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Order Matters: On the Impact of Swapping Order on an Entanglement Path in a Quantum Network

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***Abstract*—In this paper, we study the properties of path metrics of an entanglement path for a given entanglement swapping order of the path. We show how to efficiently compute the path metrics of an entanglement path for any given swapping order. We show that different entanglement swapping orders for the same path can lead to different expected throughputs. A key finding is that the binary operator corresponding to entanglement swapping along a path is not associative. We further show that the problem of computing an s-t path with maximum expected throughput under any entanglement swapping order does not have the sub-path optimality property, which is a key property most path finding algorithms such as Dijkstra's algorithm rely on. We use extensive simulations to validate our theoretical findings.**

***Keywords:* Entanglement routing, entanglement swapping, expected throughput, path metrics, quantum networks.**

1. Introduction

Quantum networks, composed of quantum repeater stations connected by quantum channels such as optical fibers or free space links, promise a smörgåsbord of advantages over classical networks [15]. Entanglements form the basis of all quantum communications: if Alice and Bob each share half of an entangled pair of qubits, known as a Bell pair, using local operations and classical communication (LOCC), Alice can use the Bell pair to teleport a data qubit to Bob, destroying the entanglement of the Bell pair as well as any superposition state of the data qubit at her site in the process. Repeaters enable remote entanglement between Alice and Bob: if Alice and Charlie share a Bell pair and Charlie and Bob share a Bell pair, using LOCC wherein Charlie applies a Bell state measurement (BSM) to her respective qubits in a process known as entanglement swapping, Alice and Bob's qubits may be projected onto a Bell pair, becoming directly entangled with each other. Given a repeater chain connected by Bell pairs between Alice and Bob, the intermediate repeaters in the path may perform entanglement swapping to generate a Bell pair between Alice and Bob.

Quantum teleportation and entanglement swapping in practice degrade the fidelity of data qubits and Bell pairs, so repeaters come in three generations, dubbed 1G, 2G, and 3G, with each generation using procedures such as heralded entanglement generation (HEG), heralded entanglement purification (HEP), and quantum error correction (QEC) to correct for photon loss and operation errors, which only compound with communication distance [7, 8]. 1G repeaters are currently in the development phase, while 2G and 3G repeaters remain theoretical concepts. Most entanglement routing studies in the area of network science assume 1G repeaters, and we do the same. Specifically, we assume 1G repeaters based on atomic ensembles and linear optics [3, 10], but omit any mention of entanglement purification by also assuming all operations, if successful, produce high-fidelity Bell pairs for simplicity's sake. For a comprehensive overview of quantum networks see [13].

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Path selection mechanisms for entanglement routing mimic many of those used in classical routing. In [14], for instance, Van Meter *et al.* adapt Dijkstra's algorithm for quantum repeater networks. The bulk of existing work on entanglement routing is concerned with optimizing metrics such as throughput and resource utilization [12, 16] just as in classical routing, while also accounting for quantum properties like fidelity [6]. Van Meter *et al.* define the throughput of a path in a quantum network as the number of Bell pairs generated per time slot of a given fidelity [14]. As in [9, 12, 16], we assume path computation, resource allocation, and all relevant LOCC are done in one time slot for a given set of source-destination repeater pairs. In order to maximize throughput for Alice and Bob, we must find paths which maximize the number of Bell pairs between them. To the best of our knowledge, the question of whether the order in which repeaters in a path perform entanglement swapping affects throughput is largely unanswered. We posit order matters.

Two well-known swapping orders of a repeater chain are the *sequential order* and the *parallel order* [4, 14]. In the sequential order, the swappings of an n-hop path are performed on the intermediate nodes in $n-1$ iterations, from left to right. In the parallel order, the swappings of an n-hop path (where n is assumed to be a power of 2) are performed on the intermediate nodes in $\log n$ iterations, where $\frac{n}{2^k}$ swappings are performed in the k-th iteration (see Section 4 for details). Assuming quantum channels can support multiple Bell pairs simultaneously using wavelength-division multiplexing (WDM), take the *width* of a repeater chain to be the minimum number of Bell pairs spanning any link in the path [12]. We study the expected throughput of a given repeater chain of arbitrary hop count and arbitrary width coupled with a swapping order in Section 4.

