CSE 591: Theoretical Aspects of CPS

Control II
Reference: Tabuada Ch 6

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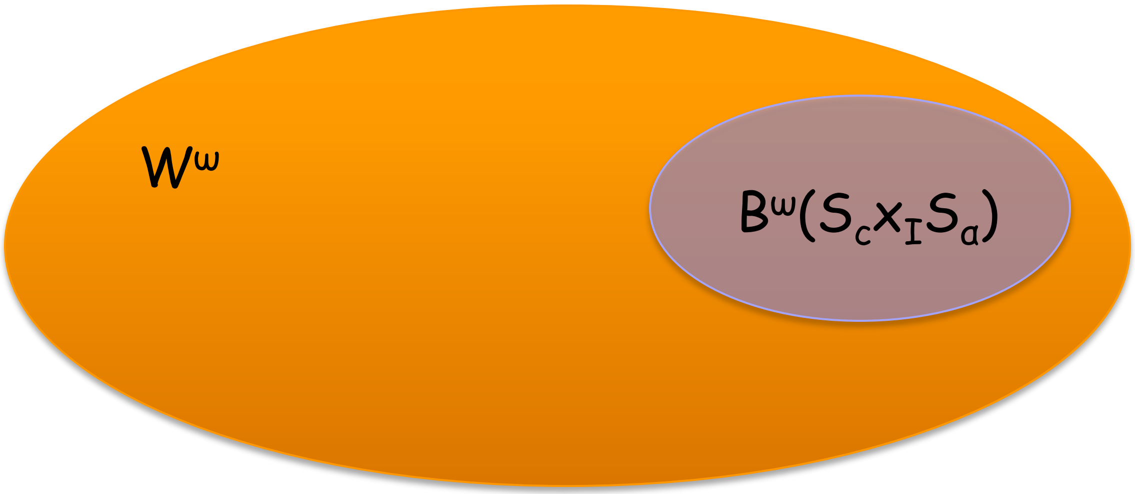
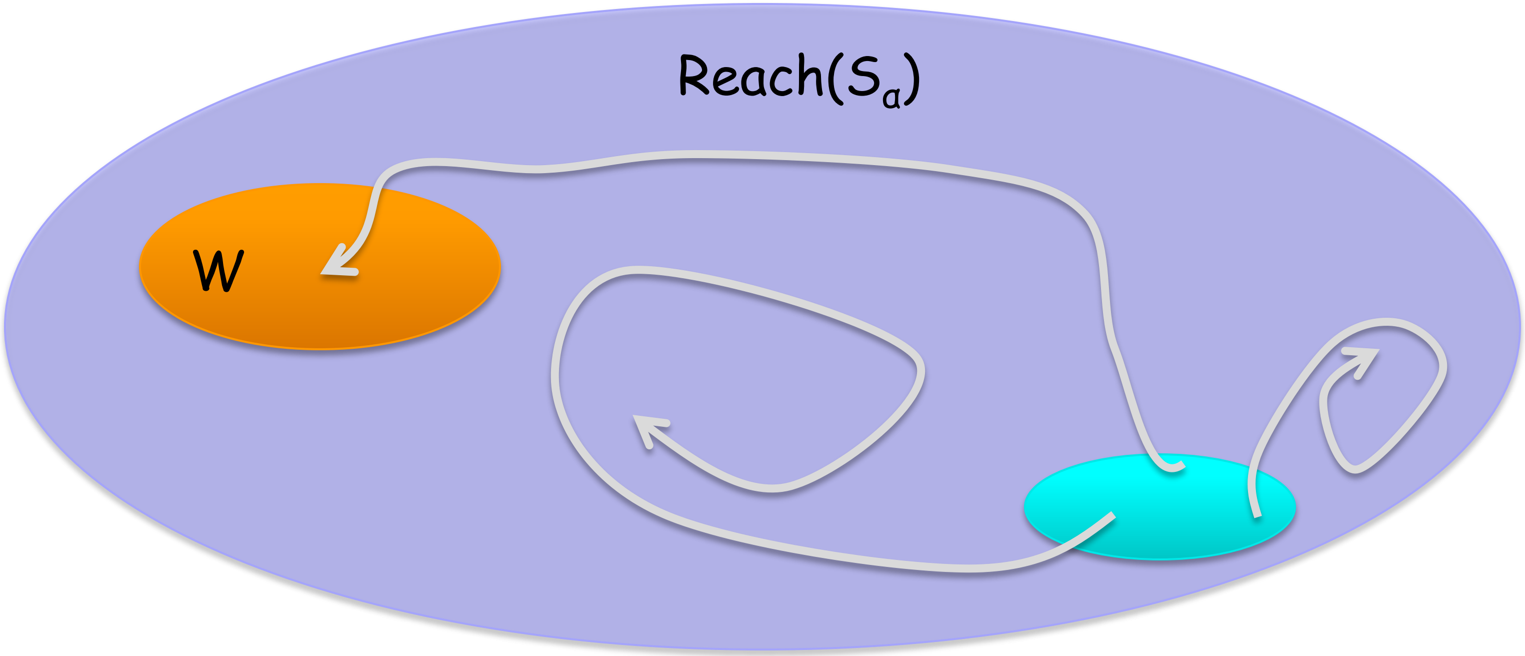
Last class: Safety games

- Can we find a controller S_c such that $B^\omega(S_c x_I S_a) \subseteq W^\omega$

