electric circuit

$$\frac{dV}{dt} = R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}I$$



$$\frac{d}{dt}\begin{bmatrix} V_C \\ I \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} V_C \\ I \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V$$

Solution to state space rep.

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

Solution:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau,$$
$$y(t) = Ce^{At}x(0) + \int_0^t e^{CA(t-\tau)} Bu(\tau)\, d\tau + Du(t).$$

Matrix exponential: $e^A := I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots$.
Easy to compute if A is diagonal.

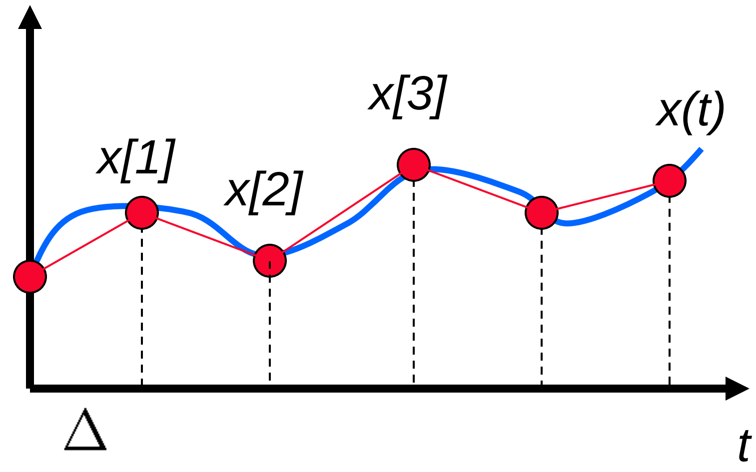
Alternative: $\mathcal{L}(e^{At}) = (sI - A)^{-1}$

Exact discretization of autonomous systems

- Consider $\dot{x} = Ax(t)$. The solution to this equation is $x(t) = e^{At}x(0)$.

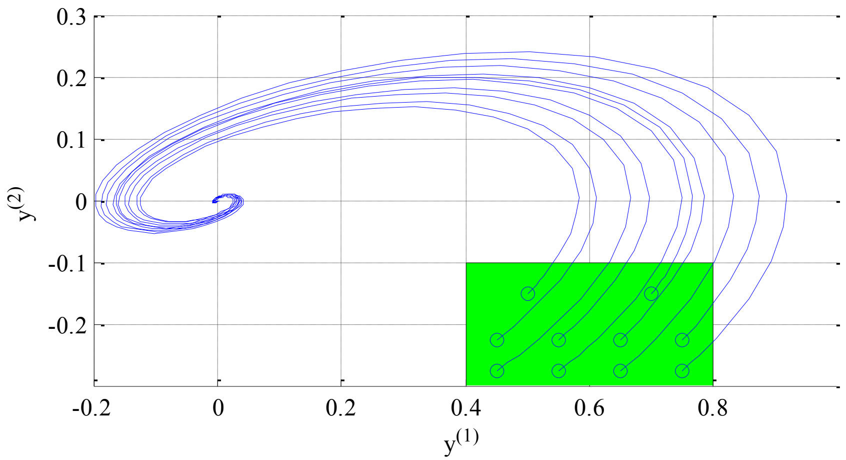
- We sample the system with sampling interval Δ. We have that

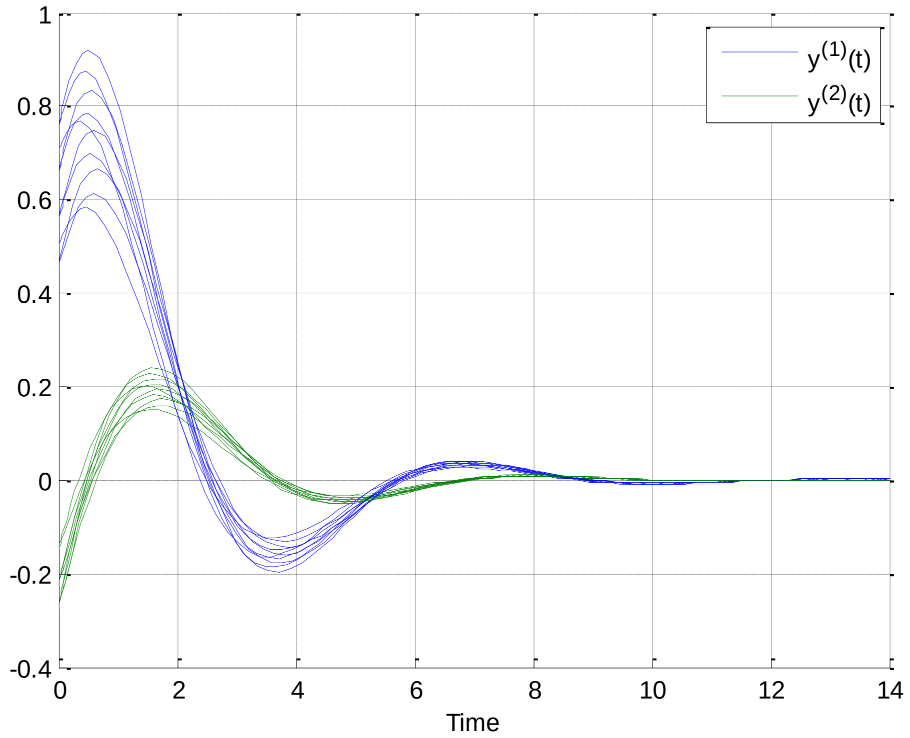
$$
\begin{aligned}
x(\Delta) &= e^{A\Delta}x(0), \\
x((k+1)\Delta) &= e^{A\Delta}x(k\Delta), \\
x[k+1] &= e^{A\Delta}x[k].
\end{aligned}
$$



