CSE 591: Theoretical Aspects of CPS

Computation of reachable sets: Tabuada Ch 7.6, References in the slides

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Reachability Problem

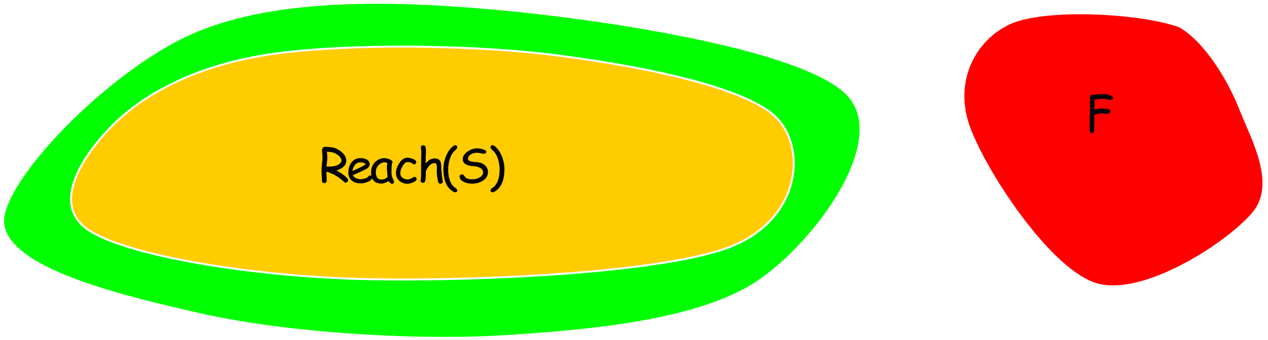
- Given a set $F \subseteq R^n$, evaluate the expressions

$$Reach_{[0,T]}(S) \cap F = \emptyset$$

$$Reach(S) \cap F = \emptyset$$

- Safety verification (F is an unsafe set)

- Exact computation difficult (impossible for most systems):
  -> Compute over-approximations the reachable set



-> S is safe

Reachability Analysis Using Numerical Techniques

- Level Sets Methods,
  [Mitchell, Tomlin and other contributors]
- CheckMate,
  [Chutinan, Krogh and other contributors]
- d/dt,
  [Dang]
- Ellipsoidal Toolbox,
  [Kurzhanski and Varaiya]
- MATISSE,
  [Girard]

Flow Pipe Approximation

- Main observation:

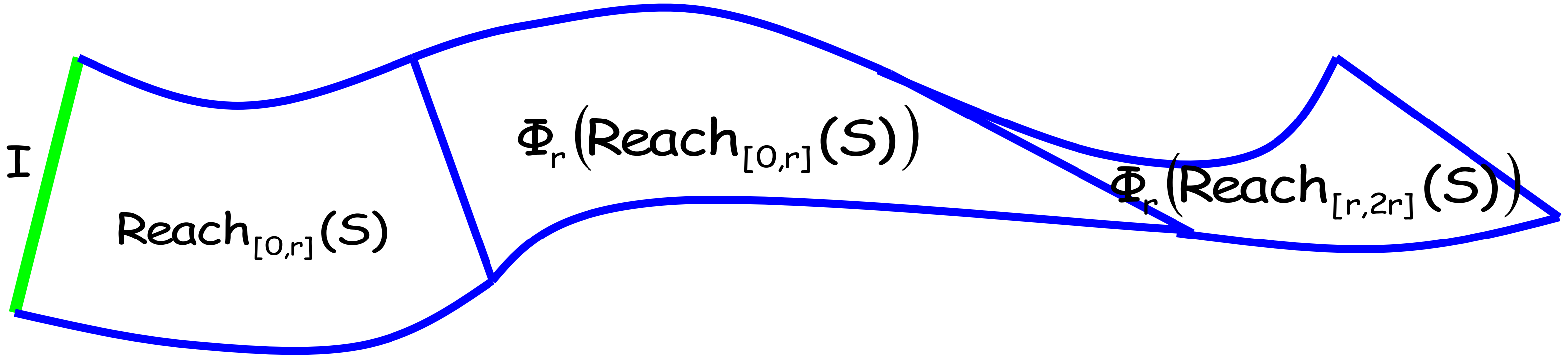
Given a time step r and T = Nr,

$$Reach_{[0,T]}(S) = \bigcup_{k=0}^{k=N-1} Reach_{[kr,(k+1)r]}(S).$$

and

$$Reach_{[kr,(k+1)r]}(S) = \Phi_r\left(Reach_{[(k-1)r,kr]}(S)\right)$$

where $\Phi_r(X)$ is the set of points reachable at time r from X.



