CSE 591: Theoretical Aspects of CPS

Multi-affine control systems: Tabuada Ch 8.4

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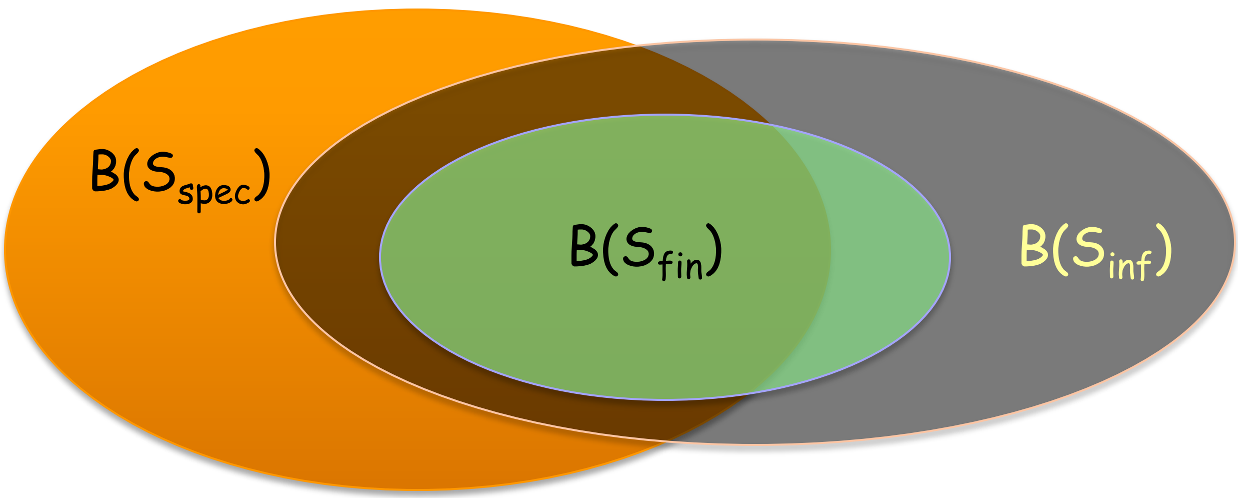
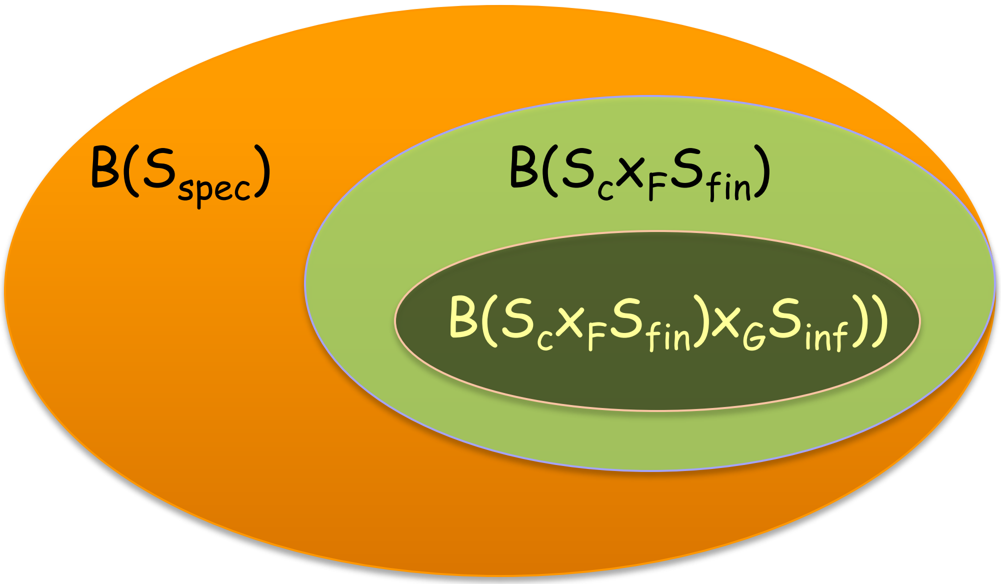
Control through Refinement