The main contributions of this paper are the following.

- We show how to efficiently compute the expected throughput of a repeater chain for any given swapping order.
- We show that the entanglement swapping operation is not associative.
- We show that the problem of computing an s-t path with maximum expected throughput does not have the sub-path optimality property, which is a key property that most path finding algorithms rely on.

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The rest of this paper is organized as follows. In Section 2, we present the system model. In Section 3, we define the metric vector of a link/path and show how to compute the metric vector resulting from a given swapping. In Section 4, we study the impact of swapping order on the resulting expected throughput of a path. In Section 5, we present simulation results which validate our theoretical findings. We conclude the paper in Section 6.

2. System Model

We model a quantum network using an undirected graph $G = (V, E; C, Q, p, q)$ with edge and vertex attributes, where

- V is the set of vertices each of which corresponds to a repeater,
- E is the set of edges each of which corresponds to a multi-mode optical fiber connecting a pair of adjacent vertices,
- $C_{uv} \subseteq \{1, 2, \ldots, W\}$ is the set of available quantum channels on edge (u, v), $\forall(u, v) \in E$,
- $Q_v \in \mathbb{N}_0$ is the number of available qubits (stored in quantum memory) for creating Bell pairs at v, $\forall v \in V$,
- $p_{uv} \in [0, 1]$ is the success probability of generating an entanglement over a channel in C_{uv}, $\forall(u, v) \in E$ (assuming the required qubits at u and v are available),
- $q_v \in [0, 1]$ is the success probability of swapping a pair of adjacent entanglements at vertex v, $\forall v \in V$.

We use the terms *vertices* (a graph theory term) and *nodes* (a networking term) interchangeably. We use the terms *edges* and *links* interchangeably. We use $<v_0, v_1, \ldots, v_n>$ to denote an ordered sequence of vertices, and use v_0-v_1-$\cdots$-v_n to denote an n-hop path with end nodes v_0, v_n and intermediate nodes $v_1, v_2, \ldots, v_{n-1}$. We abbreviate *quantum channels* as *channels*. We use W to denote the maximum number of channels a link may have. Hence $|C_{uv}| \leq W$ holds for every link $(u, v) \in E$.

A node can bind/assign each of its qubits to a quantum channel [9, 12], so that each qubit is assigned to at most one channel, and each end node of a channel is assigned at most one qubit. A channel that is assigned qubits at both of its end nodes is called a *bound channel*. Our choice of repeaters relies on the Duan-Lukin-Cirac-Zoller (DLCZ) protocol [3, 7, 10], which is as follows.

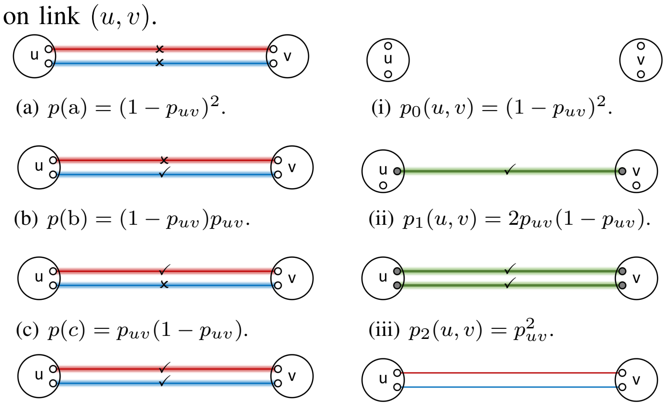
Heralded entanglement generation: To attempt to *generate an entanglement* on a bound channel on link (u, v), adjacent repeaters u and v, equipped with quantum memories (viz., atomic ensembles), simultaneously excite their respective ensemble with a laser pulse, inducing an atom to emit a photon that is entangled with said atom. These two photons are coupled to optical fibers and interfered on a polarizing beamsplitter at the midpoint between u and v, with the outputs detected by two single-photon detectors. If either detector detects a single photon, the two ensembles are projected onto an entangled state. A classical "heralding" signal is sent to u and v informing them of the result. We use p_{uv} to denote the success probability of generating a Bell pair on a bound channel on link (u, v).