Example

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0.025 & -2.5 \\ 0.5 & -1 \end{bmatrix} x(t)$$





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- Discrete time systems

Stability of LTI systems

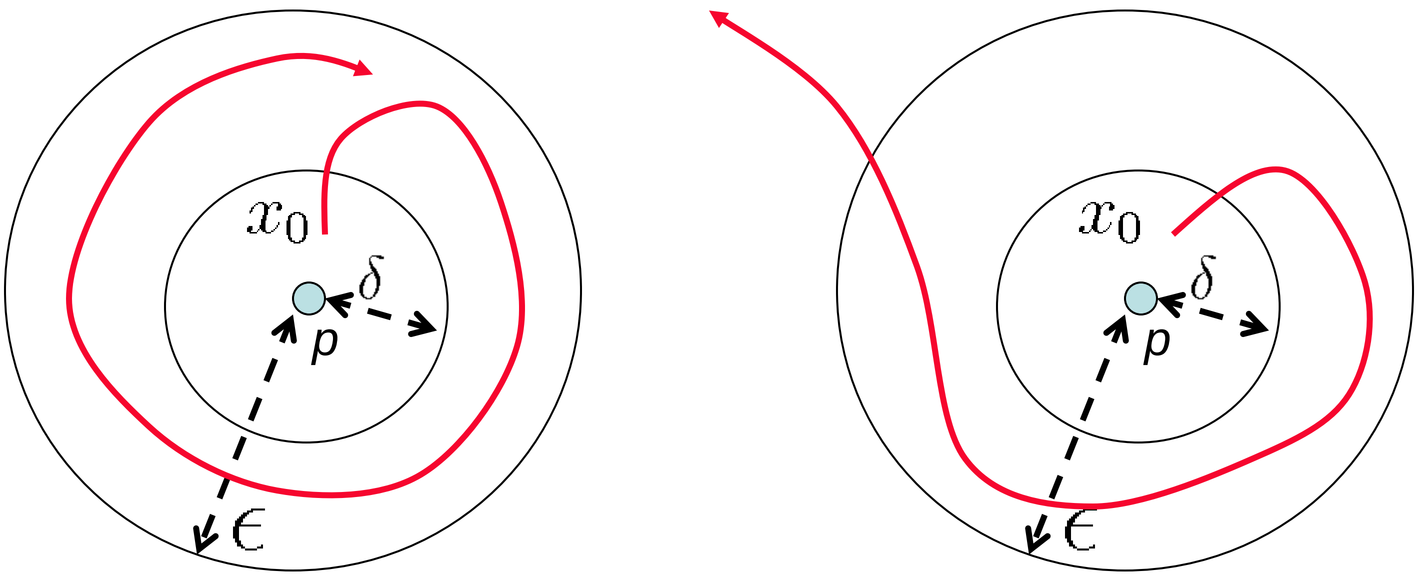
- A system is stable if with zero input, starting from any initial condition, the state trajectory converges to zero.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{At}x(0) = 0.$$

- $\mathcal{L}(e^{At}) = (sI - A)^{-1}$. The polynomial $\det(sI - A)$ is called the characteristic polynomial.
- The system is stable if and only if all the roots of the characteristic polynomial have negative real part.
- Stability also implies that bounded input will produce bounded output.

Stability of nonlinear systems

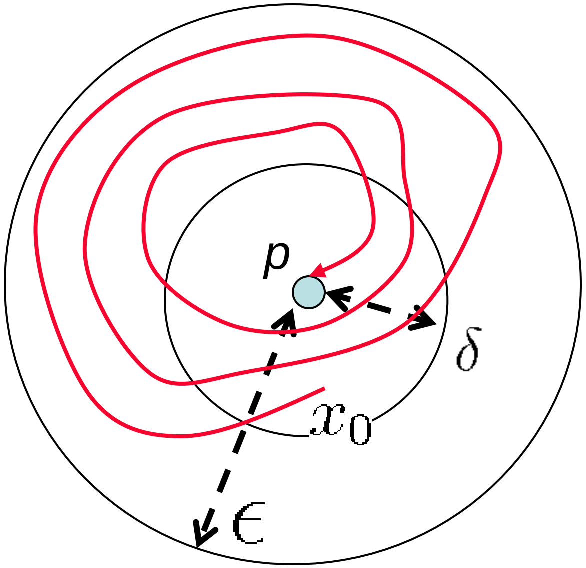
- Given $\dot{x} = f(x)$, let p be an equilibrium, i.e. $f(p) = 0$.
- The equilibrium p is stable if for any $\epsilon > 0$, there is a $\delta(\epsilon)$, such that the trajectory with initial condition x_0, with $\|x_0 - p\| < \delta(\epsilon)$ remains within ϵ distance from p.



