CSE 591: Theoretical Aspects of CPS

Signed based abstractions
References:
Tabuada Ch 7.4 & 7.5

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Last time

- O-minimal hybrid systems
- Signed based abstractions
  - A family of functions P
  - P defines a partition $\mathbb{P}$ on the state-space

Signed based abstractions

Let:

- $\Sigma=(\mathbb{R}^n,f)$ be a dynamical system
- $P=\{p_i\}_{i\in I}$ be a collection of smooth real valued functions on $\mathbb{R}^n$ (induces a partition $\mathbb{P}$)
- L be the set of initial states

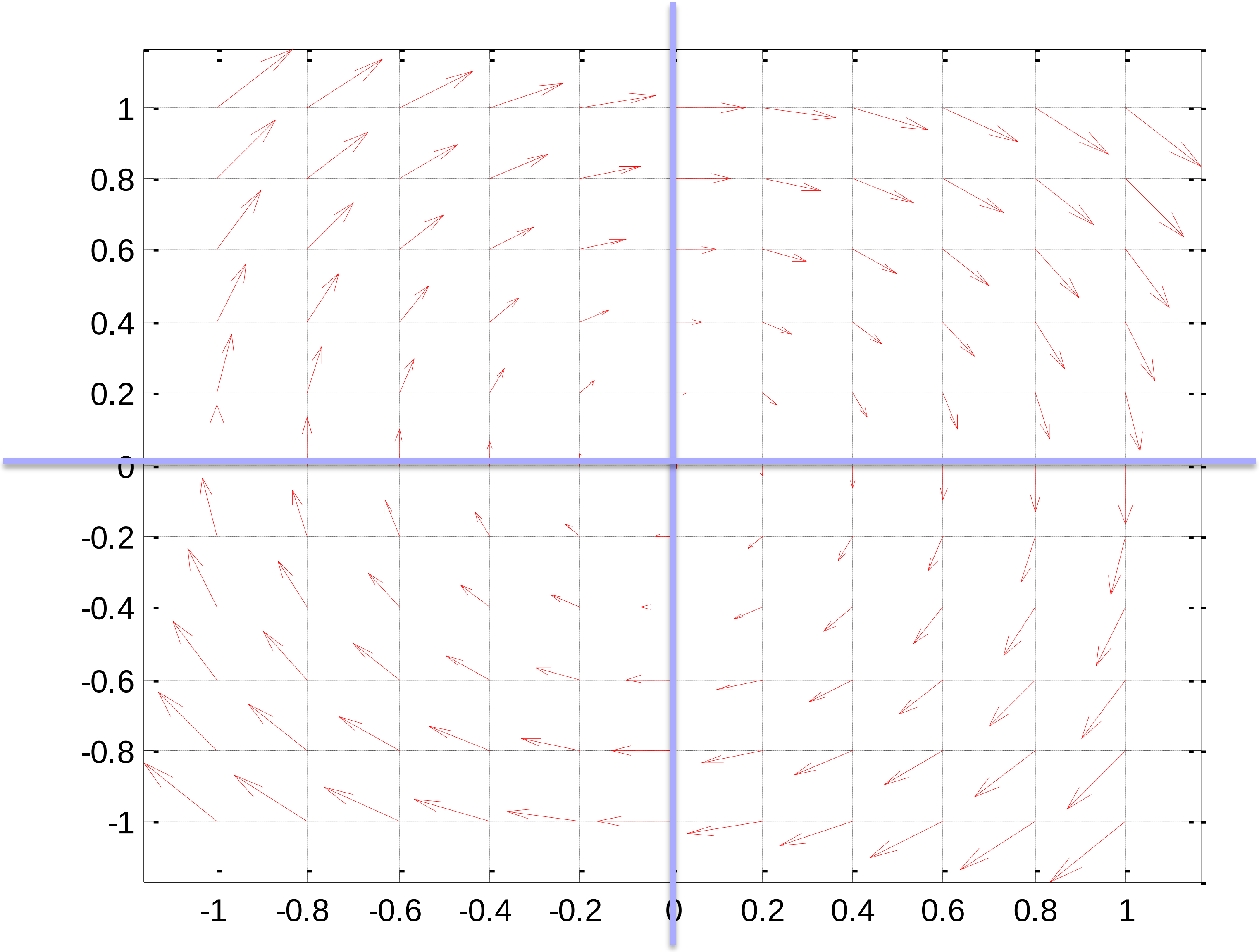
Then we can construct a finite state system $S_{PL}(\Sigma)$

- $X = \{-1,0,1\}^P$
- $X_0 = \{g\in X \mid \langle g\rangle \cap L \neq \emptyset\}$
- $U = \{*\}$
- $Y = \mathbb{R}^n / \mathbb{P}$
- $H(g) = \langle g\rangle$