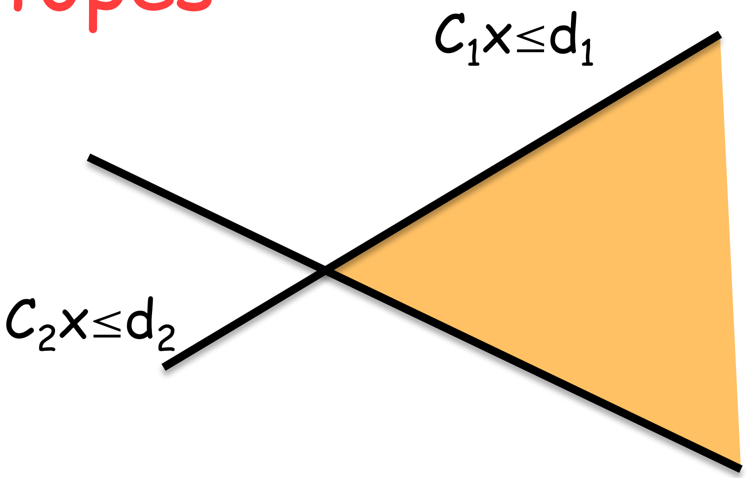
Flow Pipe Approximation

- Algorithm:
  - Over-approximate $Reach_{[0,r]}(S)$
  - Propagate the reachable set using Φ_r
- For linear systems ($f(x,u)=Ax+Bu$):

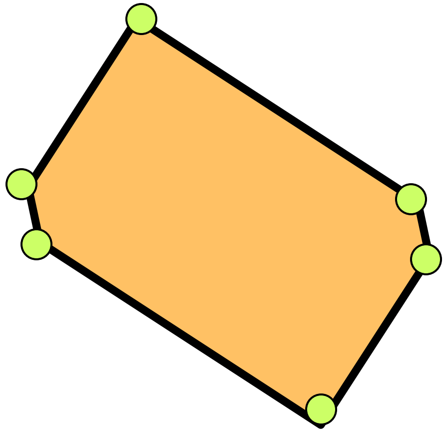
$$\Phi_r(X) = e^{rA}X \oplus V$$

- After initialization, only requires
  - linear transformations
  - Minkowski sums

Polytopes



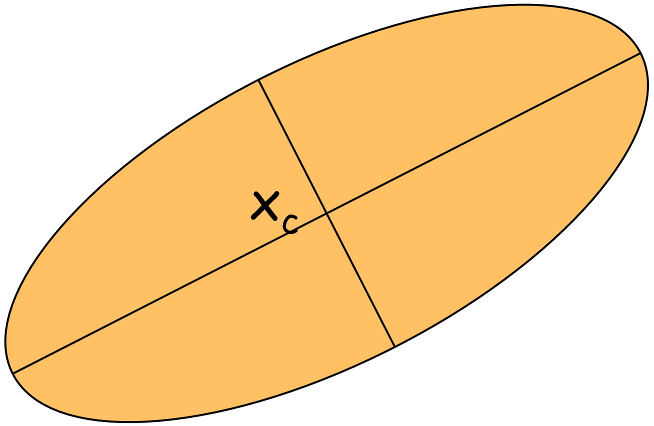
- H-Polyhedron
  - $P = \cap_{i\in\{1,p\}} \{ x\in\mathbb{R}^n \mid C_i x\leq d_i \}$
- V-Polytope
  - Convex hull of finite set of points



- H-Polyhedron vs V-Polytope
  - d-cube: 2d halfspaces vs 2^d vertices
- Matlab Toolbox for manipulating polytopes
  - MPT

Ellipsoids

- $E = \{ x \mid (x-x_c)^T A^{-1} (x-x_c) \leq 1\}$
  - A is symmetric
  - The lengths of the semi axis are given by $(\lambda_i)^{1/2}$
    λ_i are the eigenvalues of A

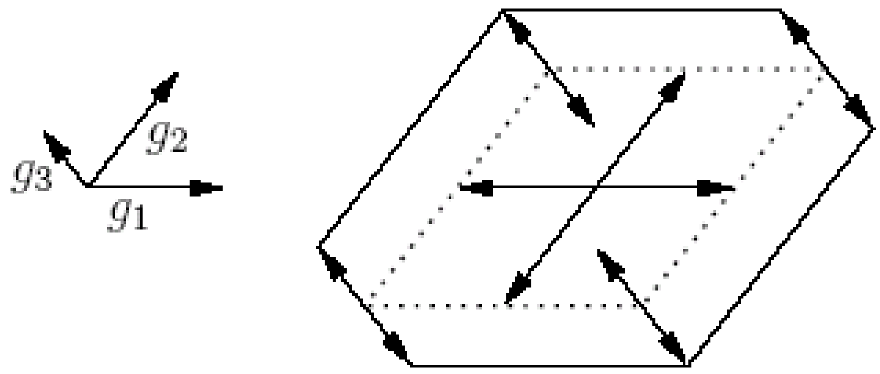


Zonotopes

- Zonotope: Minkowski sum of a finite number of segments.

