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TECHNICAL NOTE

On Convergence Properties of a Least-Distance Programming Procedure for Minimization Problems under Linear Constraints[1]

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Abstract. In Ref. 1, Bazaraa and Goode provided an algorithm for solving a nonlinear programming problem with linear constraints. In this paper, we show that this algorithm possesses good convergence properties.

Key Words. Nonlinear programming, linear constraints, algorithms, quasiconvex functions, pseudoconvex functions, global convergence.

1. Introduction

Consider the following linearly constrained nonlinear programming problem:

$$\text{(P)} \quad \text{minimize} \quad f(x), \tag{1}$$

$$\text{subject to} \quad Ax = a, \tag{2a}$$

$$Bx \leq b, \tag{2b}$$

where f, A, B, a, b are the same as in Ref. 1. We use R and R^* to denote the sets of feasible points and optimal points, respectively.

In Ref. 1, Bazaraa and Goode provide an algorithm to solve Problem (P). In this paper we make an extension of the convergence theorem of Ref. 1 (i.e., Theorem 6.1 of Ref. 1), and obtain many good convergence properties of the algorithm.

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Remark 1.1. The hypothesis $1 \leq x \leq u$ in Ref. 1, as will be seen, is not essential (see Ref. 1 for notation and the algorithm).

2. Convergence Properties of the Algorithm

In this section, we assume that $\{x_k\}$, $\{d_k\}$, $\{\bar{d}_k\}$, $\{\lambda_k\}$, $\{e_k\}$, $\{m_k\}$ are all generated by the algorithm. Moreover, we assume that

$$R_k = \{x \mid Ax = a,\ B_i x \leq b_i,\ i \in I(x_k)\}, \qquad k = 1, 2, \ldots,$$

where B_i is the ith row of B, b_i is the ith component of b.

Theorem 2.1. If the sequence $\{x_k\}$ is finite, then its last term is a Kuhn–Tucker point of Problem (P). If $\{x_k\}$ is infinite, then any one of its accumulation points is a Kuhn–Tucker point of Problem (P).

Proof. The proof of Theorem 5.1 in Ref. 1 is valid for this theorem in the absence of the hypothesis $1 \leq x \leq u$. □

From the definition of the algorithm and the theory of convex analysis (see Ref. 2, p. 41), we can prove easily the following two lemmas.

Lemma 2.1. For any k, $d_k + x_k$ is the optimal solution of the following problem:

$$R(x_k) \quad \text{minimize} \quad \|x - x_k + \nabla f(x_k)\|,$$
$$\text{subject to} \quad x \in R_k.$$

And, for any $x \in R_k$, we have

$$d_k^{\mathrm{T}}(d_k + x_k - x) \leq -\nabla f(x_k)^{\mathrm{T}}(d_k + x_k - x).$$

Lemma 2.2. For any $x \in R_k$, we have

$$\|x_{k+1} - x\|^2 \leq \|x_k - x\|^2 + 6[f(x_k) - f(x_{k+1})] - 2(\tfrac{1}{2})^{m_k} e_k \lambda_k \nabla f(x_k)^{\mathrm{T}}(x_k - x).$$

The following theorem is an extension of Theorem 6.1 of Ref. 1.

Theorem 2.2. Suppose that $\{x_k\}$ is an infinite sequence. Then:

(i) $\{f(x_k)\}$ is a monotone decreasing sequence;
(ii) if x^* is an accumulation point of $\{x_k\}$ and there exists an integer K_1 such that

$$\nabla f(x_k)^{\mathrm{T}}(x_k - x^*) \geq 0, \qquad \text{for any } k \geq K_1,$$

then $\{x_k\}$ converges to x^*;

(iii) if $\lim_{k\to\infty} \|x_k\| = +\infty$ and there exist $\bar{x} \in R$ and an integer K_2 such that

$$\nabla f(x_k)^{\mathrm{T}}(x_k - \bar{x}) \geq 0, \qquad \text{for any } k \geq K_2,$$

then

$$\lim_{k\to\infty} f(x_k) = -\infty.$$

Proof. (i) is true by the definition of the algorithm. We now begin to prove (ii). From (i), it follows that $\{f(x_k)\}\downarrow f(x^*)$. Hence,

$$\sum_{k=1}^{\infty} (f(x_k) - f(x_{k+1})) < +\infty. \tag{3}$$

Since

$$\nabla f(x_k)^{\mathrm{T}}(x_k - x^*) \geq 0, \qquad \text{for } K \geq K_1,$$

we have that

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + 6[f(x_k) - f(x_{k+1})] \tag{4}$$

holds for $k \geq K_1$ by Lemma 2.2. From (3), (4), and the fact that x^* is an accumulation point of $\{x_k\}$, we conclude that $\{x_k\}$ converges to x^*.