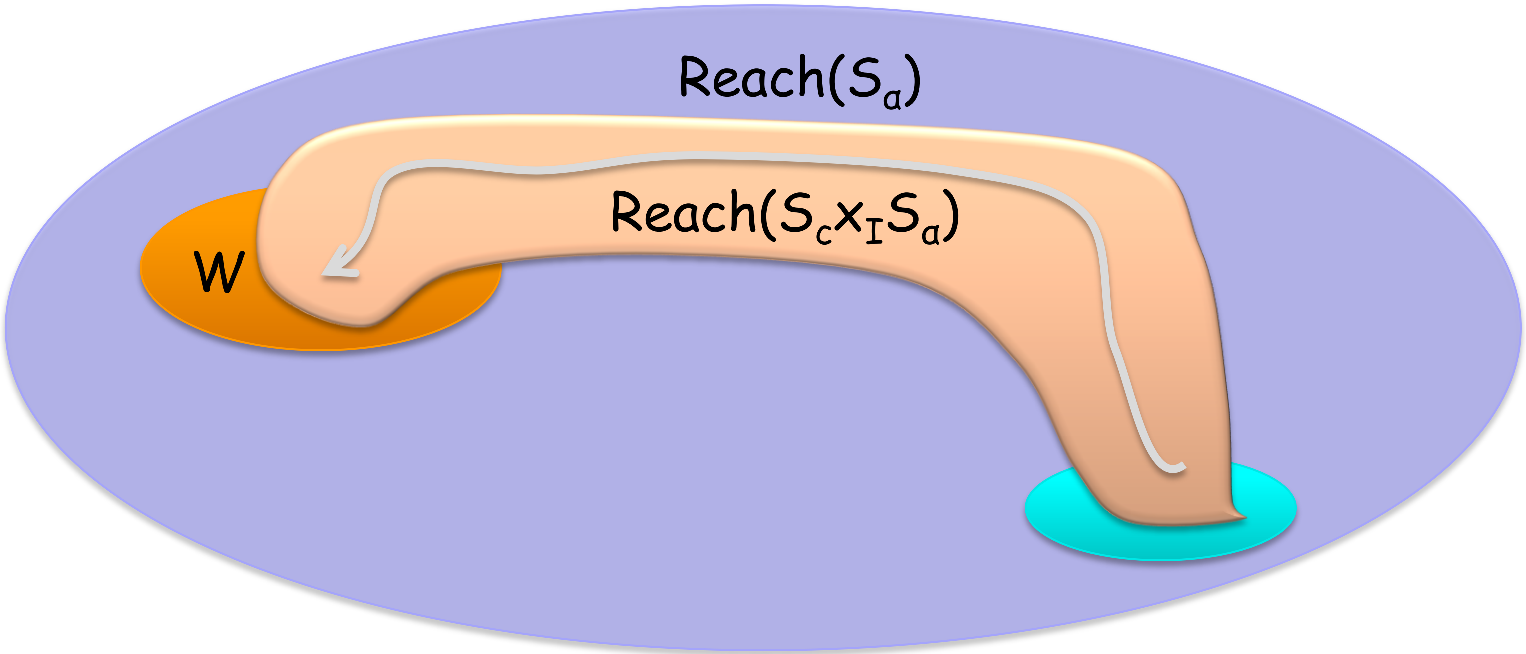


Reachability games



Reachability games

- Can we find a controler S_c such that the output trajectories of $S_c x_F S_a$ reach a set W?



Reachability game

- Let S_a with $Y_a = X_a$ and $H_a = 1_{X_a}$ and $W \subseteq X_a$ be the set of states. Does it exist controller S_c s.t.
  1. S_c is feedback composable with S_a
  2. For every maximal behavior $y \in B(S_c x_F S_a) \cup B^{\omega}(S_c x_F S_a)$ there exists $k \geq 0$ s.t. $y_k \in W$

- A reachability game is solvable when S_c exists

Reachability game

- Fixed-point operator $G_W : 2^X \rightarrow 2^X$:
  $G_W(Z) = \{x \in X \mid x \in W \text{ or } \exists u \in U(x). \emptyset \neq Post_u(x) \subseteq Z\}$

- G_W is order preserving

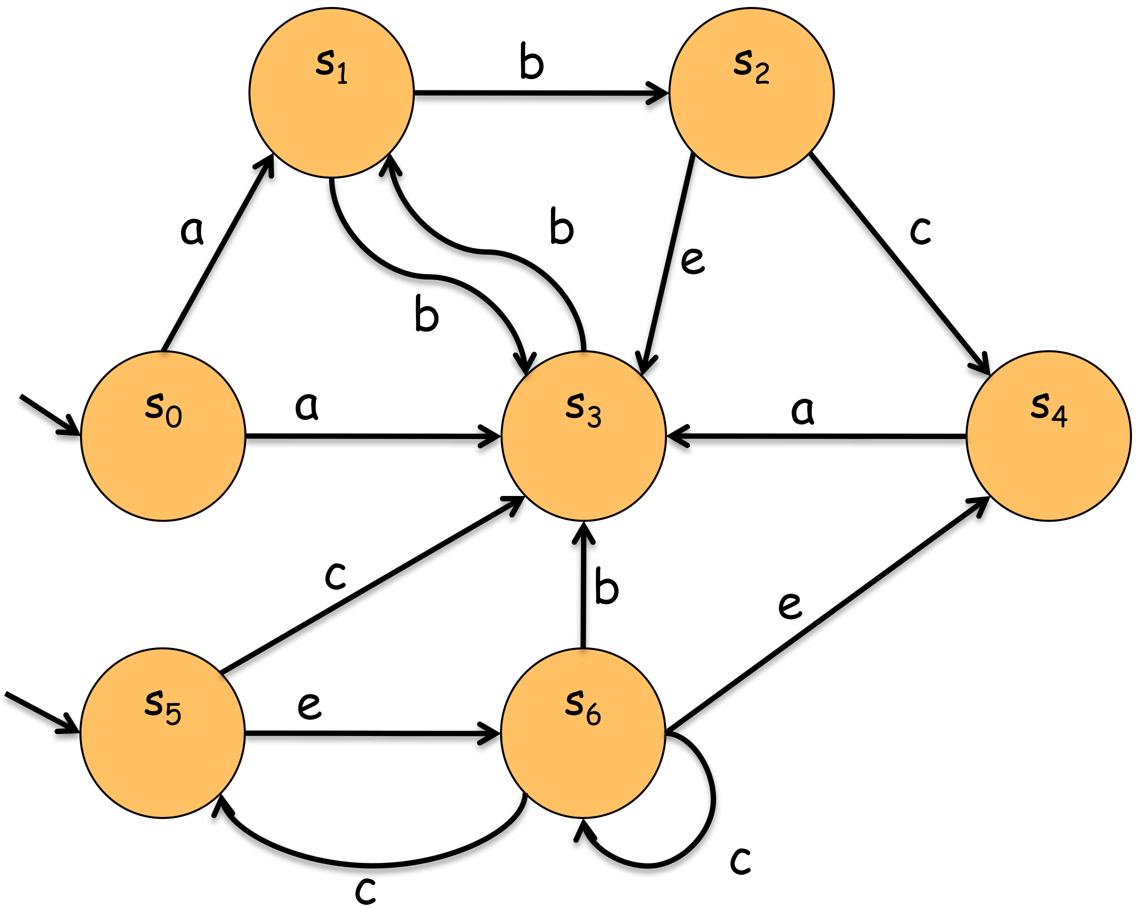
- Given a fixed-point Z of G_W, we can construct a controller