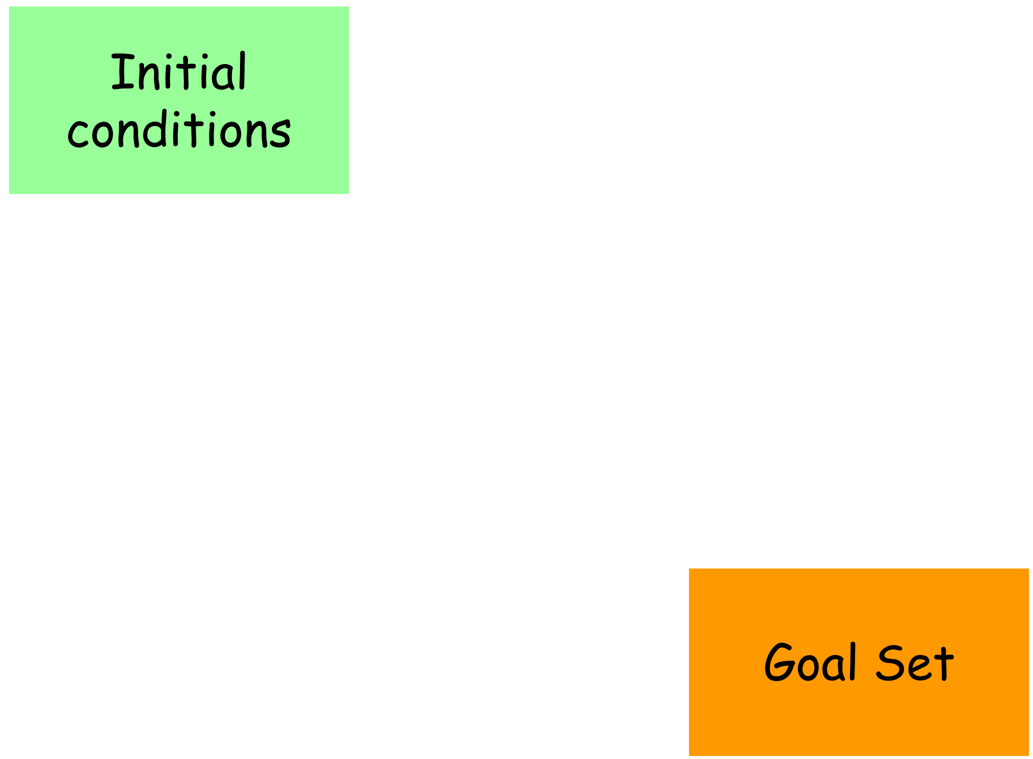
- Given a infinite-state system S_{inf}, find a controller $S_{c'}$ such that $S_{c'} x_F S_{inf}$ satisfies a specification S_{spec}
- In some cases, we can find a finite-state system S_{fin} such that $S_{fin} \preceq_{AS} S_{inf}$
- We know how to design a controller S_c for S_{fin}. How can we use S_c to control S_{inf}?