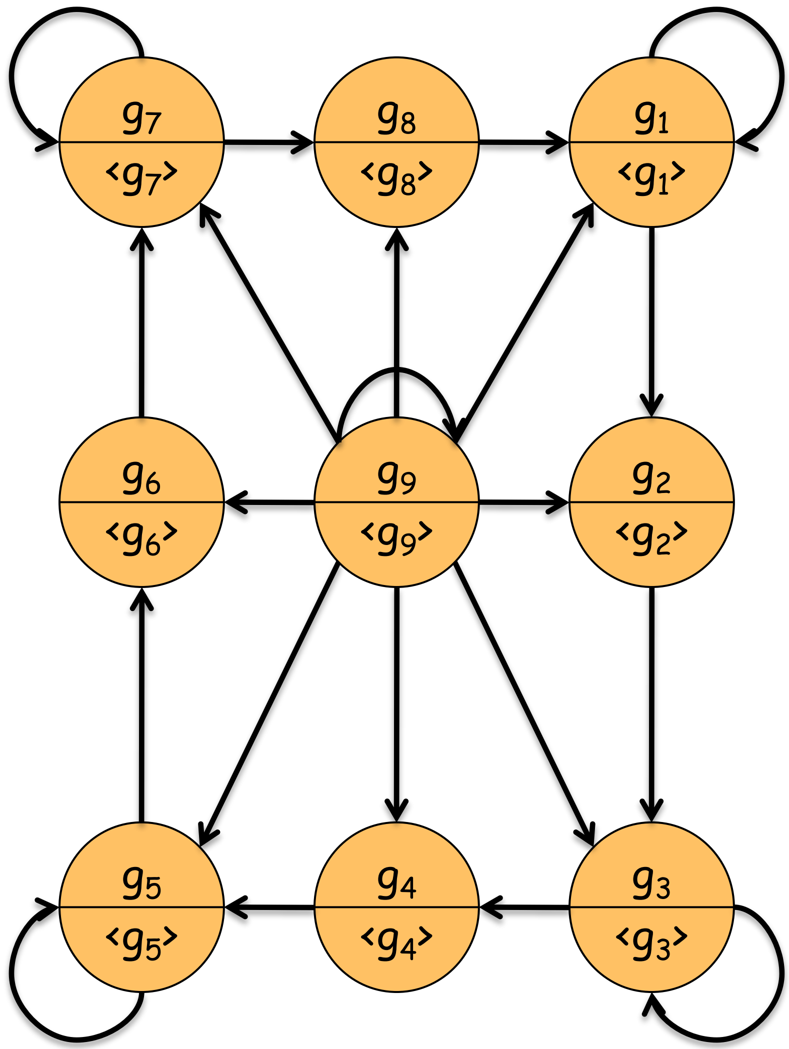
Signed based abstractions

- $g \rightarrow g'$ if for every $i \in I$ any of the following holds:
  1. $g(p_i)=1$ implies any of the following
     1. $sign(L_f p_i(\langle g \rangle)) \subseteq \{1,0\}$ and $g'(p_i)=1$
     2. $sign(L_f p_i(\langle g \rangle)) \supseteq \{1,0\}$ and $g'(p_i) \in \{1,0\}$
  2. $g(p_i)=0$ implies any of the following
     1. $sign(L_f p_i(\langle g \rangle)) = \{1\}$ and $g'(p_i)=1$
     2. $sign(L_f p_i(\langle g \rangle)) = \{-1\}$ and $g'(p_i)=-1$
     3. $sign(L_f p_i(\langle g \rangle)) = \{-1,1\}$ and $g'(p_i) \in \{-1,1\}$
     4. $sign(L_f p_i(\langle g \rangle)) \supseteq \{0\}$ and $g'(p_i) \in \{-1,0,1\}$
  3. $g(p_i)=-1$ implies any of the following
     1. $sign(L_f p_i(\langle g \rangle)) \supseteq \{1\}$ and $g'(p_i) \in \{0,-1\}$
     2. $sign(L_f p_i(\langle g \rangle)) \subseteq \{-1,0\}$ and $g'(p_i)=-1$

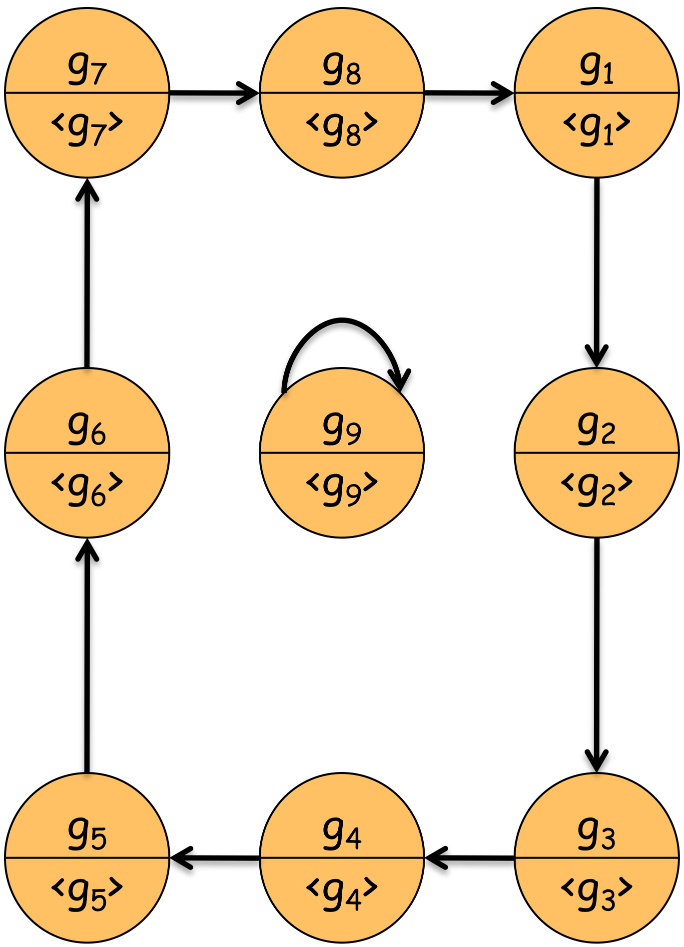
Example 7.20



Example 7.20: $S_{PL}(\Sigma)$



Example 7.20: $S_{\mathbb{P}L}(\Sigma)$



Proposition 7.21

Let:

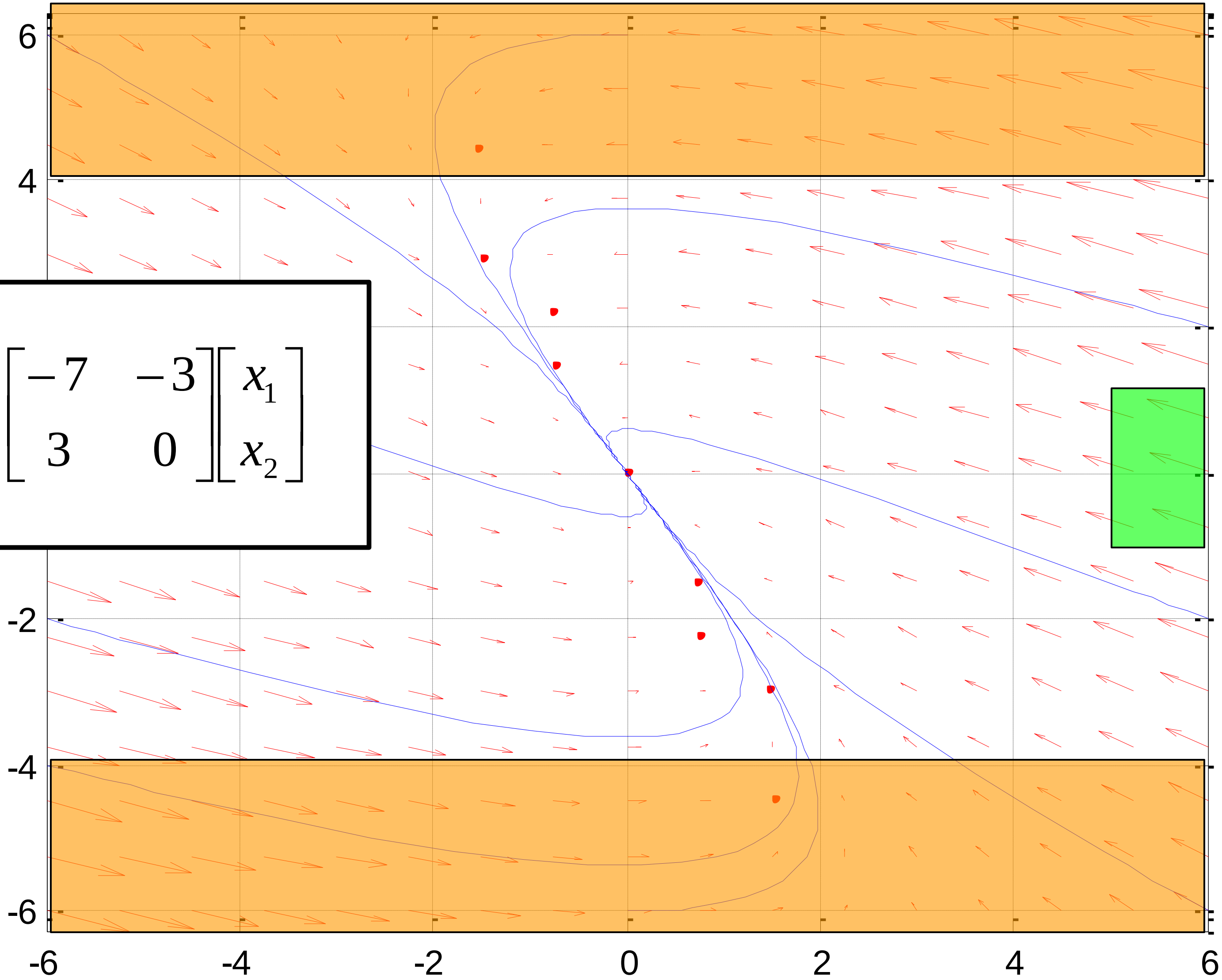
- $\Sigma = (\mathbb{R}^n, f)$ be a dynamical system
- $P = \{p_i\}_{i \in I}$ be a collection of smooth real valued functions on $\mathbb{R}^n$ (induces a partition $\mathbb{P}$)
- L be the set of initial states