- Theorem 6.10
  A reachability game with goal set W is solvable iff the least fixed-point Z of G_W satisfies $Z \cap X_0 \neq \emptyset$.
  $$Z = \lim_{i \rightarrow \infty} G^i_W(\emptyset)$$

Example 6.11 - $W = \{s_4\}$

$Z=\emptyset$

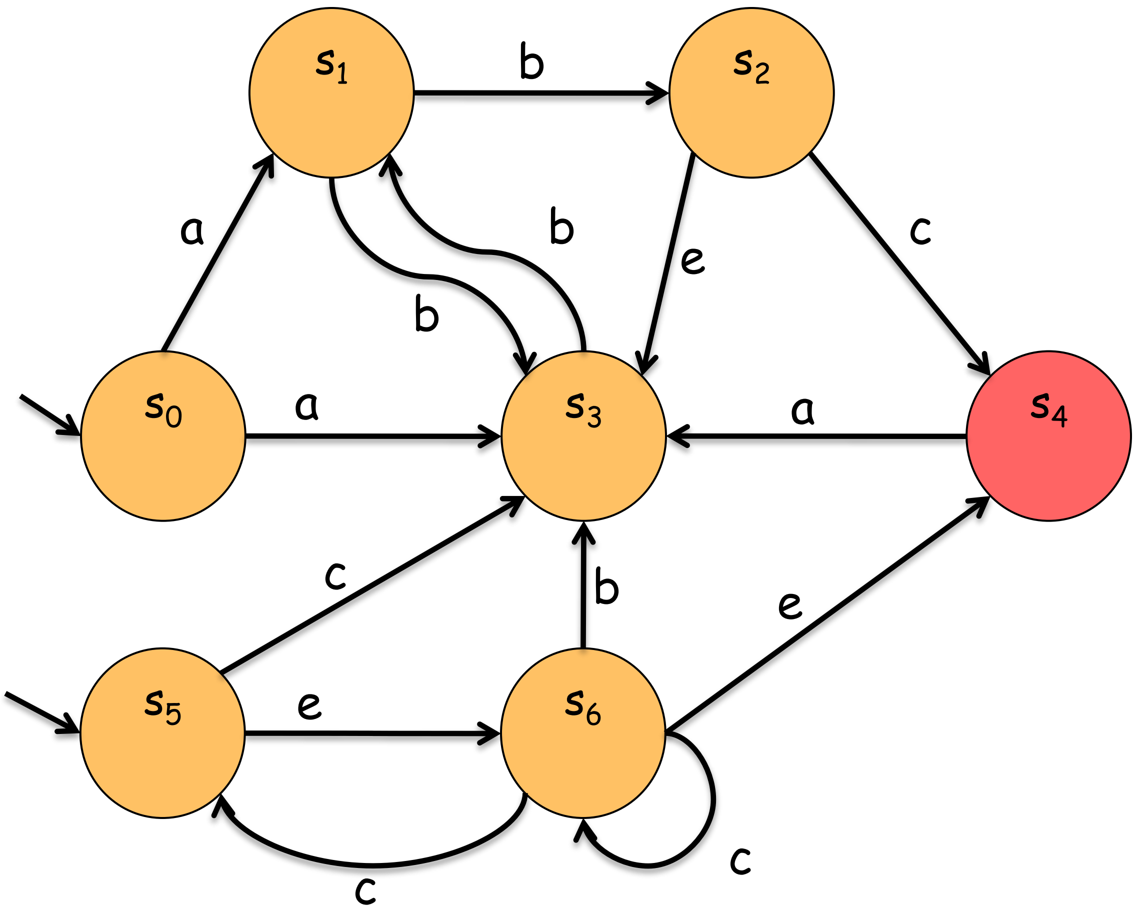
$G_W(\emptyset)$?



$G_W(Z) = \{x \in X \mid x \in W \text{ or } \exists u \in U(x).\emptyset \neq Post_u(x) \subseteq Z\}$