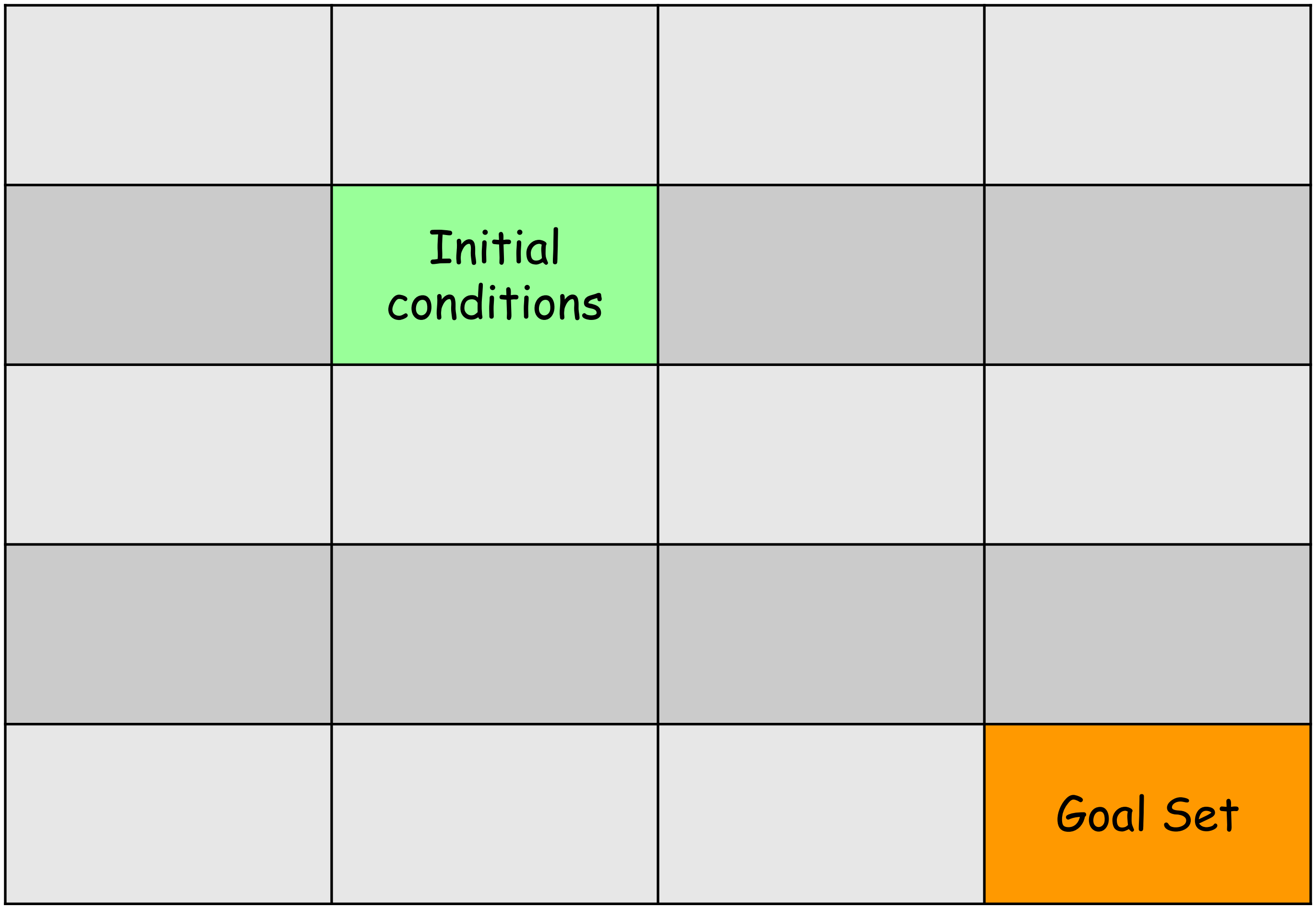
Control through Refinement



Control problem



Control problem



Affine maps

- A function $f : \mathbb{R} \to \mathbb{R}$ is affine if $\forall x,y \in \mathbb{R}$ and $\forall \alpha,\beta \in \mathbb{R}$ s.t. $\alpha+\beta=1$, we have $f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$

- Alternative def.: $f((1-\lambda)x+\lambda y) = (\alpha-\lambda)f(x)+\lambda f(y)$ with $\lambda \in \mathbb{R}$

- Let $f : A_1 \times \ldots \times A_n \to \mathbb{R}$ then $f_{âi} : A_i \to \mathbb{R}$ is a map obtained from f by fixing all the other coordinates

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is multi-affine if $\forall i$ $f_{âi} : \mathbb{R} \to \mathbb{R}$ is affine

- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is multi-affine if $\forall i$ $f_{âi} : \mathbb{R}^n \to \mathbb{R}$ is multi-affine