Entanglement swapping: We use $e(u, v)$ to denote an entanglement between nodes u and v, where u and v do not have to be adjacent. Two entanglements are said to be adjacent if they share a common end node. Let $e(x, v)$ and $e(v, y)$ be two adjacent entanglements. After v receives heralding signals from x and y indicating the successful generation of $e(x, v)$ and $e(v, y)$, v attempts *entanglement swapping of* $e(x, v)$ *and* $e(v, y)$ by applying a laser pulse to its atomic excitations stored in the ensembles corresponding to $e(x, v)$ and $e(v, y)$, converting them into photons which are then interfered on a beamsplitter. The detection of a single photon in either detector heralds the successful projection of x and y's ensembles onto an entangled state $e(x, y)$. We use q_v to denote the success probability of entanglement swapping at v. Note that $q_v \leq 0.5$ for BSMs based on linear optics, but we allow q_v to take on arbitrary values in $[0, 1]$ for experimental purposes.

In practical implementations of the DLCZ protocol, after generating entanglement in the form of a single delocalized excitation shared between the desired end nodes, we would post-select for two-photon entanglement, which is more useful. We refer to [3, 10] for a description of this procedure.

3. Success Probabilities of Entanglement Generation and Entanglement Swapping

Since link (u, v) has $|C_{uv}|$ available channels, node u has Q_u available qubits, and node v has Q_v available qubits, we can attempt to generate $w = \min\{|C_{uv}|, Q_u, Q_v\}$ entanglements on link (u, v).



(a) $p(\text{a}) = (1 - p_{uv})^2$.

(b) $p(\text{b}) = (1 - p_{uv})p_{uv}$.

(c) $p(c) = p_{uv}(1 - p_{uv})$.

(d) $p(d) = p_{uv}^2$.

(i) $p_0(u, v) = (1 - p_{uv})^2$.

(ii) $p_1(u, v) = 2p_{uv}(1 - p_{uv})$.

(iii) $p_2(u, v) = p_{uv}^2$.

(iv) A link with two channels.

Fig. 1: (iv): link (u, v) with two available channels; nodes u and v each have 2 qubits. (a): entanglement attempt failed on both channels. (b): entanglement attempt successful on blue channel only. (c): entanglement attempt successful on red channel only. (d): entanglement attempt successful on both channels. (i): probability of having 0 entanglement on (u, v) is the probability of case (a). (ii): probability of having 1 entanglement on (u, v) is the probability of cases (b) and (c). (iii): probability of having 2 entanglements on (u, v) is the probability of case (d).

Theorem 1: Assume that we attempt to generate w entanglements on w different available channels on link (u, v), where $w = \min\{|C_{uv}|, Q_u, Q_v\}$. The probability of successfully generating exactly k entanglements on link (u, v) is

$$p_k(u, v) = \begin{cases} \binom{w}{k} p_{uv}^k (1 - p_{uv})^{w-k}, & k = 0, 1, \ldots, w \\ 0, & k > w \end{cases} \quad (1)$$

where p_{uv} is the success probability of generating an entanglement on a single channel on link (u, v). $\square$

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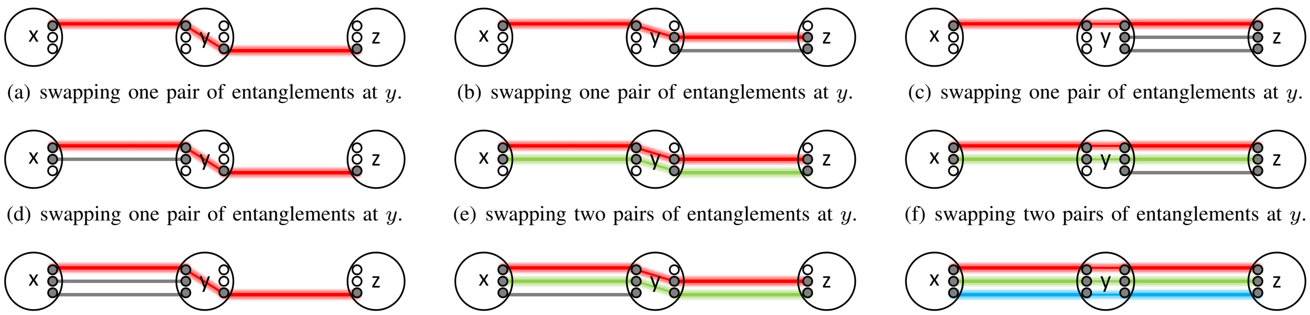