Example 6.11 - $W = \{s_4\}$

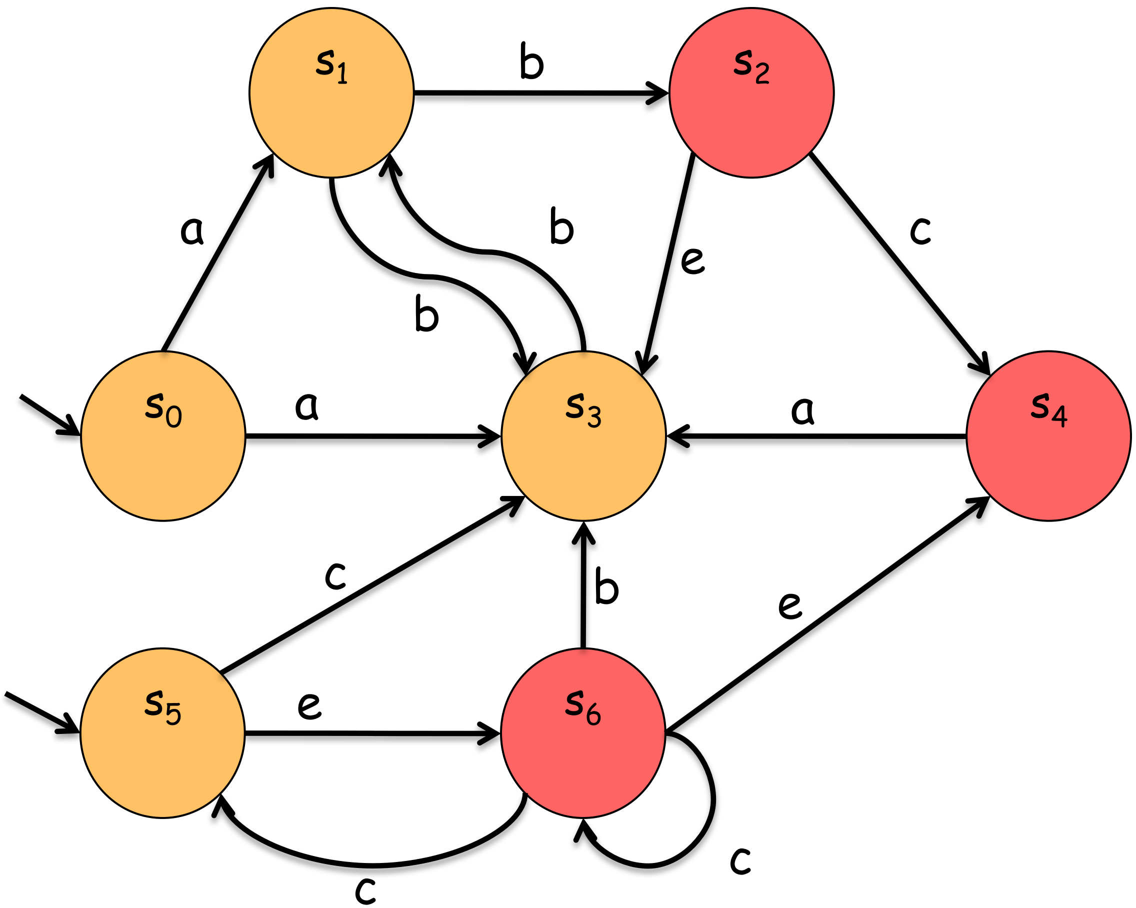
$G_W(\emptyset)=\{s_4\}$



$G_W(Z) = \{x \in X \mid x \in W \text{ or } \exists u \in U(x). \emptyset \neq Post_u(x) \subseteq Z\}$

Example 6.11 - $W = \{s_4\}$

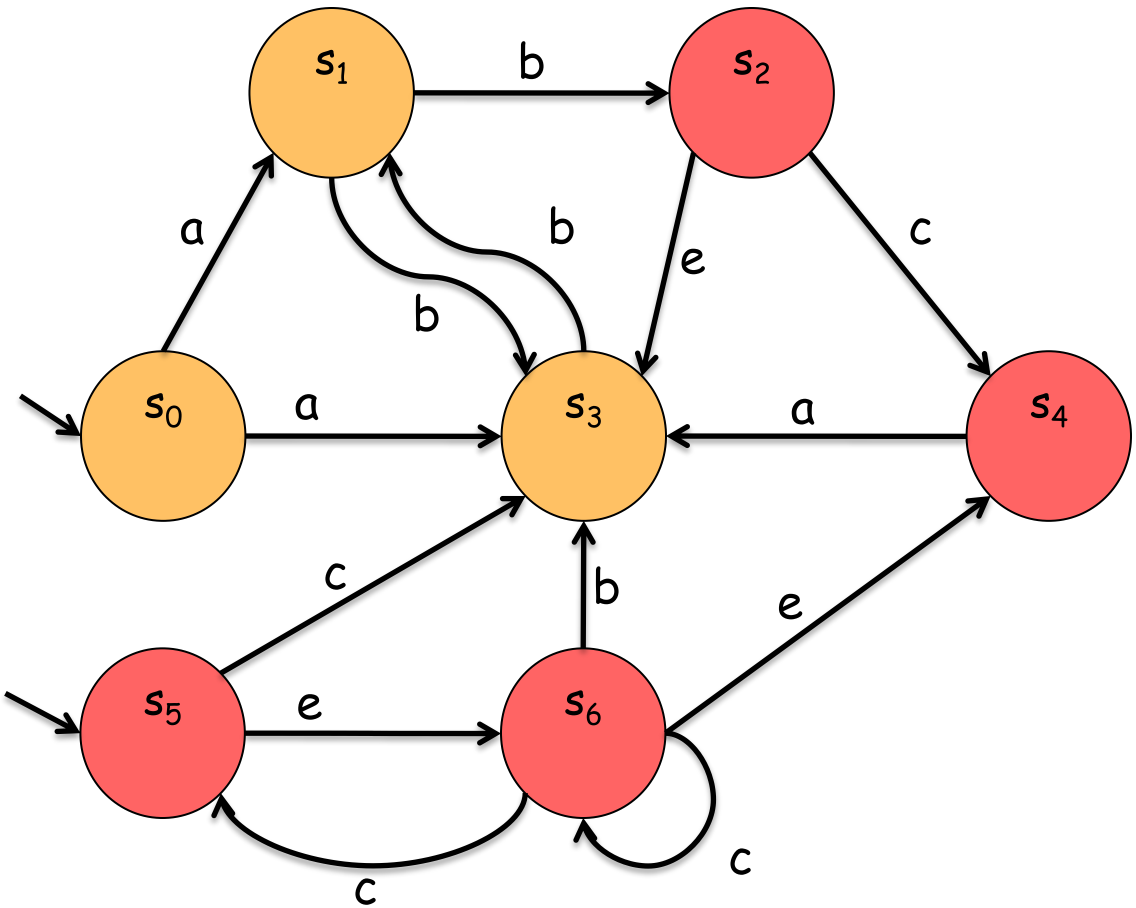
$G^2_W(\emptyset) = \{s_2, s_4, s_6\}$



$G_W(Z) = \{x \in X \mid x \in W \text{ or } \exists u \in U(x). \emptyset \neq Post_u(x) \subseteq Z\}$