Then the relation $R \subseteq \mathbb{R}^n x\{-1,0,1\}^P$

- $(x,g) \in R$ if $x \in \langle g \rangle$

is a simulation relation from $S_{PL}(\Sigma)$ to $S_{\mathbb{P}L}(\Sigma)$

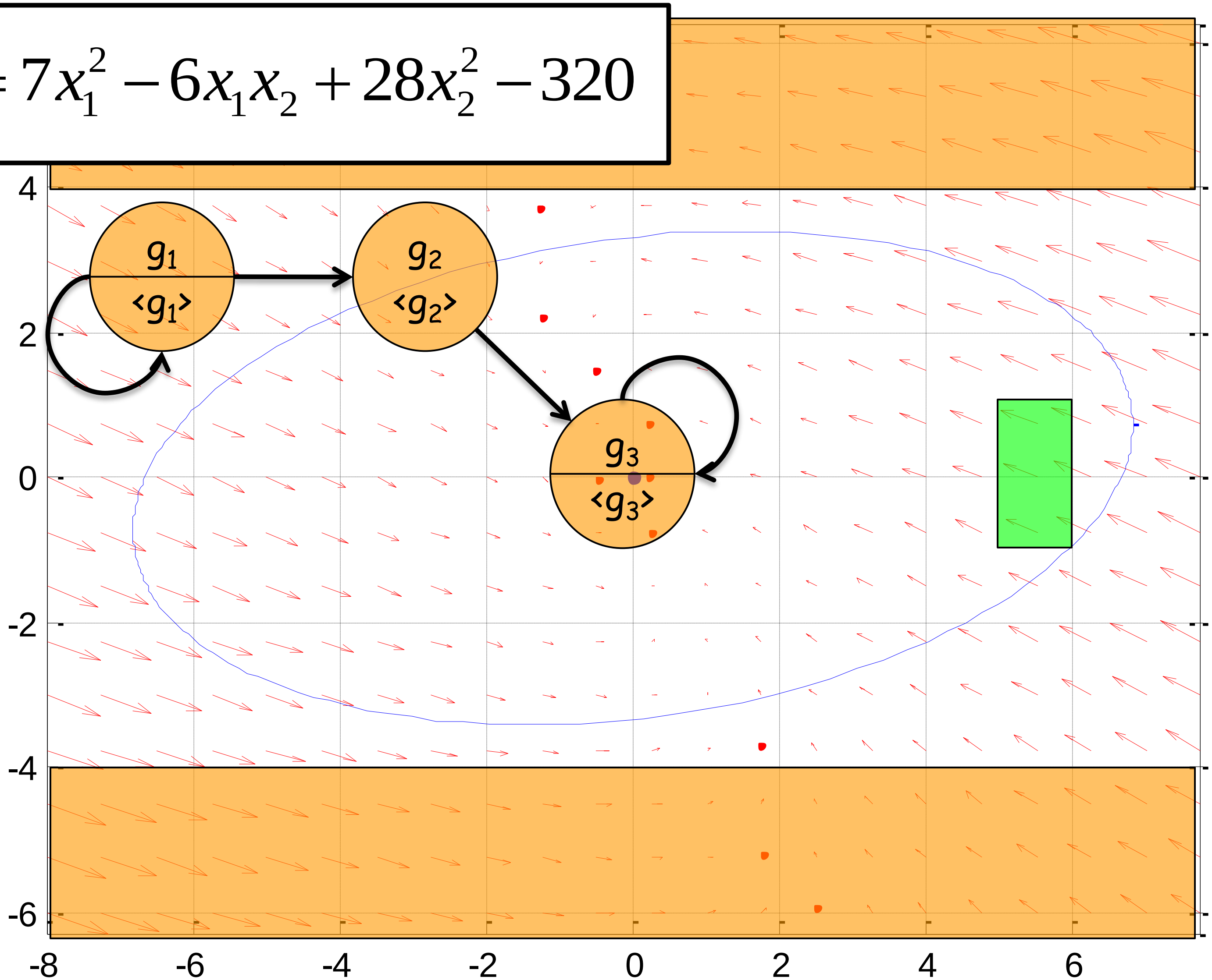
Example 7.22



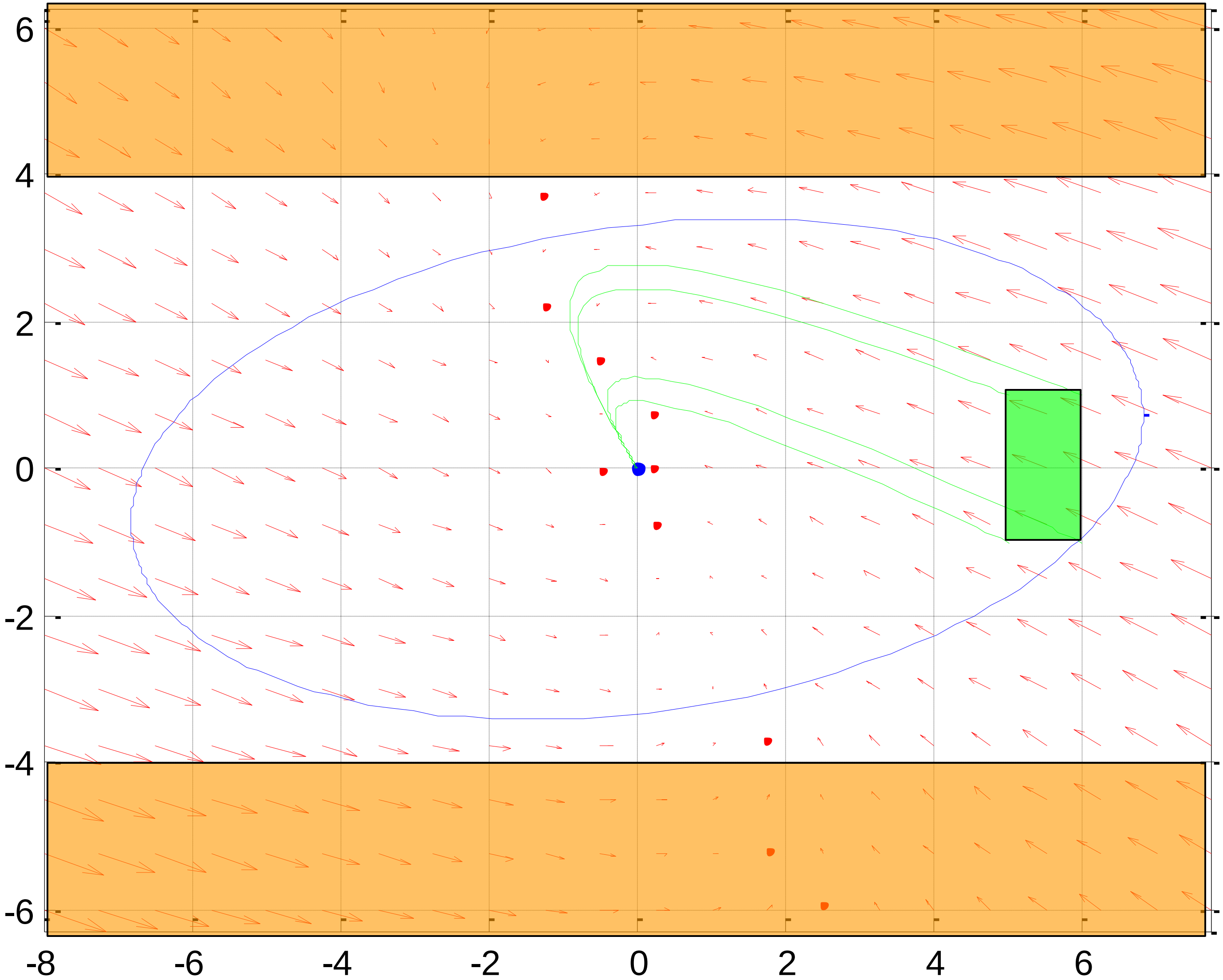
$$f(x) = \begin{bmatrix} -7 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 7.22