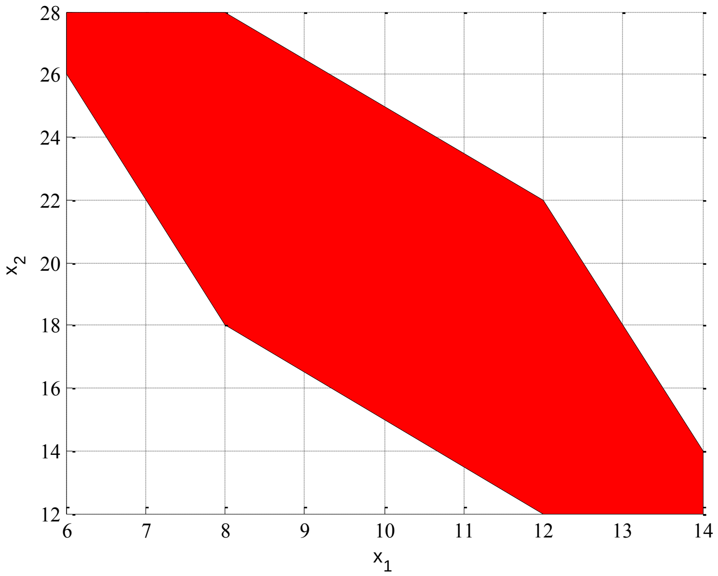
$$Z = \left\{ x \in R^n, x = c + \sum_{i=1}^{i=p} x_i g_i, -1 \le x_i \le 1 \right\}.$$

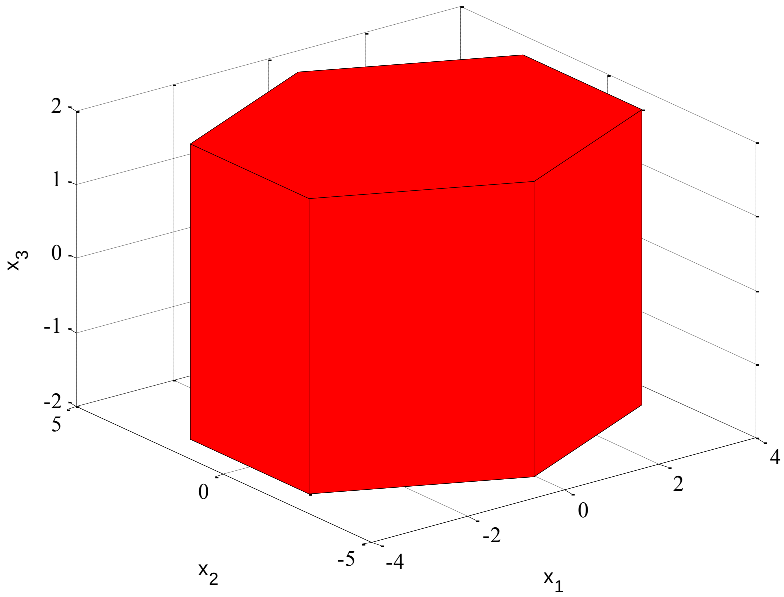
- c is the center of the zonotope, $\{g_1,\ldots,g_p\}$ are the generators. The ratio p/n is the order of the zonotope.



Two dimensional zonotope with 3 generators

Examples





$c = [10, 20]^T$

$g_1 = [2,-3]^T$

$g_2 = [1,0]^T$

$g_3 = [0,1]^T$

$g_4 = [-1,4]^T$

$c = [0, 0, 0]^T$

$g_1 = [3^{1/2},1,0]^T$

$g_2 = [3^{1/2},-1,0]^T$

$g_3 = [0,2,0]^T$

$g_4 = [0,0,2]^T$

Some Properties of Zonotopes

- The encoding of a zonotope has a polynomial complexity with the dimension.

- The set of zonotopes is closed under linear transformation

$$Z = (c, < g_1, \ldots, g_p >), LZ = (Lc, < Lg_1, \ldots, Lg_p >).$$

- The set of zonotopes is closed under the Minkowski sum

$$Z_1 = (c_1, < g_1, \ldots, g_p >), Z_2 = (c_2, < h_1, \ldots, h_q >).$$

$$Z_1 \oplus Z_2 = (c_1 + c_2, < g_1, \ldots, g_p, h_1, \ldots, h_q >).$$

- Exactly what we need for our reachability algorithm

Flow Pipe Approximation

- Choice of the representation of the reachable set:
  - closed under linear maps and Minkowski sums
    - -> Polytopes [Krogh; Dang]
    - -> Accurate
    - -> Not scalable (exponential complexity)
  - scalable representations
    - -> Ellipsoids [Kurzhanski and Varaiya]
    - -> Oriented rectangular hull [Krogh]
    - -> Not closed under linear maps and Minkowski sums
    - -> Additional computations and approximations
  - Zonotopes [Girard]
    - -> Closed under linear maps and Minkowski sums
    - -> Accurate
    - -> Scalable (polynomial complexity/dimension)

Linear Systems

Linear systems: $\dot{x} = Ax + Bu$.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

If $u(t) \in \mathbb{R}^m$: $\mathrm{im}[B \; AB \; \cdots \; A^{n-1}B]$

If $\|u(t)\|_\infty \leq \mu$,

$$\left\| \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau \right\|_\infty \leq \int_0^t \left\| e^{(t-\tau)A}Bu(\tau) \right\|_\infty d\tau$$