Example 6.11 - $W = \{s_4\}$

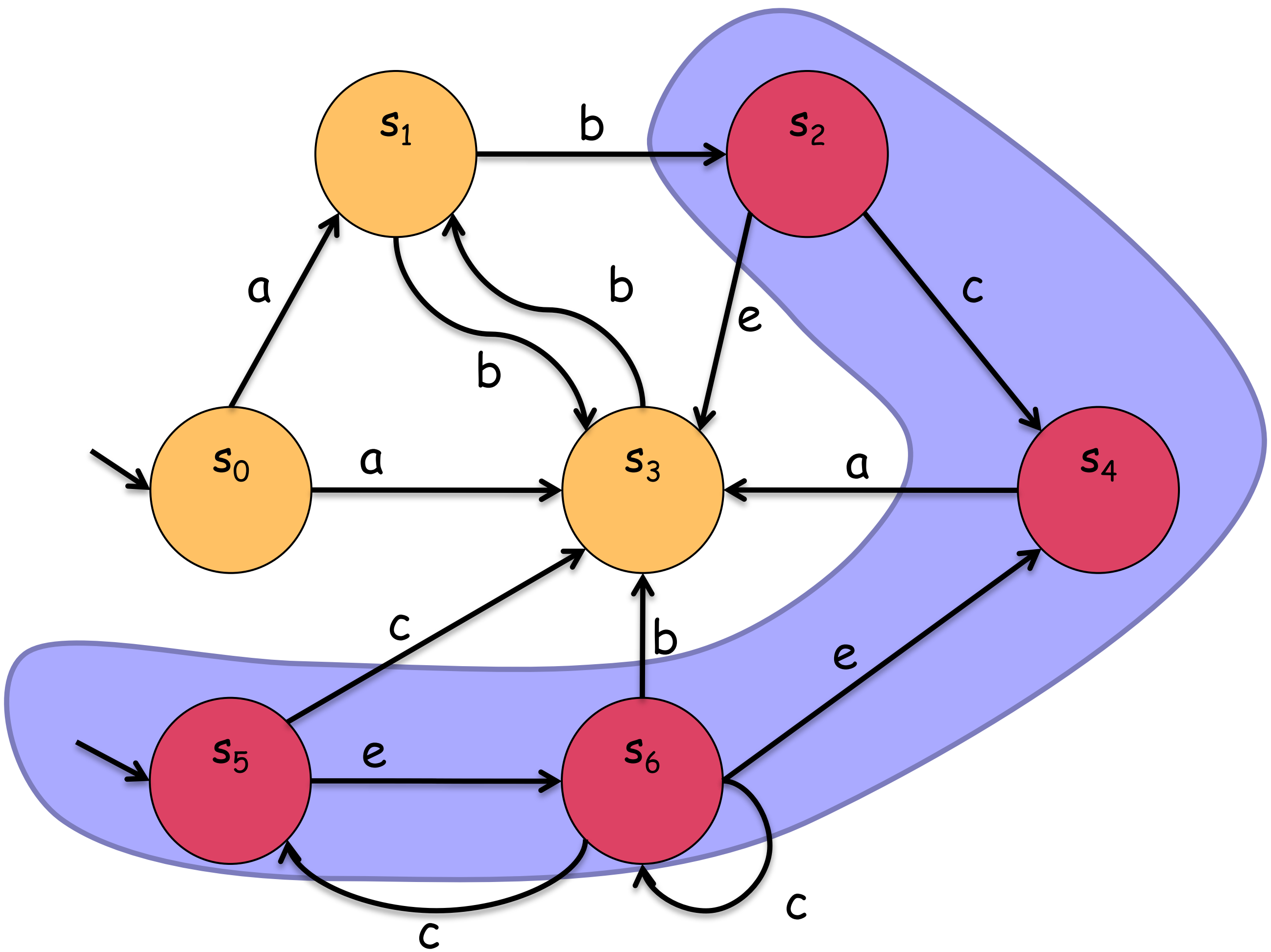
$G^3{}_W(\emptyset) = \{s_2, s_4, s_5, s_6\}$



$G_W(Z) = \{x \in X \mid x \in W \text{ or } \exists u \in U(x).\emptyset \neq Post_u(x) \subseteq Z\}$

Example 6.11 - W = $\{s_4\}$

$G^4{}_W(\emptyset)=$
$\{s_2,s_4,s_5,s_6\}$

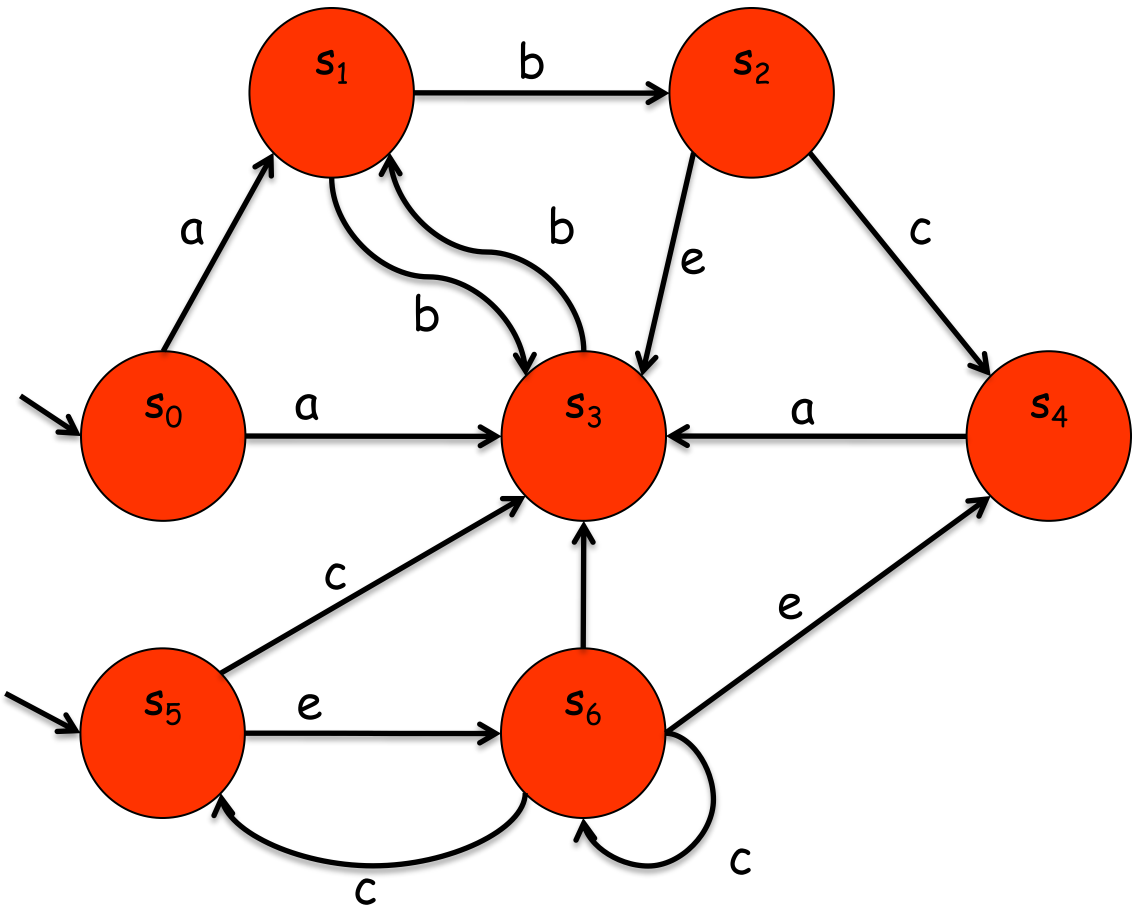


$G_W(Z) = \{x\in X \mid x\in W \text{ or } \exists u\in U(x).\emptyset\neq Post_u(x)\subseteq Z\}$

Example 6.11 - $W = \{s_4\}$

$Z=X$

$G_W(X)$?