$$p(x) = 7x_1^2 - 6x_1x_2 + 28x_2^2 - 320$$



Example 7.22

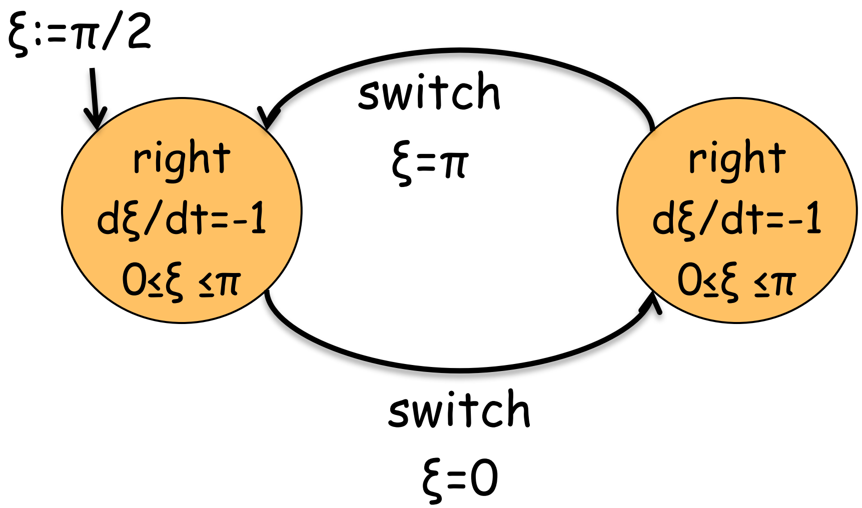


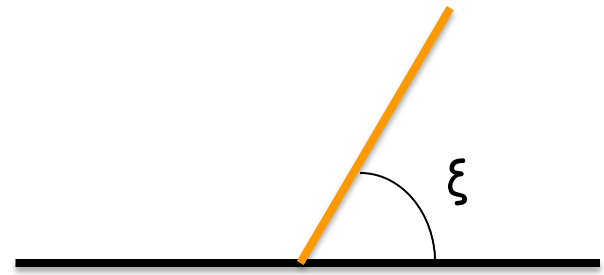
Closure wrt sign of Lie derivative

- A set P of smooth real valued functions is closed wrt to the sign of the Lie derivative when
  - for every $p \in P$, and
  - for every $k \in \mathbb{N}$
- the sign of $L_f^k p$ is constant on every set <g> and can be determined by the knowledge of g.