Linear Systems

$$\int_0^t \left\| e^{(t-\tau)A} B u(\tau) \right\|_\infty d\tau \leq \int_0^t \left\| e^{(t-\tau)A} B \right\|_\infty \| u(\tau) \|_\infty d\tau$$

$$\leq \int_0^t \left\| e^{(t-\tau)A} B \right\|_\infty \mu d\tau$$

$$= \mu \int_0^t \left\| e^{(t-\tau)A} B \right\|_\infty d\tau$$

$$= \mu \cdot \beta$$

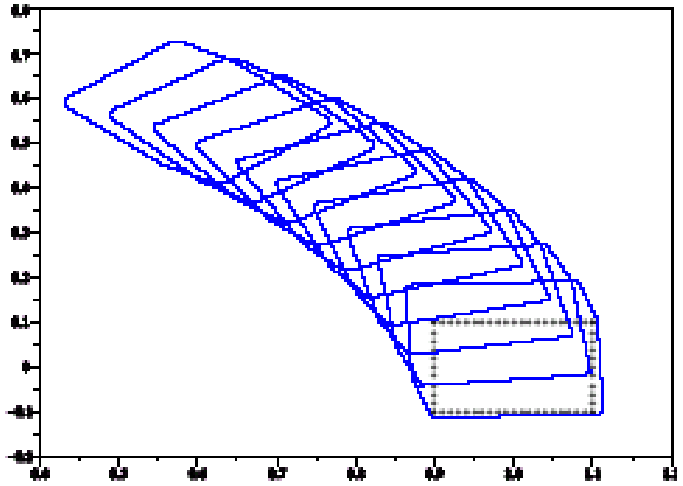
Thus $\int_0^t e^{A(t-\tau)} B u(\tau) d\tau$ is approximated with a rectangle.

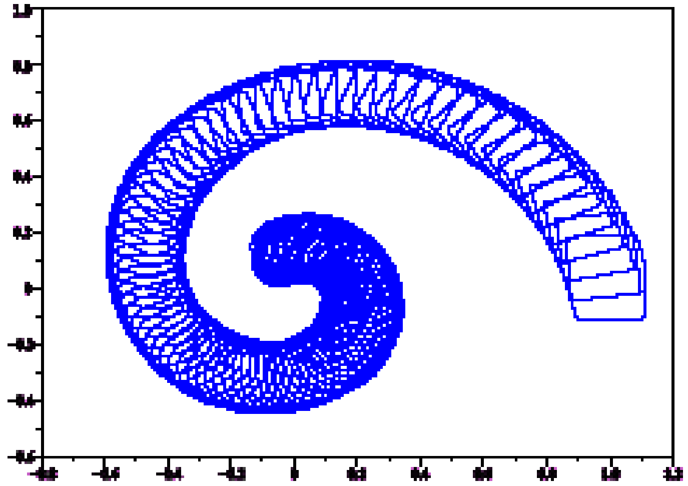
Reachability Algorithm with Zonotopes

APPROXIMATE $\text{Reach}_{[0,T]}(S)$

1. $r = \frac{T}{N}$
2. Obtain a zonotope Q_0 as an over approximation of $\text{Reach}_{[0,r]}(S)$.
3. $R_0 = Q_0$.
4. $Q_{i+1} = e^{Ar} Q_i \oplus \square$.
5. $R_{i+1} = R_i \cup Q_{i+1}$.
6. $\text{Reach}_{[0,T]}(S) \approx R_N$