$G_W(Z) = \{x \in X \mid x \in W \text{ or } \exists u \in U(x). \emptyset \neq Post_u(x) \subseteq Z\}$

Behavioral games

- (Behavior inclusion hame) Consider S_a and the specification S_b with $Y_a = Y_b$. Does there exist controller S_c s.t.
  1. S_c is feedback composable with S_a
  2. $S_c x_F S_a$ is nonblocking
  3. $S_c x_F S_a \preceq_B S_b$
- A behavior inclusion game is solvable when S_c exists
- If S_b is output deterministic, then behavioral games can be reduced to similarity games

Similarity games

- (Simulation game) Consider S_a and the specification S_b with $Y_a = Y_b$. Does there exist controller S_c s.t.
  1. S_c is feedback composable with S_a
  2. $S_c x_F S_a$ is nonblocking
  3. $S_c x_F S_a \preceq_S S_b$

- A simulation game is solvable when S_c exists

Simulation game

- Fixed-point operator $F_C : 2^{X_a \times X_b} \rightarrow 2^{X_a \times X_b}$ with def:
  $(x_a,x_b) \in F_C(Z)$ for some $Z \subseteq X_a \times X_b$ if
  - $H_a(x_a)=H_b(x_b)$
  - $(x_a,x_b) \in Z$
  - $\exists u_a \in U_a(x_a)$. $\forall x_a' \in Post_{ua}(x_a)$ there exists $x_b \xrightarrow[b]{u_b} x_b'$ with $(x_a',x_b') \in Z$

- F_C is order preserving

- Given a fixed-point Z of F_C, we can construct a controller

- Theorem 6.15
  A simulation game is solvable iff the greatest fixed-point Z of F_C satisfies $Z \cap (X_{a0} \times X_{b0}) \neq \emptyset$.