Some results

- Proposition 8.17
  Let E be an n-rectangle in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a multi-affine function, then:
  $$x \in E \Longrightarrow f(x) = \Sigma_{v \in V(E)} \lambda_v f(v) \text{ with } \Sigma_{v \in V(E)} \lambda_v = 1$$

- Proposition 8.18
  Let E be an n-rectangle in $\mathbb{R}^n$ and $g : V(E) \to \mathbb{R}^m$.
  There exists a unique multi-affine function $f|_{V(E)} = g$

Multi-affine control system

- A system $\Sigma=(\mathbb{R}^n,U,f)$ is multi-affine control system if

$$f(x,u) = g(x)+B(u)$$

with $B\in\mathbb{R}^{n\times m}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a multi-affine function

- We write $\Sigma=(\mathbb{R}^n,U,g,B)$

Problem: Rectangular invariant

- Consider $\Sigma=(\mathbb{R}^n, U, g, B)$ and E an n-rectangle
- Does there exist a multi-affine feedback control law $k : \mathbb{R}^n \to \mathbb{R}^m$ s.t. any solution ξ of $(\mathbb{R}^n, U, g+Bk)$ satisfies

$$\xi(0) \in E \Longrightarrow \xi(t) \in E \text{ for all } t?$$

Problem: Control to facet

- Let $\Sigma=(\mathbb{R}^n,U,g,B)$, E an n-rectangle and a facet F of E
- Does there exist a multi-affine feedback control law $k : \mathbb{R}^n \to \mathbb{R}^m$ s.t. any solution ξ of $(\mathbb{R}^n,U,g+Bk)$ satisfies

$$\xi(0)\in E \implies \exists \tau>0 \text{ s.t.}$$

$$\xi(t)\in E \text{ for all } t\in[0,\tau)$$

$$\xi(\tau)\in F$$

$$\xi(\tau)\notin E\cup F \text{ for } t\in(\tau,\tau+\varepsilon]$$

Theorem 8.22

- Consider $\Sigma=(\mathbb{R}^n,U,g,B)$ and E an n-rectangle
- The controller invariance problem is solvable if
- $\forall v \in V(E)\ .\ U_E(v) = \bigcap_{F \in F(v)} \{u \in \mathbb{R}^m \mid \eta_F^T (g(v)+Bu)<0 \} \neq \emptyset$