(a) swapping one pair of entanglements at y. (b) swapping one pair of entanglements at y. (c) swapping one pair of entanglements at y.

(d) swapping one pair of entanglements at y. (e) swapping two pairs of entanglements at y. (f) swapping two pairs of entanglements at y.

(g) swapping one pair of entanglements at y. (h) swapping two pairs of entanglements at y. (i) swapping three pairs of entanglements at y.

Fig. 2: We attempt $m = \min\{L, R\}$ entanglement swappings at node y.

Proof. Since we only attempt to generate w entanglements on link (u, v), the probability of having more than w entanglements is 0.

Let $k \in \{0, 1, \ldots, w\}$ be fixed. Let $C_k = \{c_1, c_2, \ldots, c_k\}$ be a subset of the w channels over which we attempt to generate entanglements. The probability of having these k attempts succeed and the remaining $w - k$ attempts fail is $p_{uv}^k (1 - p_{uv})^{w-k}$. There are $\binom{w}{k}$ different choices of C_k. This proves the theorem. ■

Fig.1 illustrates Theorem 1 for the case $w = 2$. Fig. 1(a) illustrates case (a) where the attempts on both channels are unsuccessful. The probability for case (a) is $(1 - p_{uv})^2$. Fig. 1(b) illustrates case (b) where the attempt on the first channel is unsuccessful, but the attempt on the second channel is successful. The probability for case (b) is $(1 - p_{uv})p_{uv}$. Fig. 1(c) illustrates case (c) where the attempt on the first channel is successful, but the attempt on the second channel is unsuccessful. The probability for case (c) is $p_{uv}(1 - p_{uv})$. Fig. 1(d) illustrates case (d) where the attempts on both channels are successful. The probability for case (d) is p_{uv}^2. Scenario (i) consists of case (a), where there is 0 entanglement between u and v, illustrated in Fig. 1(i). The probability for scenario (i) is $p_0(u, v) = p(a) = (1 - p_{uv})^2$. Scenario (ii) consists of cases (b) and (c), where there is 1 entanglement between u and v, illustrated in Fig. 1(ii). The probability for scenario (ii) is $p_1(u, v) = p(b) + p(c) = 2p_{uv}(1 - p_{uv})$. Scenario (iii) consists of case (d), where there are 2 entanglements between u and v, illustrated in Fig. 1(iii). The probability for scenario (iii) is $p_2(u, v) = p(d) = p_{uv}^2$.

Assume that there are L entanglements $e_i(x, y)$, $i = 1, \ldots, L$, connecting nodes x and y, and R entanglements $e_j(y, z)$, $j = 1, \ldots, R$, connecting nodes y and z. We can perform up to $m = \min\{L, R\}$ entanglement swappings at node y, i.e., swap $e_i(x, y)$ and $e_i(y, z)$ for $i = 1, \ldots, m$. Fig. 2 illustrates all cases for $L \in \{1, 2, 3\}$ and $R \in \{1, 2, 3\}$. For example, in Fig. 2(c), we attempt to swap the only entanglement on link (x, y) with the 1st entanglement (of the 3 entanglements) on link (y, z). In Fig. 2(h), we attempt two swaps: swapping the 1st entanglement on link (x, y) with the 1st entanglement on link (y, z), and swapping the 2nd entanglement on link (x, y) with the 2nd entanglement on link (y, z).