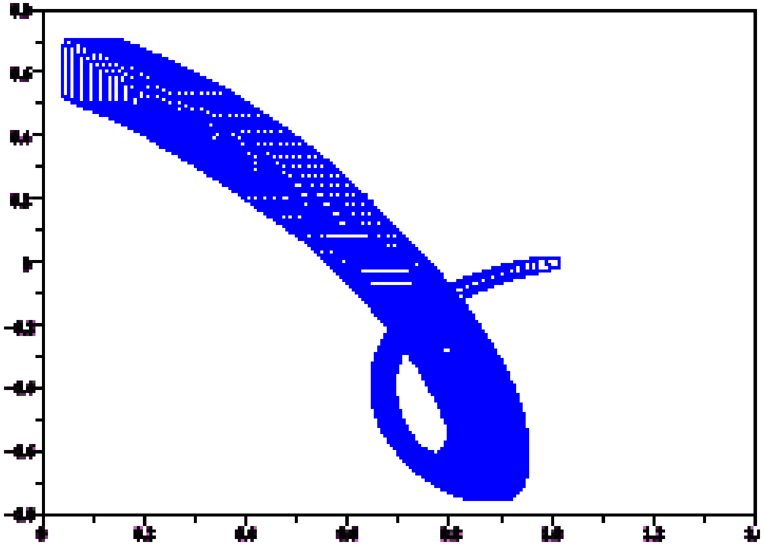
Two dimensional example

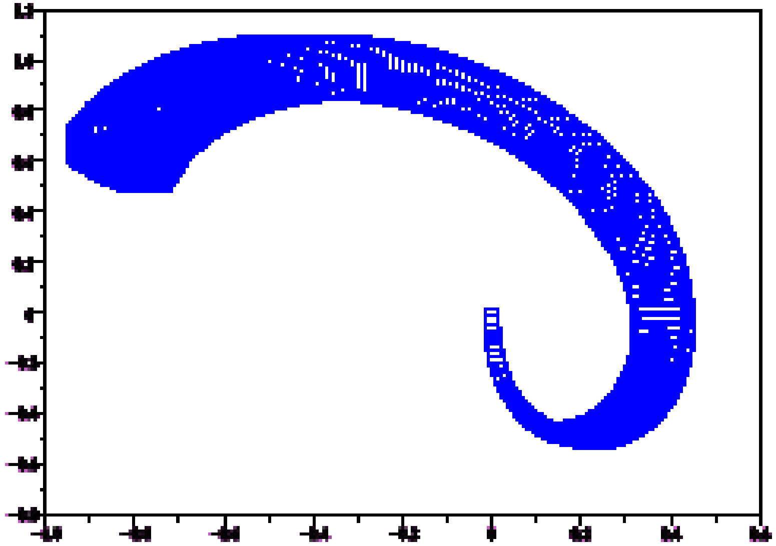




Reachable set on the interval [0,2],
100 iterations.

Five dimensional example





Projections of the reachable set on the interval [0,1], 200 iterations.

Hybrid Systems

We consider the class of hybrid systems that consists of:

1. A finite set Q of modes.
2. In each mode q, the continuous dynamics is given by a linear system:

$$\dot{x}(t) = A_q x(t) + B_q u(t),\ u(t) \in U_q$$

3. Switching conditions (Guards) are given by linear inequalities:

$$q \to q' \Leftrightarrow x(t) \in S_{q,q'} = \left\{ m_{q,q'} \le h_{q,q'} \cdot x \le M_{q,q'} \right\}$$

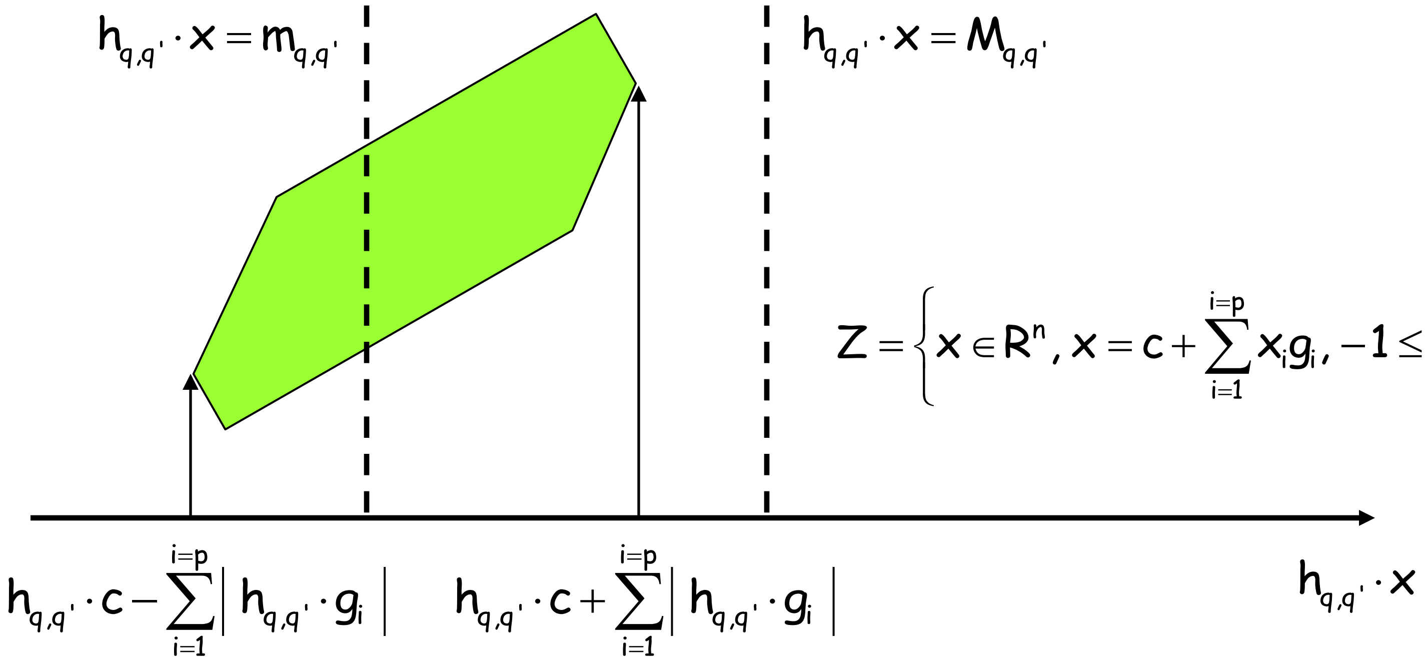
Reachability of Hybrid Systems

Following the classical scheme for reachability of hybrid systems:

- In each mode, the reachability analysis of the continuous dynamics is handled by our algorithm.
- Processing of discrete transitions requires:
  1. Detection of the intersection of a zonotope with a guard.
  2. Computation of this intersection
     - The intersection of zonotope with a band is not a zonotope.
     - Over-approximation algorithms.