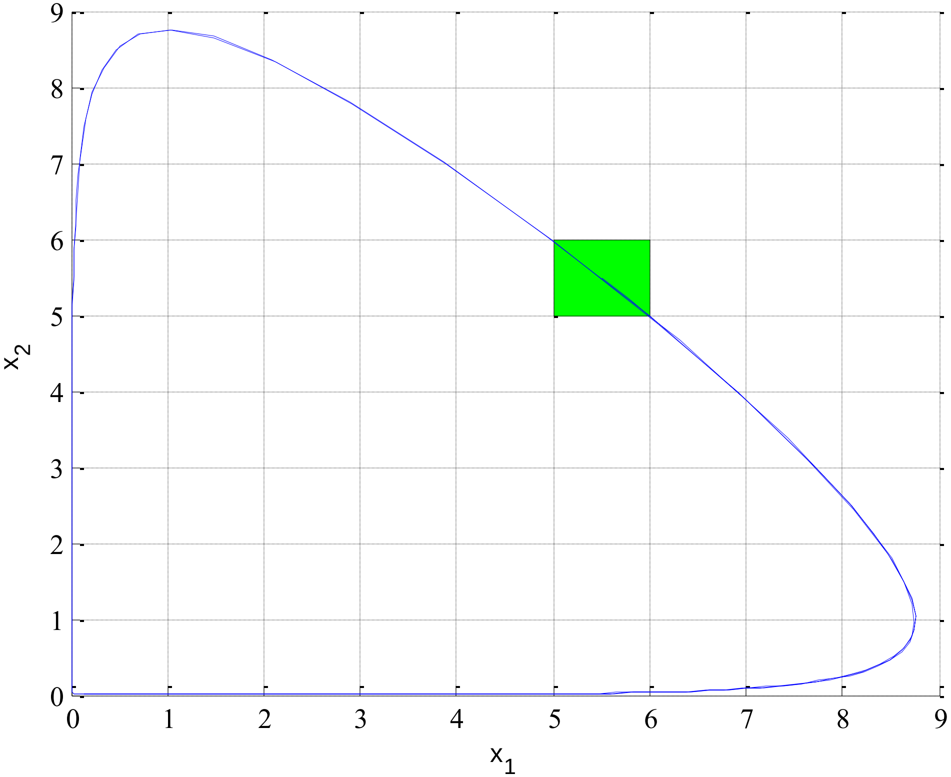
Theorem 8.23

- Consider $\Sigma=(\mathbb{R}^n,U,g,B)$, E an n-rectangle and F be a facet
- The control to facet problem is solvable if
- $\forall v \in V(E)\ .\ U_E(v) = \bigcap_{G\in F(v)} \{u\in\mathbb{R}^m \mid n_G^T\,(g(v)+Bu)<0\} \neq \emptyset$ with $F\notin F(v)$
- $\forall v \in V(E)\ .\ U_E(v) = \bigcap_{G\in F(v), G\neq F} \{u\in\mathbb{R}^m \mid n_G^T\,(g(v)+Bu)<0$ and $n_F^T\,(g(v)+Bu)>0\} \neq \emptyset$