stable

Stability of nonlinear systems

- The equilibrium p is asymptotically stable if for any $\epsilon > 0$, there is a $\delta(\epsilon)$, such that the trajectory with initial condition x_0, with $\|x_0 - p\| < \delta(\epsilon)$ remains within ϵ distance from p and converge to p.

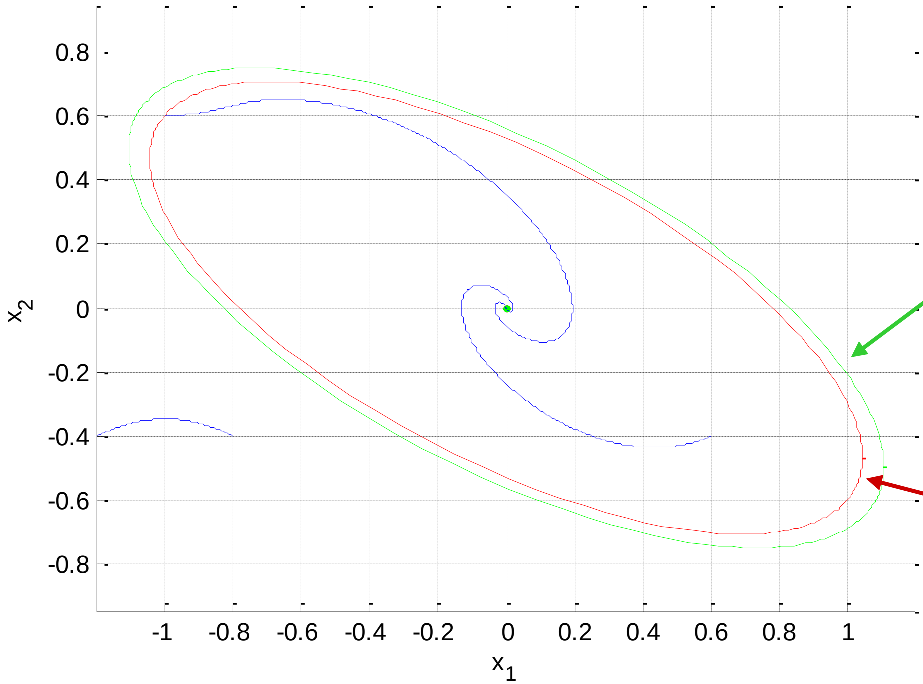


Asymptotically stable

Lyapunov functions

- A smooth function V(x) is called positive definite if $V(p) = 0$ and $V(x) > 0$ if $x \neq p$.
- A smooth function V(x) is called positive semidefinite if $V(p) = 0$ and $V(x) \geq 0$ if $x \neq p$.
- If there exists a positive definite function $V(x)$ such that $\frac{d}{dt}V(x(t))$ is negative semidefinite, then p is stable.
- If there exists a positive definite function $V(x)$ such that $\frac{d}{dt}V(x(t))$ is negative definite, then p is asymptotically stable.

Example



$$\Omega = \{x \in \mathbb{R}^2 \mid V(x) \leq 0.34\}$$

$$V(x) = x^T P x$$

$$P = \begin{bmatrix} 0.4946 & 0.4834 \\ 0.4834 & 1.0774 \end{bmatrix}$$

$$x(t) \in \{x \in \mathbb{R}^2 \mid V(x) \leq V(x(0))\}$$

$$P_e = V(x(0))P^{-1}$$

$$\|x(t)\| \leq \sqrt{\lambda_{\max}(P_e)}$$

$$\dot{x}(t) = Ax(t) - b\,\mathrm{sat}(cx(t)), \quad s_3(t) = cx(t)$$

Contents

Reachability

- Given a system $\dot{x} = f(x, u)$, and the set of initial condition I, i.e. $x(0) \in I$.
- A state ξ is reachable at time $t = T$ if there exist an initial state $x_0 \in I$ and input $\tilde{u}(t)$ such that the state trajectory starting from x_0 at time $t = 0$, when given input $\tilde{u}$ satisfies $x(T) = \xi$.

Reachability of linear systems

- Given a linear system $\dot{x} = Ax + Bu$, where $x(t) \in \mathbb{R}^n$ and the initial condition $x(0) = 0$.
- The reachable set of the system is a linear space

$$\mathcal{R} = \mathrm{im}[B \ AB \ A^2B \ \cdots \ A^{n-1}B]$$

- Any state in $\mathcal{R}$ is reachable at any time.