The probability of having exactly k successful entanglements connecting nodes x and z from these m attempts is

$$\binom{m}{k} q_y^k (1 - q_y)^{m-k}, k = 0, 1, \ldots, m \tag{2}$$

where q_y is the success probability of performing a single entanglement swapping at node y.

Suppose we know the probability of having exactly k entanglements connecting nodes x and y, and the probability of having exactly k entanglements connecting nodes y and z, for $k = 0, 1, 2, \ldots, W$. Can we obtain the probability of having exactly k entanglements connecting nodes x and z, after performing entanglement swapping at node y? The following theorem answers this question.

Theorem 2: Let $w \in \{1, \ldots, W\}$. Assume that the probability of having exactly k entanglements connecting nodes x and y is $p_k(x, y)$, where $p_k(x, y) = 0$ for $k > w$. Also assume that the probability of having exactly k entanglements connecting nodes y and z is $p_k(y, z)$, where $p_k(y, z) = 0$ for $k > w$. After performing w entanglement swappings at node y, the probability of having exactly k entanglements connecting x and z is

$$\begin{aligned} p_k(x, z) = & \sum_{i=k}^{w} p_i(x, y) \sum_{j=k}^{i-1} p_j(y, z) \binom{j}{k} q_y^k (1 - q_y)^{j-k} \\ & + \sum_{i=k}^{w} p_i(x, y) \sum_{j=i}^{w} p_j(y, z) \binom{i}{k} q_y^k (1 - q_y)^{i-k} \end{aligned} \tag{3}$$

for $k = 1, 2, \ldots, w$. We also have $p_k(x, z) = 0$ for $k > w$, and $p_0(x, z) = 1 - \sum_{k=1}^{w} p_k(x, z)$. □

Proof. Since $p_k(x, y) = 0$ and $p_k(y, z) = 0$ for $k > w$, we have $p_k(x, z) = 0$ for $k > w$. This also implies $p_0(x, z) = 1 - \sum_{k=1}^{w} p_k(x, z)$.

Let k be an integer in $\{1, 2, \ldots, w\}$. In order to have k entanglements connecting x and z, there must be $i \geq k$ entanglements connecting x and y, and $j \geq k$ entanglements connecting y and z. For each $i \in [k, w]$, if $j \in [k, i-1]$, we perform j (the minimum of $\{i, j\}$) entanglement swappings at y; if $j \in [i, w]$, we perform i (the minimum of $\{i, j\}$) entanglement swappings at y. This leads to formula (3). ■

For any two nodes x and y, we use $\mathbf{e}(x, y)$ to denote the set of entanglements connecting x and y, and use $\mathbf{p}(x, y)$ to denote the $(W+1)$-dimensional column vector $(p_0(x, y), p_1(x, y), \ldots, p_W(x, y))^T$. We call $\mathbf{p}(x, y)$ the *metric vector* of $\mathbf{e}(x, y)$, as it characterizes the properties of $\mathbf{e}(x, y)$ as a random variable. Initially, neither $\mathbf{e}(x, y)$ nor $\mathbf{p}(x, y)$ is defined for any pair of nodes x and y. After the entanglement generation attempts on link (u, v), $\mathbf{e}(u, v)$ becomes defined and denotes the resulting set of entanglements connecting u and v, and also $\mathbf{p}(u, v)$ becomes defined and is computed according to equation (1).

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When $\mathbf{p}(x, y)$ and $\mathbf{p}(y, z)$ are both defined (which implies that both $\mathbf{e}(x, y)$ and $\mathbf{e}(y, z)$ are defined), and we attempt to perform entanglement swapping at node y, then $\mathbf{e}(x, z)$ becomes defined and denotes the resulting set of entanglements connecting nodes x and z, and also $\mathbf{p}(x, z)$ becomes defined and is computed according to Theorem 2. Here nodes x and y (y and z, respectively) do not have to be adjacent.