Event Detection

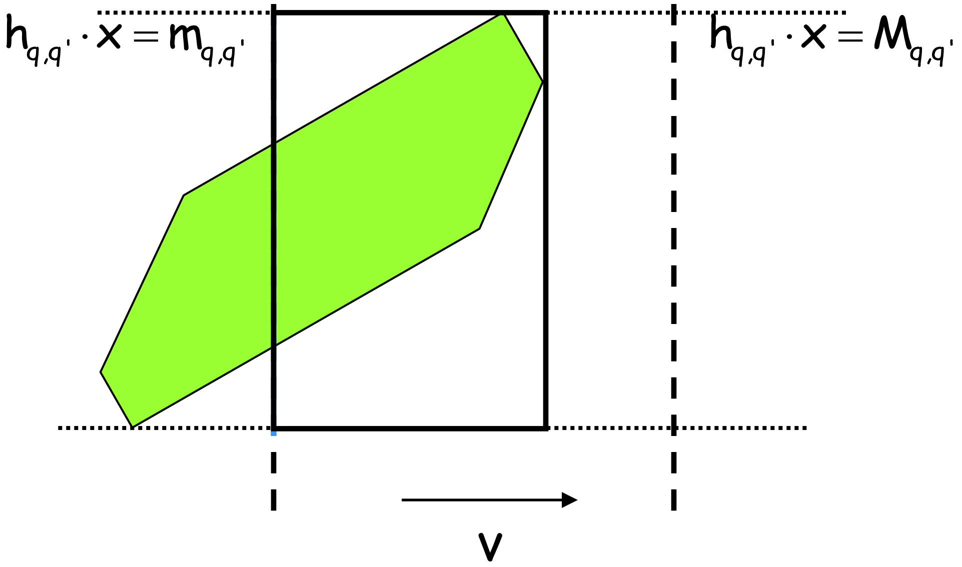
- Detection of the intersection of a zonotope with a guard.



$$\text{Intersection} \Leftrightarrow \left[h_{q,q'} \cdot c - \sum_{i=1}^{i=p} \left| h_{q,q'} \cdot g_i \right|, h_{q,q'} \cdot c + \sum_{i=1}^{i=p} \left| h_{q,q'} \cdot g_i \right| \right] \cap \left[m_{q,q'}, M_{q,q'} \right] \neq \varnothing$$

Computing the Intersection

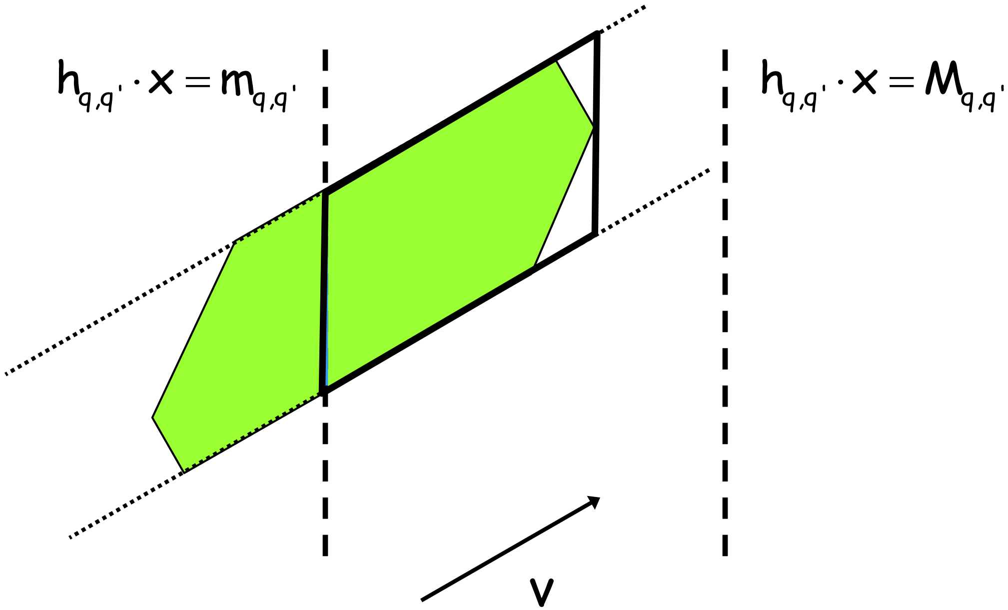
- Over-approximation by projection and bloating:



- The over-approximation I is
  - a zonotope: $I = P_v Z \oplus [\alpha, \beta] v.$
  - included in the guard

Computing the Intersection

- You can project in an other direction:



- Find the direction which results in the *best* over-approximation.