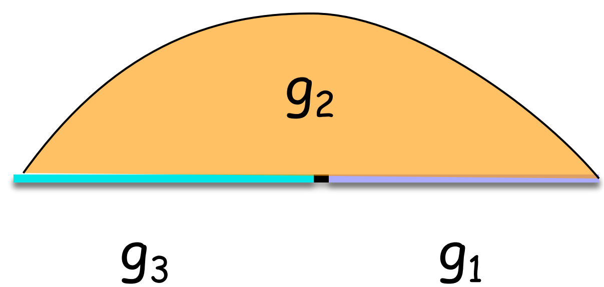
Example 7.26

one clock: dξ/dt = 1

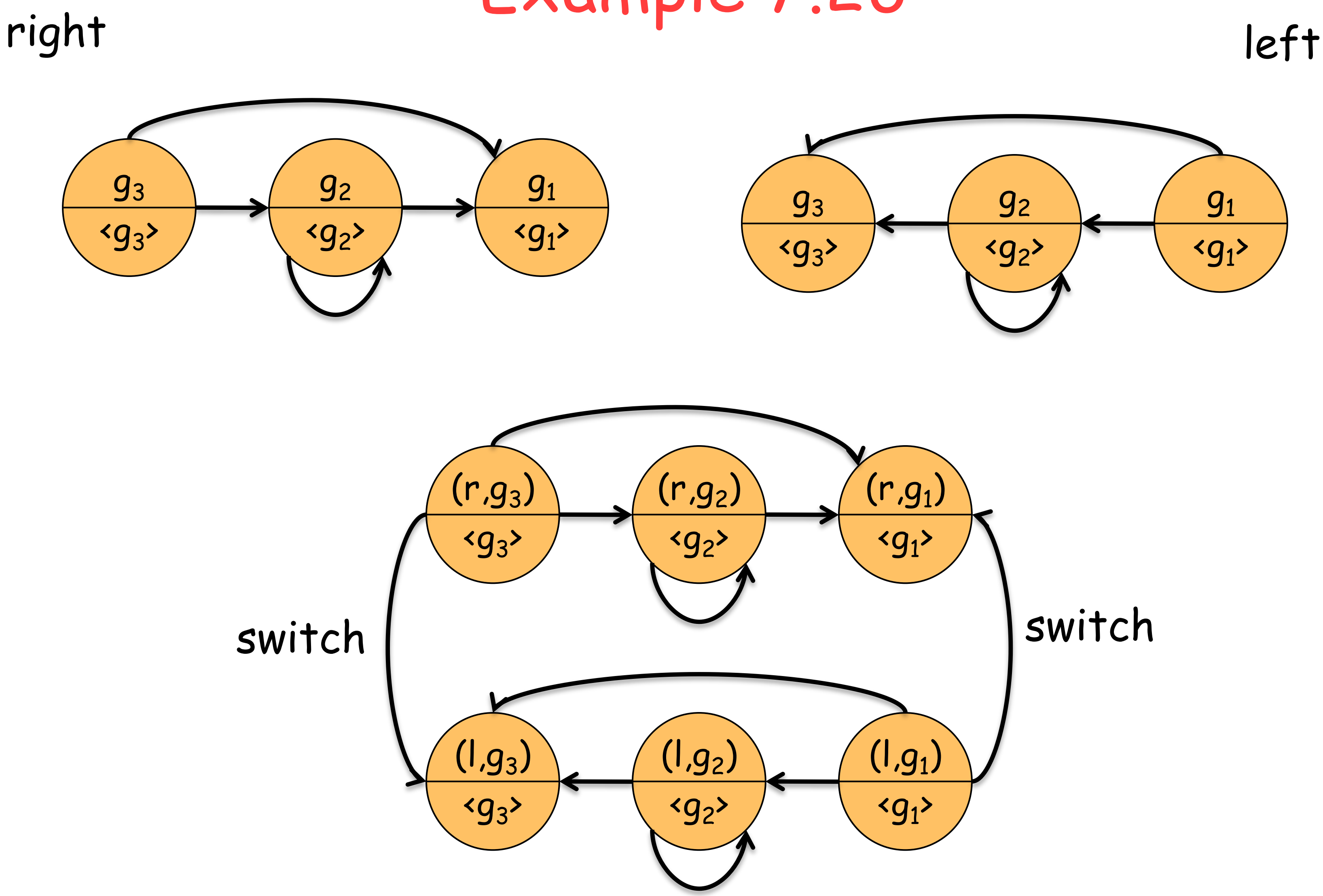




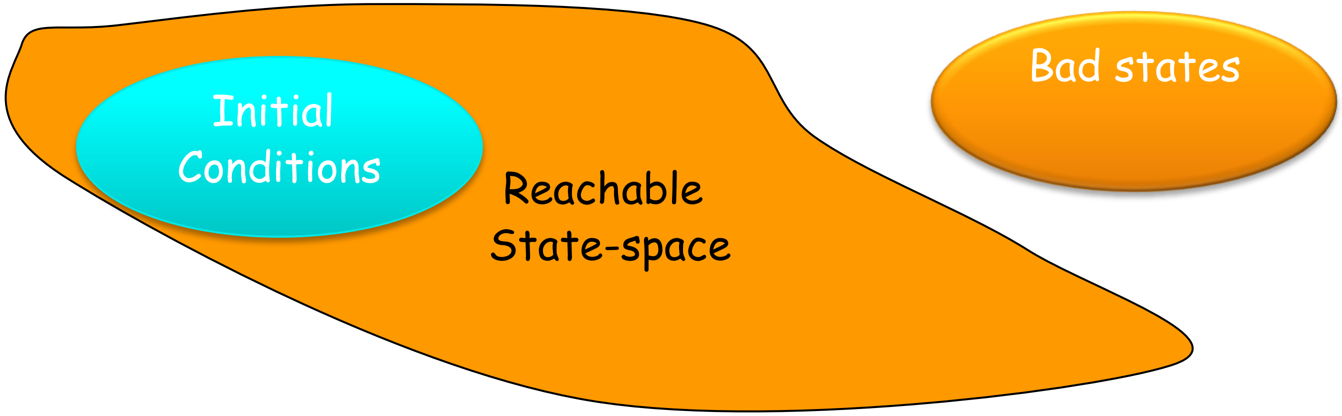
$p_1(x) = x$
$p_2(x) = x - \pi$



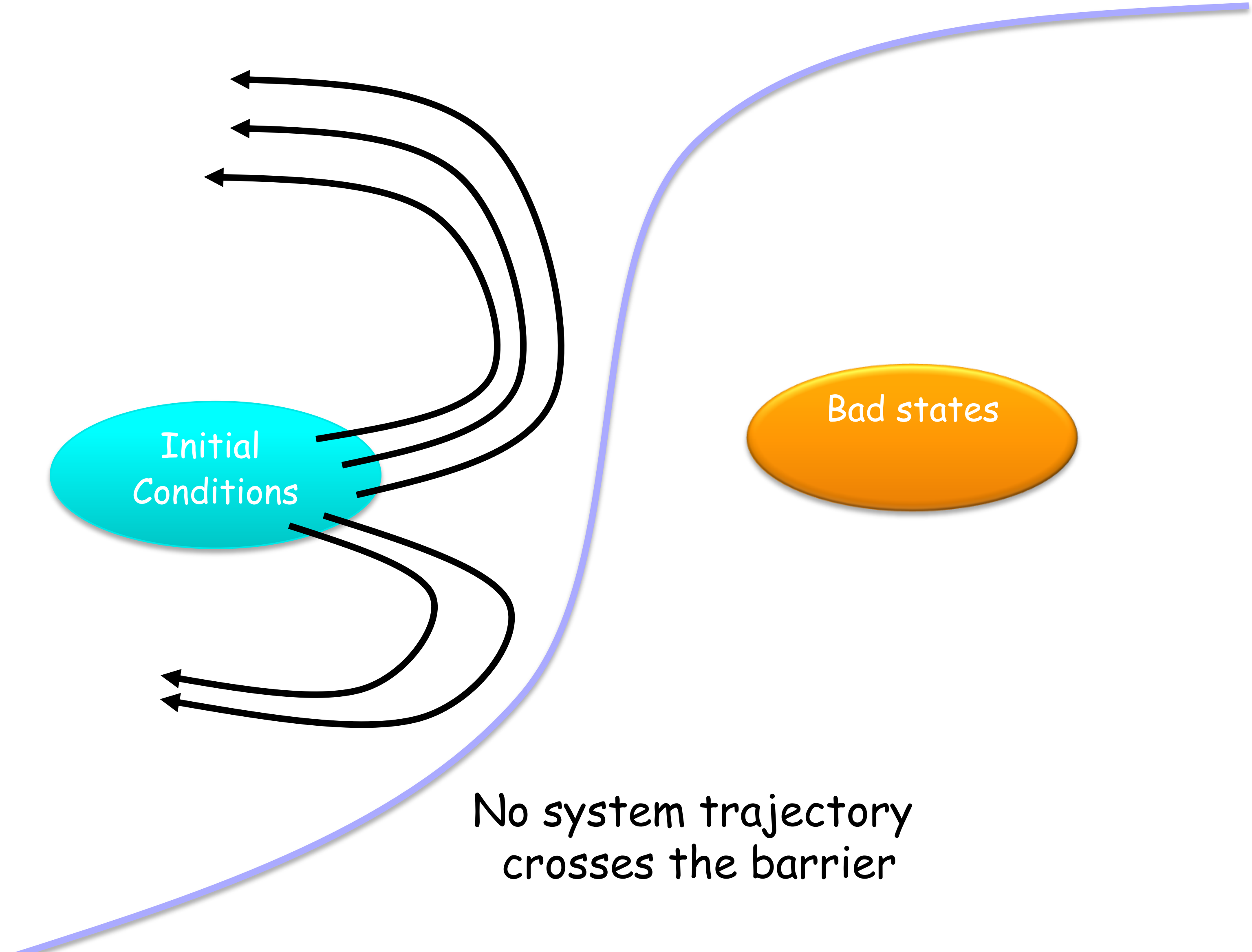
Example 7.26


right
g_3
$\langle g_3 \rangle$
g_2
$\langle g_2 \rangle$
g_1
$\langle g_1 \rangle$
left
g_3
$\langle g_3 \rangle$
g_2
$\langle g_2 \rangle$