$$Z = \lim_{i \to \infty} F^i_C(X_a \times X_b)$$

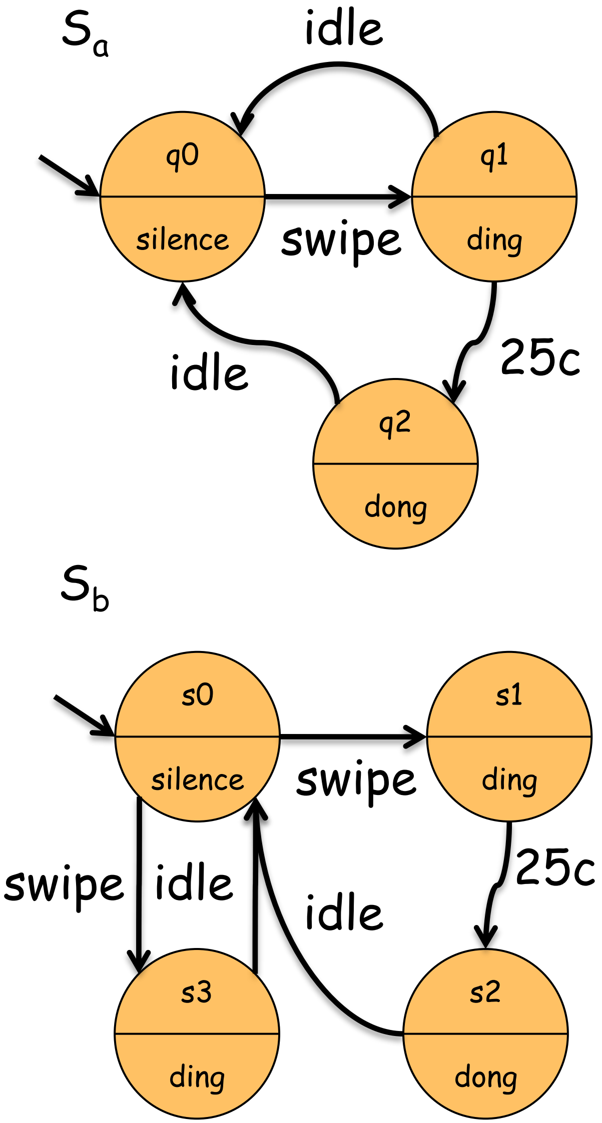
Remarks

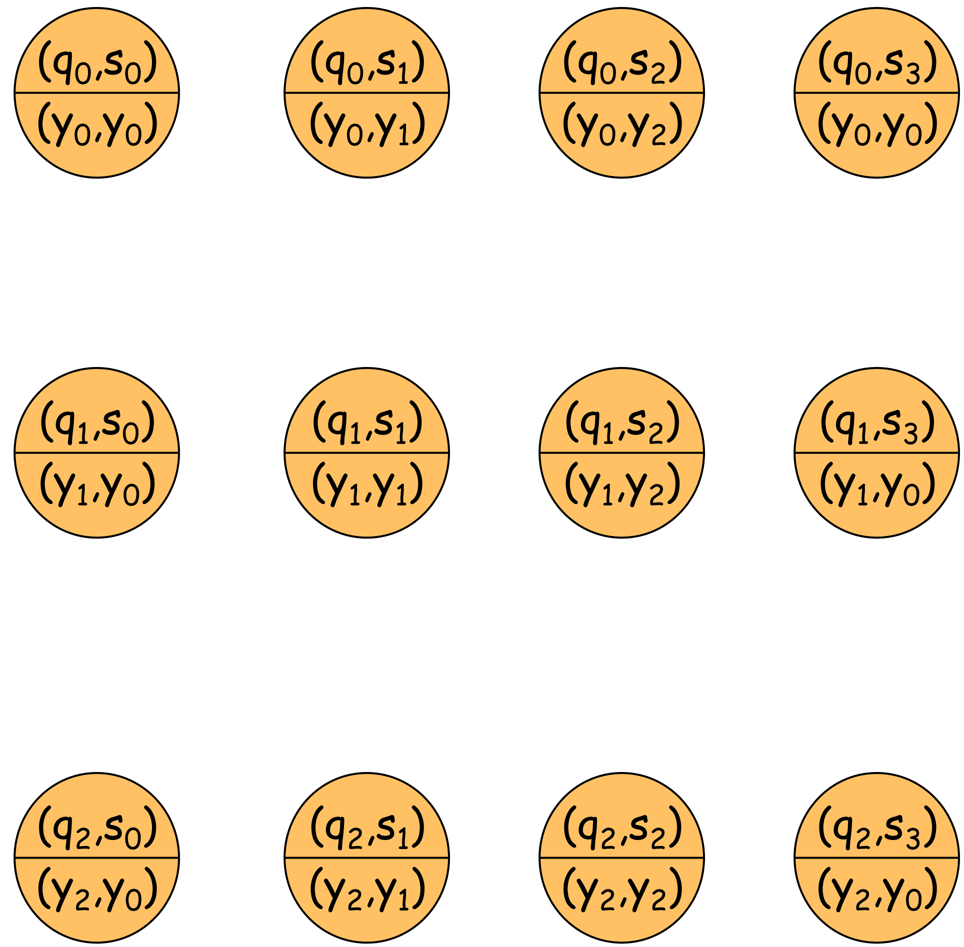
- Proposition 6.17:
  The controller determined by the greatest fixed-point is the least restrictive controller

- If S_a cannot be initialized, then we can replace $Z \cap (X_{a0} \times X_{b0}) \neq \emptyset$ with $X_{a0} = \pi_a(Z \cap (X_{a0} \times X_{b0}))$