We finally prove (iii). From

$$\lim_{k\to\infty} \|x_k\| = +\infty,$$

we obtain

$$\lim_{k\to\infty} \|x_{k+1} - \bar{x}\|^2 = +\infty.$$

Because

$$\nabla f(x_k)^{\mathrm{T}}(x_k - \bar{x}) \geq 0, \qquad \text{for any } k \geq K_2,$$

by Lemma 2.2, we have

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_{K_2} - \bar{x}\|^2 + 6[f(x_{K_2}) - f(x_{k+1})], \qquad \text{for every } k \geq K_2. \tag{5}$$

Hence,

$$\lim_{k\to\infty} f(x_k) = -\infty,$$

and the proof is complete. □

Section 6 of Ref. 1 gives a proof to the assertion that, if $\{x_k\}$ has an accumulation point $\bar{x}$ which satisfies the second-order sufficiency optimality conditions, then the whole sequence of iterates converges to $\bar{x}$. But that proof cannot be shifted to the present case, because it makes use of the hypothesis $1 \leq x \leq u$. We now prove the above assertion without using the hypothesis $1 \leq x \leq u$.

Definition 2.1. $\bar{x}$ is said to satisfy the second-order optimality conditions for Problem (P) if $\bar{x}$ is a Kuhn–Tucker point of Problem (P), $f(x)$ is twice differentiable at $\bar{x}$, and there is a positive number γ such that, whenever

$$\nabla f(\bar{x})^{\mathrm{T}} d \leq 0, \qquad Ad = 0, \qquad B_i d \leq 0, \qquad i \in J(\bar{x}), \quad \|d\| = 1,$$

we have

$$d^{\mathrm{T}} H(\bar{x}) d > \gamma,$$

where

$$J(\bar{x}) = \{i \mid B_i \bar{x} = b_i\},$$

and $H(\bar{x})$ is the Hessian of $f(x)$ at $\bar{x}$.

Theorem 2.3. Suppose that $\bar{x}$ satisfies the second-order optimality conditions for Problem (P). Then, there exists a number $\epsilon > 0$ such that

$$0 < \|x - \bar{x}\| \leq \epsilon \Rightarrow \nabla f(x)^{\mathrm{T}}(x - \bar{x}) > 0, \qquad x \in R. \tag{6}$$

Proof. By Lemma 6.1 of Ref. 1, there exists a number $\theta > 0$ such that

$$\nabla f(\bar{x})^{\mathrm{T}} d \leq \theta, \qquad Ad = 0,$$

$$B_i d \leq 0,\ i \in J(\bar{x}),\ \|d\| = 1 \Rightarrow d^{\mathrm{T}} H(\bar{x}) d \geq \gamma/2. \tag{7}$$

Because the set

$$\{d \mid \|d\| = 1, Ad = 0, B_i d \leq 0, i \in J(\bar{x}), \nabla f(\bar{x})^{\mathrm{T}} d \geq \theta\}$$

is compact and $f(x)$ is continuously differentiable, there exists a positive number ϵ_1 such that

$$x \in R, \qquad \|x - \bar{x}\| \leq \epsilon_1, \qquad \|d\| = 1, \qquad Ad = 0, B_i d \leq 0,$$

$$i \in J(\bar{x}), \qquad \nabla f(\bar{x})^{\mathrm{T}} d > \theta \Rightarrow \nabla f(x)^{\mathrm{T}} d \geq \theta/2. \tag{8}$$

Since $f(x)$ is twice differentiable at $\bar{x}$, there exists a positive number $\epsilon > \epsilon_1$ such that

$$0 < \|x - \bar{x}\| \leq \varepsilon$$

$$\Rightarrow \|[\nabla f(x) - \nabla f(\bar{x})]/\|x - \bar{x}\| - H(\bar{x})(x - \bar{x})/\|x - \bar{x}\|\,\| \leq \gamma/4, \qquad x \in R. \tag{9}$$

Suppose that $x \in R$ and that $0 < \|x - \bar{x}\| \leq \epsilon$. Then, one and only one of the following two cases holds:

(i) $\nabla f(\bar{x})^T(x - \bar{x})/\|x - \bar{x}\| > \theta$;
(ii) $0 \leq \nabla f(\bar{x})^T(x - \bar{x})/\|x - \bar{x}\| \leq \theta$.