The operations and computations associated with entanglement swappings define two binary operators $\oplus$ and $\otimes$, where

$$\mathbf{e}(x, z) = \mathbf{e}(x, y) \oplus \mathbf{e}(y, z) \tag{4}$$

denotes that $\mathbf{e}(x, z)$ is obtained from entanglement swapping of $\mathbf{e}(x, y)$ and $\mathbf{e}(y, z)$ at node y, and

$$\mathbf{p}(x, z) = \mathbf{p}(x, y) \otimes \mathbf{p}(y, z) \tag{5}$$

denotes that $\mathbf{p}(x, z)$ is computed from $\mathbf{p}(x, y)$ and $\mathbf{p}(y, z)$ according to formula (3). We will further study the properties of these two operators in Sections 4 and 5.

4. Impact of Entanglement Swapping Order

Efficient algorithms for computing a *good* routing path for a source-destination pair are fundamental in both traditional computer networks [1, 2] and quantum networks [9, 12]. A path connecting source node s to destination node t is known as an s-t path. Path finding algorithms rely on a *path metric* that *characterizes the quality of the path*.

Path metrics are commonly known as *path lengths* [11], although they do not always carry the literal meaning of *length*. The path metric (path length) is computed from the link metrics of the links on the path. Depending on the application, the link metrics contribute to the path metric in different ways.

In the *minimum delay path* problem, each link has a metric known as the *link delay* which measures the transmission delay of the link, and each path has a metric known as the *path delay* which measures the transmission delay of the path. In this case, the path delay is computed as the *summation* of the link delays over the links on the path. In the *most reliable path* problem, each link has a metric known as the *link reliability*, and each path has a metric known as the *path reliability*. In this case, the path reliability is computed as the *product* of the link reliabilities over the links on the path.

An important fact common to the above path finding problems is the following. Let $\kappa(u, v)$ denote the metric of a link (u, v), and $\kappa(\pi)$ denote the metric of a path $\pi = v_0\text{-}v_1\text{-}v_2\text{-}\cdots\text{-}v_n$. Then there is a binary operator $\odot$ such that

$$\kappa(\pi) = \kappa(v_0, v_1) \odot \kappa(v_1, v_2) \odot \cdots \odot \kappa(v_{n-1}, v_n), \tag{6}$$

where $\alpha \odot \beta \stackrel{\text{def}}{=} \alpha + \beta$ for the minimum delay path problem, and $\alpha \odot \beta \stackrel{\text{def}}{=} \alpha \times \beta$ for the most reliable path problem. In both path finding problems, the binary operator $\odot$ is *associative*:

$$(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma), \forall\ \alpha,\ \beta,\ \gamma. \tag{7}$$

The associativity (7) of the operator $\odot$ implies the following sub-path optimality property.

Definition 1 (Sub-path Optimality): A path finding problem has the *sub-path optimality property* if for any s-t path π and any intermediate node x on π, the optimality of π implies the optimality of π^{sx} as well as the optimality of π^{xt}, where π^{sx} is the portion of π from s to x and π^{xt} is the portion of π from x to t. □

If the sub-path optimality property holds, one can compute an optimal path by *extending optimal partial paths from the source node hop-by-hop in a greedy manner until the destination node is reached*, just like Dijkstra's algorithm [2].

For entanglement routing in a quantum network, a well-known path metric is the *expected throughput* [12] of an entanglement path π connecting nodes s and t, which can be computed by the following formula

$$EXT^{\pi} = \sum_{k=1}^{W} k \times p_k(s, t), \tag{8}$$

where $p_k(s, t)$ is the probability of having exactly k entanglements connecting s and t, after the necessary entanglement swappings at all intermediate nodes on path π.

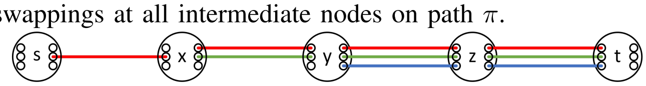

Fig. 3: Entanglement swapping order matters.