Direction of Projection

- Computation of the best direction is feasible but difficult
- Heuristics:
  - direction as weighted sum of generators (C. Le Guernic):

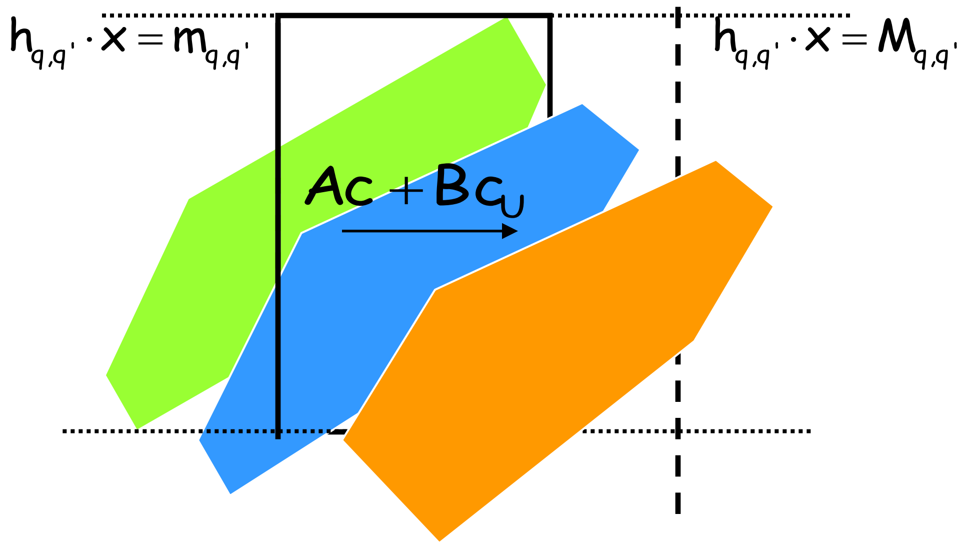
$$v = \sum_{i=1}^{i=p} \frac{|g_i \cdot h_{q,q'}|}{\|g_i\|} g_i.$$

  - use the dynamics of the system:

$$v = Ac + Bc_U.$$

Direction of Projection

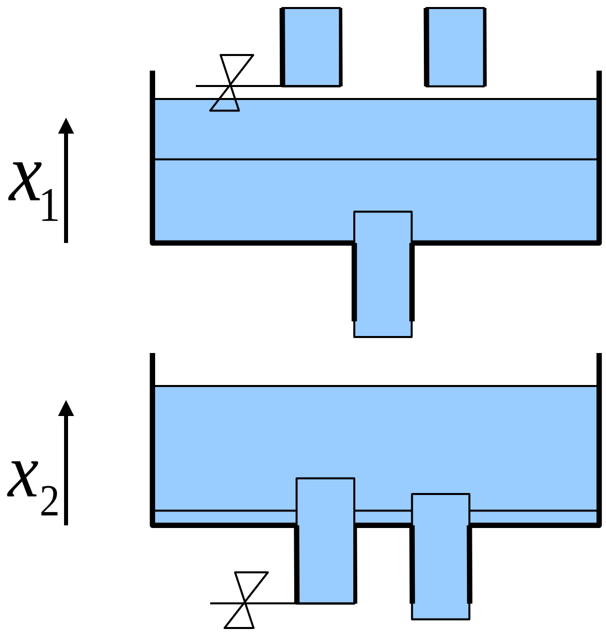
- *Dynamic* heuristic:

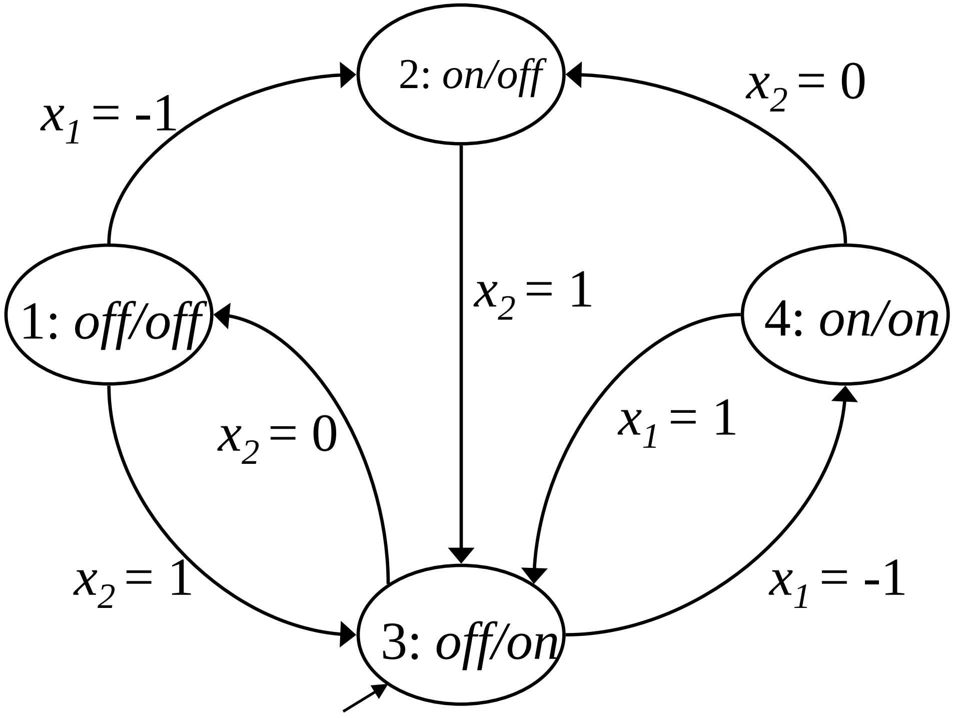


- A large part of the over-approximation at step k is actually reachable at steps k+1, k+2…