We have $\nabla f(x)^T(x - \bar{x}) > 0$ from (8) in case (i) and the same from (9) and (7) in case (ii). This completes the proof. □

Corollary 2.1. Suppose that $\bar{x}$ is an accumulation point of $\{x_k\}$ and that $\bar{x}$ satisfies the second-order optimality conditions for Problem (P). Then, there exists an integer K_1 such that

$$\nabla f(x_k)^T(x_k - \bar{x}) \geq 0, \qquad \text{for any } k \geq K_1.$$

Hence, $\{x_k\}$ converges to $\bar{x}$.

Proof. This is a consequence of Theorem 2.3, a result of Ref. 1 (see Lemma 6.2 in Ref. 1), and Theorem 2.2. □

The following two theorems show that the algorithm has many good convergence properties when $f(x)$ is either quasiconvex or pseudoconvex.

Theorem 2.4. Suppose that $f(x)$ is quasiconvex on R. Then:

(i) $\{f(x_k)\}$ is a monotone decreasing sequence;
(ii) if $d_K = 0$ for some K, then x_K is a Kuhn–Tucker point of Problem (P); otherwise, $\{x_k\}$ is infinite and we have the property below;
(iii) if $\bar{x}$ is an accumulation point of $\{x_k\}$, then $\bar{x}$ is a Kuhn–Tucker point of Problem (P), and the whole sequence $\{x_k\}$ converges to $\bar{x}$;
(iv) if $\{x_k\}$ has no accumulation point, then $R^* = \emptyset$ and

$$\{f(x_k)\} \downarrow \inf\{f(x) \mid x \in R\}.$$

Proof. (i) is obvious. (ii) is obtained from Theorem 2.1. We now begin to prove (iii). Since $\bar{x}$ is an accumulation point of $\{x_k\}$, we know that $\bar{x}$ is a Kuhn–Tucker point of Problem (P) and that $\{f(x_k)\} \downarrow f(\bar{x})$. Therefore, by the quasiconvexity of $f(x)$, $\nabla f(x_k)^T(x_k - \bar{x}) \geq 0$ holds for each integer k. By Theorem 2.2, $\{x_k\}$ converges to $\bar{x}$.

Finally, we prove (iv). Suppose that $\{x_k\}$ has no accumulation point. We need to show that $R^* = \emptyset$ and that $\{f(x_k)\} \downarrow \inf\{f(x) \mid x \in R\}$.

If any one of the two is false, then there exists some $x^* \in R$ such that $f(x_k) > f(x^*)$ holds for each integer k. By the quasiconvexity of $f(x)$, $\nabla f(x_k)^T(x_k - x^*) \geq 0$ must hold for each integer k. By Lemma 2.2, we obtain

$$\|x_{k+1} - x^*\|^2 \leq \|x_1 - x^*\|^2 + 6[f(x_1) - f(x^*)]. \tag{10}$$

Thus, $\{x_k\}$ is bounded. The contradiction completes the proof. □

Theorem 2.5. Suppose that $f(x)$ is pseudoconvex on R. Then:

(i) if $d_K = 0$ for some K, then x_K is a Kuhn–Tucker point of Problem (P); otherwise, $\{x_k\}$ is infinite and we have the property below;

(ii) $\{f(x_k)\}\downarrow\inf\{f(x) \mid x \in R\}$;

(iii) the necessary and sufficient condition that $R^* \neq \emptyset$ is that $\{x_k\}$ is bounded;

(iv) if $R^* \neq \emptyset$, then $\{x_k\}$ converges to some $x^* \in R^*$.

Proof. Since $f(x)$ is quasiconvex, the assertions of Theorem 2.4 hold. Since $f(x)$ is pseudoconvex, R^* is the set of Kuhn–Tucker points of Problem (P). This completes the proof. □

References

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