g_1
$\langle g_1 \rangle$
(r,g_3)
$\langle g_3 \rangle$
(r,g_2)
$\langle g_2 \rangle$
(r,g_1)
$\langle g_1 \rangle$
switch
switch
(l,g_3)
$\langle g_3 \rangle$
(l,g_2)
$\langle g_2 \rangle$
(l,g_1)
$\langle g_1 \rangle$

Safety verification problem



Barrier certificates



No system trajectory crosses the barrier

Theorem 7.27

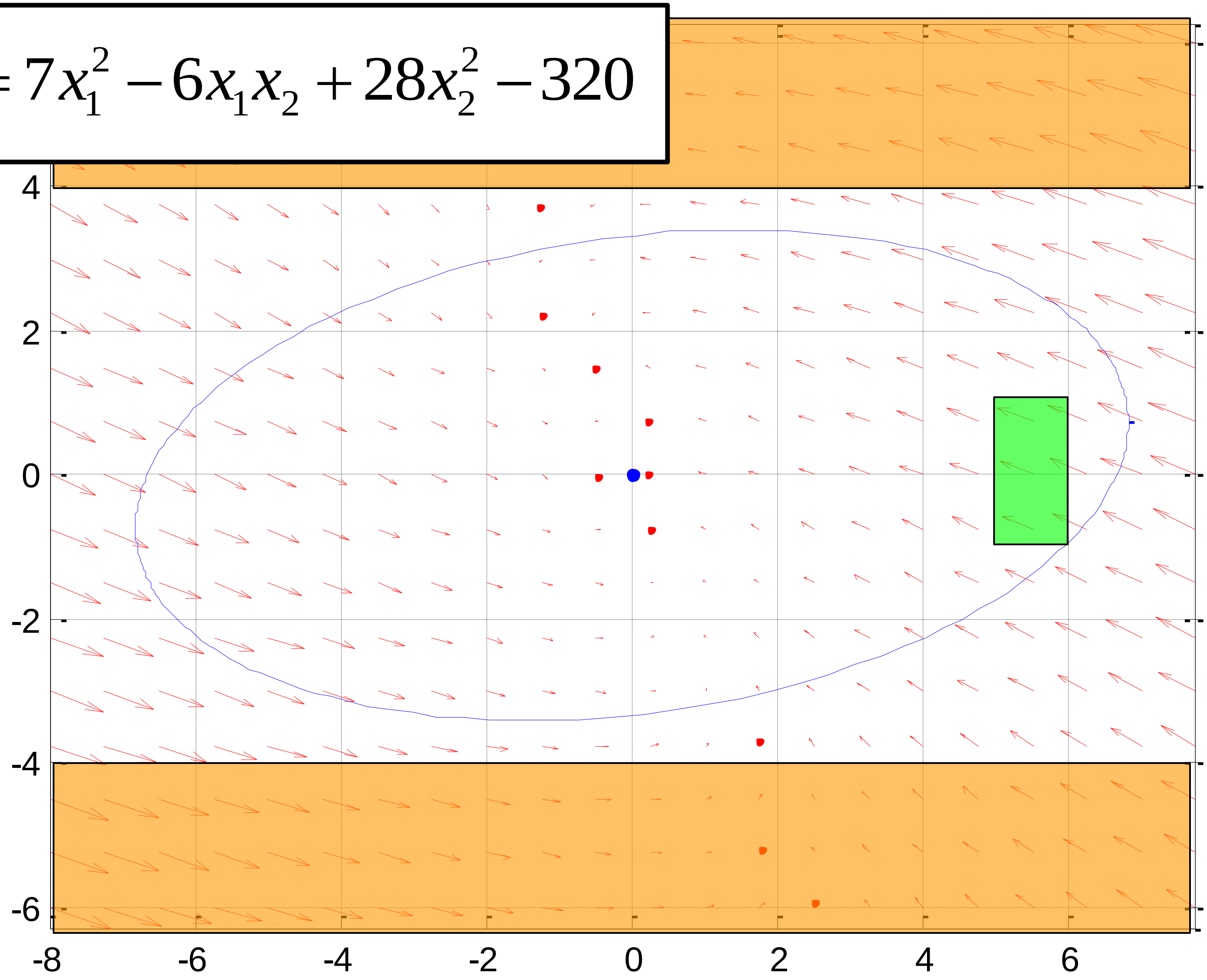
Let:

- $\Sigma = (\mathbb{R}^n, f)$ be a dynamical system
- $L \subseteq \mathbb{R}^n$ be the set of initial states
- $W \subseteq \mathbb{R}^n = Y$ be a set of unsafe outputs
- Q be a finite equivalence relation on $\mathbb{R}^n$ respecting L and $\pi_Q^{-1}(B)$
- If there exists a smooth function $E : \mathbb{R}^n \to \mathbb{R}$ s.t.
  1. $E(x) \leq 0$ for $x \in L$
  2. $E(x) > 0$ for $x \in \pi_Q^{-1}(B)$
  3. $(L_f E)(x) \leq 0$ for $x \in \mathbb{R}^n$

Then $Reach(S_{QL}(\Sigma)) \cap \pi_Q(B) = \varnothing$

Example 7.28

$$E(x) = 7x_1^2 - 6x_1x_2 + 28x_2^2 - 320$$



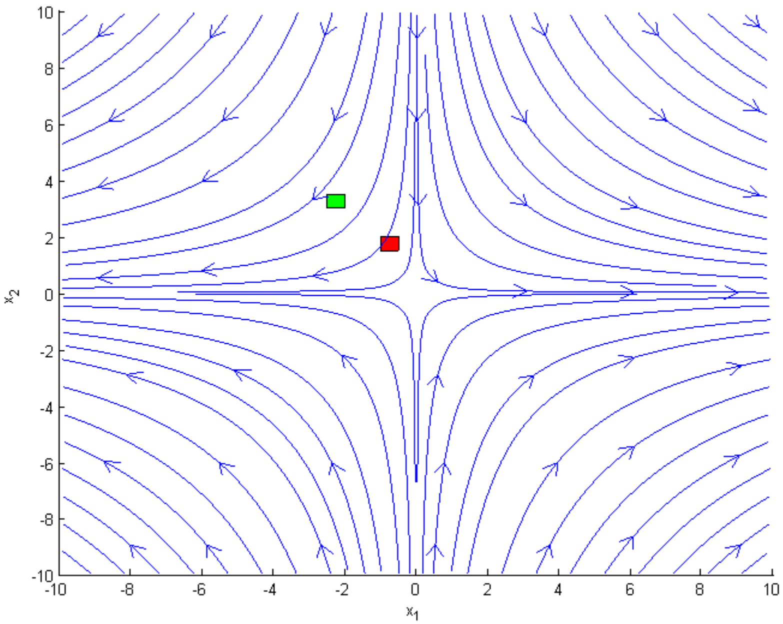
Example

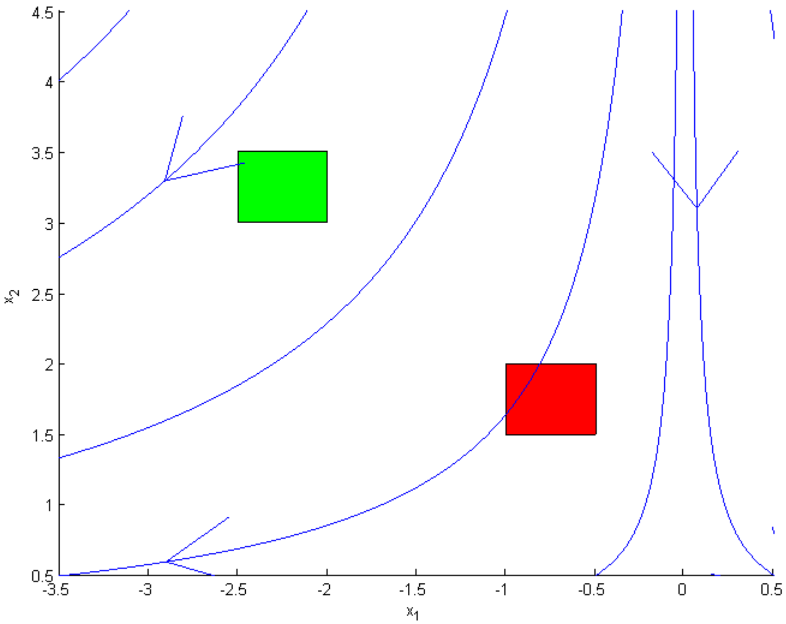
- System:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = f(x)$$

- Constraints:
  - L = {x| $-2.5 \leq x_1 \leq -2$, $3 \leq x_2 \leq 3.5$}
  - W = {x| $-1 \leq x_1 \leq -0.5$, $1.5 \leq x_2 \leq 2$}

Example





Example

Computation using SOSTOOLS

$V(x_1,x_2)$ = 12.595-.28299e-4 x_1+.12889e-4*x_2-

1.759*x_1^2+5.6059*x_1*x_2+1.6468*x_2^2