Example

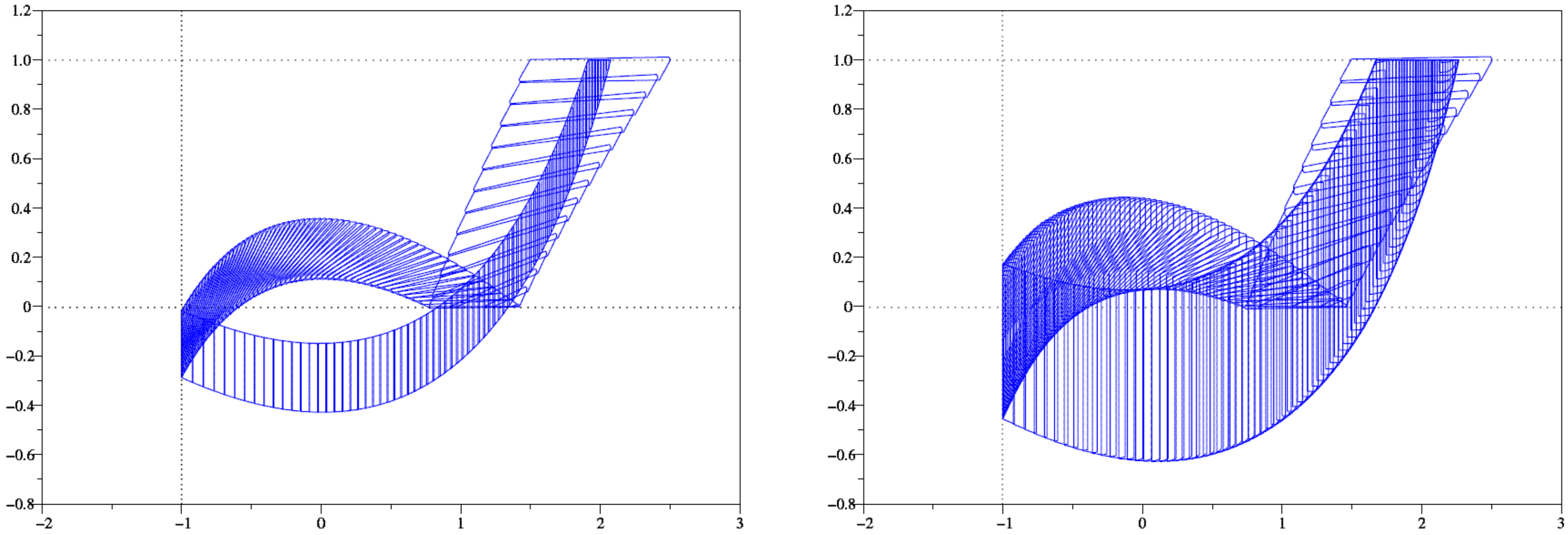
Two tank system:





Want to check robustness of periodic behavior.

Example



Reachable set of the two tank system for $\mu = 0.01$ and $\mu = 0.1$

Hybrid reachability needs to be tested for large scale examples.

Reachability of Nonlinear Systems

Two approaches for reachability analysis of nonlinear systems:

- Hybridization approach [Asarin, Dang, Girard]:
  - state space is partitioned
  - in each region, linear conservative approximation of the nonlinear vector field
  - accurate approximation
- Trajectory piecewise linearization [Han, Krogh]:
  - at each time step, vector field linearized around the center of the zonotope
  - efficient computations

MATISSE

- MATISSE is a MATLAB toolbox.
- Developed by Antoine Girard and George Pappas at UPenn.
- Main purpose is to compute abstraction/reduction of constrained linear systems, based on approximate bisimulation.
- Contains a functionality to compute the reachable set of a constrained linear system.

Constrained Linear Systems

- Constraints and reachable set are expressed as zonotopes.

- Constrained linear systems are systems of the form:

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx,\\ x(0) \in I&, u(t) \in U.\end{aligned}$$

- The set I and U are zonotopes.

Usage

- A constrained linear system (CLS) is defined as a 5-tuple, (A,B,C,U,I).

```
S = cls(A,B,C,U,I);
```

- Example:

```
A=[0 1 0;-1 0 0; 0 0 -1];B=[1;0;1];
C=[1 0 0;0 1 1];
U=inhull(-1,1);
I=inhull([9 9 9]',[10 10 10]');
S=cls(A,B,C,U,I);
```

Reachable set

- Reachable set is computed using the function reach_set. The function returns two arrays of zonotopes.

```
[Rx,Ru]=reach_set(S,dt,N);
```

- S is a CLS, dt is the time step, N is the number of intervals. The end time of the reachability algorithm is thus N x dt.
- Then, a 2-dimensional cross-section of the reachability set can be plotted using:

```
plot_reach(Rx,Ru,P,'b');
```

Plotting the reachable set

- Plotting the reachability set in 2D:

```
plot_reach(Rx,Ru,P,'b');
```

- P is a 2xm matrix that defines the projection from output space to $\mathbb{R}^2$.

- The color of the plot is defined by the last option. In this case 'b' means blue, 'r' means red, etc

Example