Example 6.16

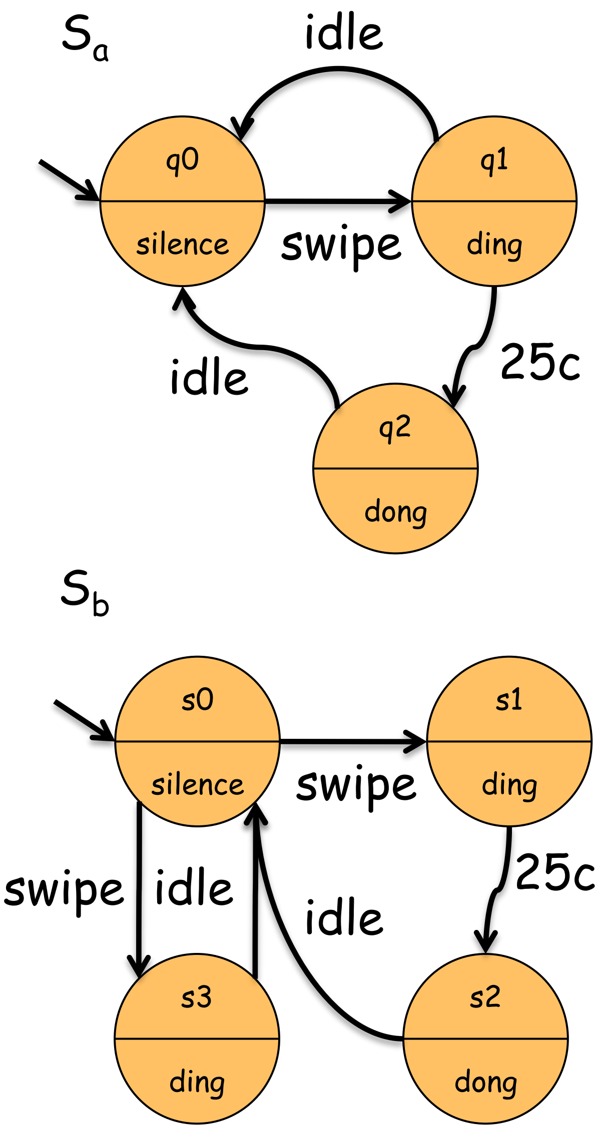
y_0 = silence, y_1 = ding, y_2 = dong

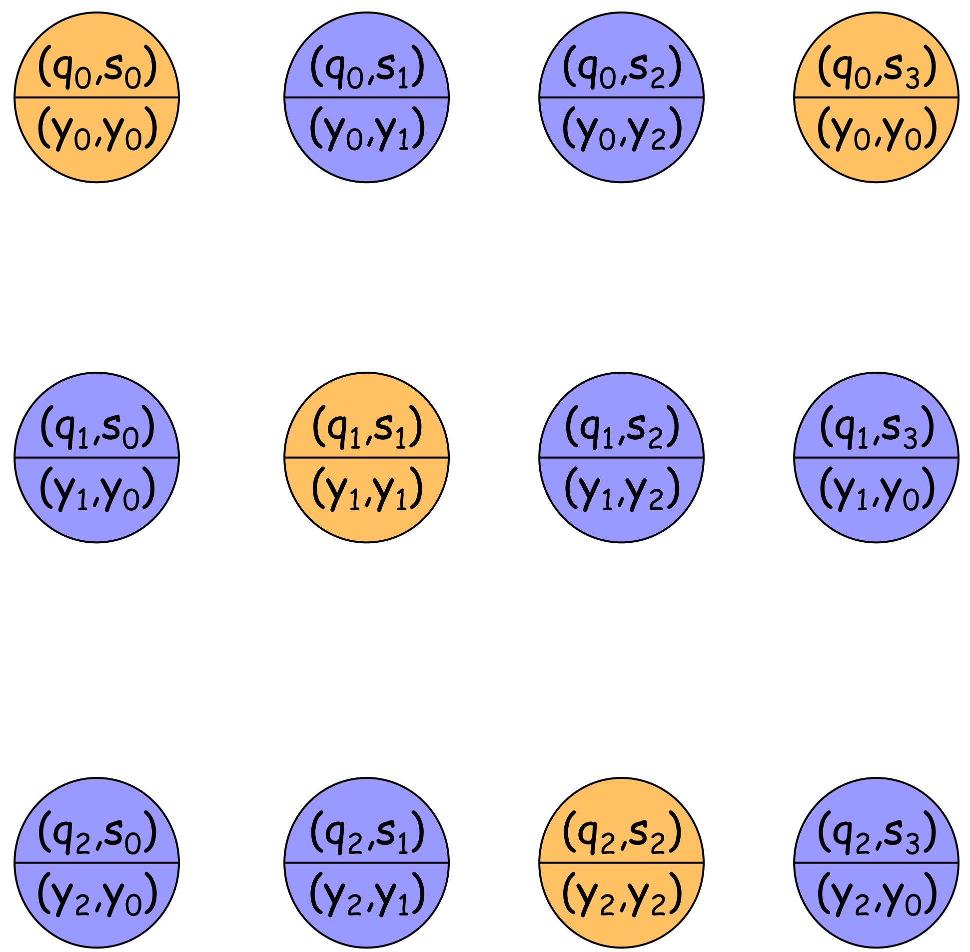




Example 6.16

y_0 = silence, y_1 = ding, y_2 = dong

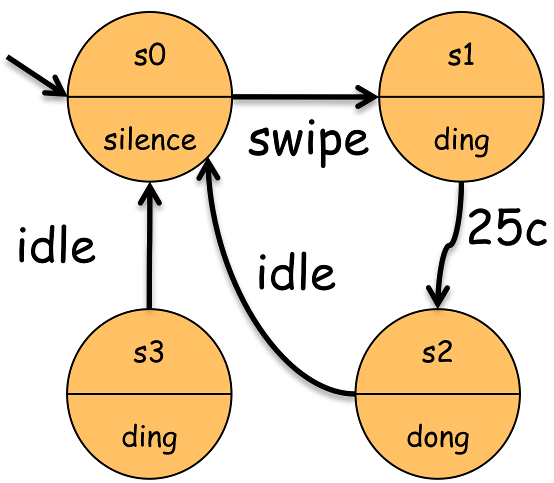


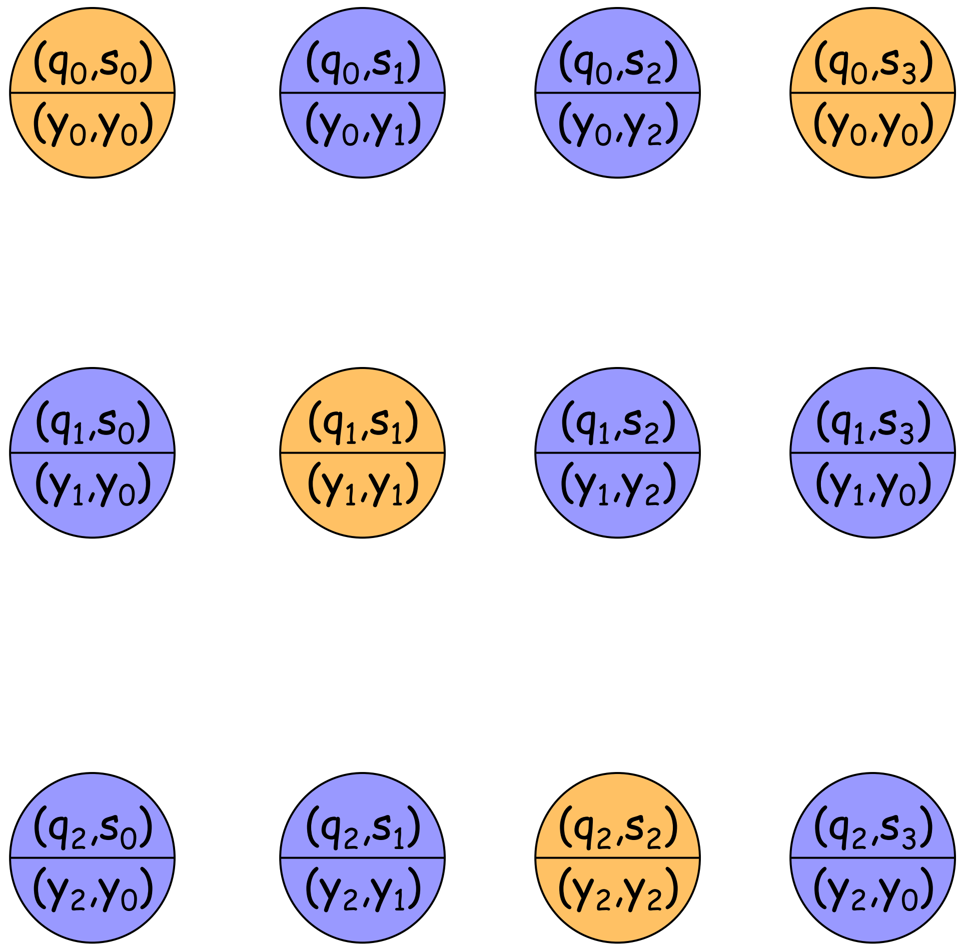


Example 6.16

y_0 = silence, y_1 = ding, y_2 = dong

$S_c x_F S_b$





Similarity games

- (Bisimulation game) Consider S_a and the specification S_b with $Y_a = Y_b$. Does there exist controller S_c s.t.
  1. S_c is feedback composable with S_a
  2. $S_c \times_F S_a \cong_S S_b$

- A bisimulation game is solvable when S_c exists

Bisimulation game

- Fixed-point operator $G_C : 2^{X_a \times X_b} \rightarrow 2^{X_a \times X_b}$ with def:
  $(x_a,x_b) \in G_C(Z)$ for some $Z \subseteq X_a \times X_b$ if
  - $H_a(x_a)=H_b(x_b)$
  - $(x_a,x_b) \in Z$
  - For every $x_b \xrightarrow[b]{u_b} x_b'$ there exists $u_a \in U_a(x_a)$ such that
    - there exists $x_a' \in Post_{ua}(x_a)$ there exists with $(x_a',x_b') \in Z$
    - for every $x_a'' \in Post_{ua}(x_a)$ there exists $x_b \xrightarrow[b]{u_b} x_b''$ with $(x_a'',x_b'') \in Z$

- G_C is order preserving

- Given a fixed-point Z of G_C, we can construct a controller

- Theorem 6.19
  A simulation game is solvable iff the greatest fixed-point Z of G_C satisfies $X_{b0} = \pi_b(Z \cap (X_{a0} \times X_{b0}))$.

$$Z = \lim_{i \rightarrow \infty} G^i_C(X_a \times X_b)$$