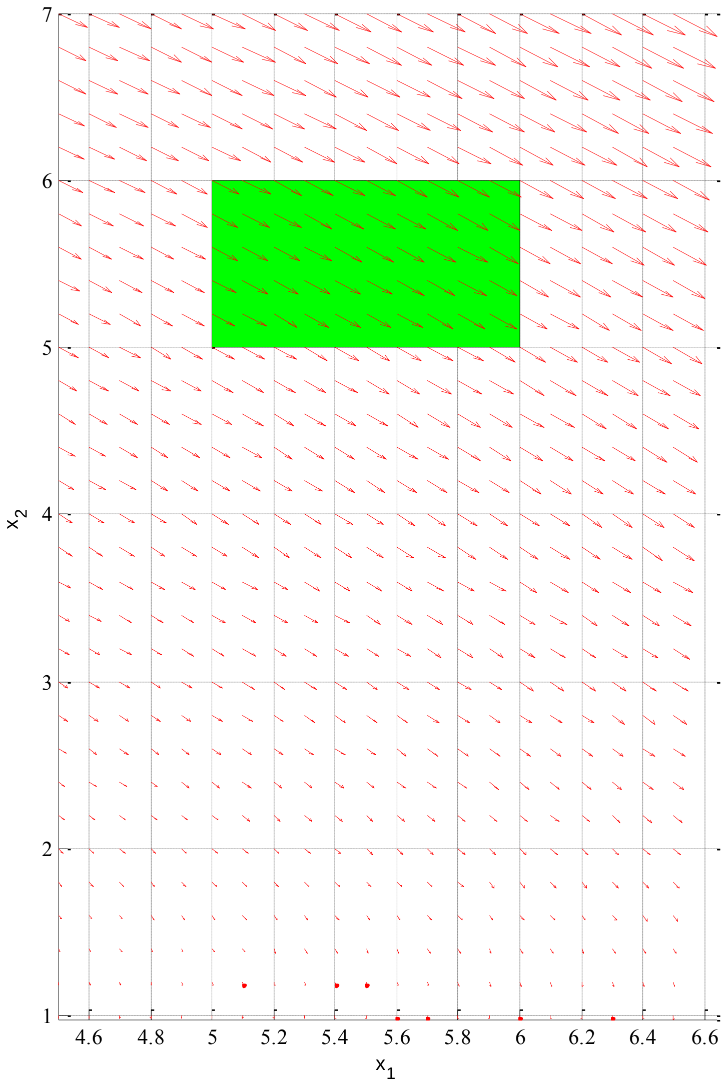
Example 8.24

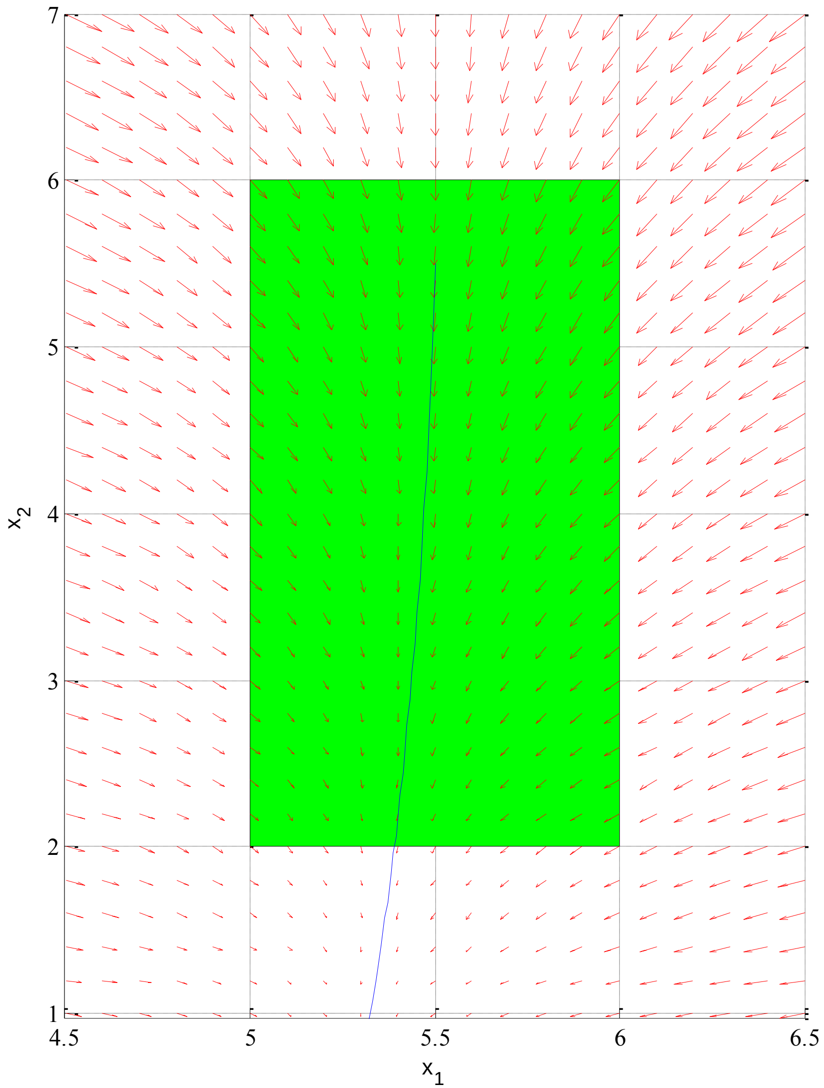
$$\dot{\xi}_1 = -\xi_1 + \xi_1\xi_2 - v$$
$$\dot{\xi}_2 = \xi_2 + \xi_1\xi_2$$

ξ_1 predators
ξ_2 prays



Example 8.24


7
6
5
4
3
2
1
x2
4.6
4.8
5
5.2
5.4
5.6
5.8
6
6.2
6.4
6.6
x1


7
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x2
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6.5
x1

Finite state system

- $\Sigma=(\mathbb{R}^n,U,g,B)$ and let P be a partition that completes $\mathcal{E}$
- The system $S_\varepsilon=\{X_\varepsilon,U_\varepsilon,\rightarrow_\varepsilon,Y_\varepsilon,H_\varepsilon)$, where
  1. $X_\varepsilon = \mathcal{E}$
  2. $U_\varepsilon = \mathcal{E}$
  3. $(x,x',x')\in\rightarrow_\varepsilon$ if
     1. $x=x'$ and the control invariance problem can be solved
     2. x,x' share a facet F and the control to facet problem is solvable
  4. $Y_\varepsilon = \mathbb{R}^n/Q$
  5. $H_\varepsilon(E)=P(E)$