In order to use (8), we need to compute $\mathbf{p}(s, t)$, which we call the *path metric vector* of path π. The example in Fig. 3 shows that $\mathbf{p}(s, t)$ **depends on not only the s-t path π, but also the order in which the entanglement swappings are performed on path π.**

In Fig. 3, we have a 4-hop path $\pi = s\text{-}x\text{-}y\text{-}z\text{-}t$. The number of available channels on links (s, x), (x, y), (y, z), and (z, t) are 1, 2, 3, and 3, respectively. $W = 3$, $p_{uv} = p = 0.5$ for all links, $q_v = q = 0.8$ for all nodes. For this 4-hop path, we need to perform entanglement swapping at nodes x, y, and z in some order to get entanglements connecting s and t. There are 5 distinct ways to associate 3 applications of the binary operator $\otimes$. Applying Theorems 1 and 2, we get

$$\Big[[\mathbf{p}(s,x) \otimes \mathbf{p}(x,y)] \otimes \mathbf{p}(y,z)\Big] \otimes \mathbf{p}(z,t) = \begin{pmatrix} 0.853 \\ 0.147 \\ 0 \\ 0 \end{pmatrix}, \tag{9}$$

$$\Big[\mathbf{p}(s,x) \otimes \mathbf{p}(x,y)\Big] \otimes \Big[\mathbf{p}(y,z) \otimes \mathbf{p}(z,t)\Big] = \begin{pmatrix} 0.843 \\ 0.157 \\ 0 \\ 0 \end{pmatrix}, \tag{10}$$

$$\Big[\mathbf{p}(s,x) \otimes [\mathbf{p}(x,y) \otimes \mathbf{p}(y,z)]\Big] \otimes \mathbf{p}(z,t) = \begin{pmatrix} 0.847 \\ 0.153 \\ 0 \\ 0 \end{pmatrix}, \tag{11}$$

$$\mathbf{p}(s,x) \otimes \Big[[\mathbf{p}(x,y) \otimes \mathbf{p}(y,z)] \otimes \mathbf{p}(z,t)\Big] = \begin{pmatrix} 0.845 \\ 0.155 \\ 0 \\ 0 \end{pmatrix}, \tag{12}$$

$$\mathbf{p}(s,x) \otimes \Big[\mathbf{p}(x,y) \otimes [\mathbf{p}(y,z) \otimes \mathbf{p}(z,t)]\Big] = \begin{pmatrix} 0.841 \\ 0.159 \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

where we rounded the values to three decimal places for ease of reading. Note that the right-hand sides in equations (9)-(13) are different. This observation proves the following theorem.

Theorem 3: The binary operator $\otimes$ is **not** associative. □

Equations (9)-(13) correspond to 5 ways to associate the 3 $\otimes$ operations. They correspond to the following 5 swapping orders: $\mathcal{O}_1 = <x, y, z>$, $\mathcal{O}_2 = <x, z, y>$, $\mathcal{O}_3 = <y, x, z>$, $\mathcal{O}_4 = <y, z, x>$, and $\mathcal{O}_5 = <z, y, x>$, respectively. Swapping order $\mathcal{O}_6 = <z, x, y>$ is missing from the above. It is *equivalent* to the swapping order $\mathcal{O}_2 = <x, z, y>$ as both correspond to the same association order, in equation (10). The swapping orders $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$, and $\mathcal{O}_5$ lead to *different expected throughput values* 0.147, 0.157, 0.153, 0.155, and 0.159, respectively, *of the same path* $\pi = <s, x, y, z, t>$.

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Summarizing the above analyses/observations, we have

Theorem 4: Let π be an s-t path with hop-count $H(\pi)$. Let $\mathcal{O}$ be a swapping order of π. The metric vector $\mathbf{p}(s,t)$ corresponding to order $\mathcal{O}$ can be computed in $O(H(\pi)\times W^2)$ time. Let $\mathbf{p}^{\pi,\mathcal{O}}(s,t)$ denote the metric vector $\mathbf{p}(s,t)$ corresponding to swapping order $\mathcal{O}$ of path π. The *expected throughput of path* π *under swapping order* $\mathcal{O}$ is

$$EXT^{\pi}(\mathcal{O}) = \sum_{k=1}^{W} k \times p_k^{\pi,\mathcal{O}}(s,t). \tag{14}$$

In general, $EXT^{\pi}(\mathcal{O})$ is a non-constant function of $\mathcal{O}$. □

Proof. Computing $\binom{w}{k}$ for $w = 0,1,\ldots,W$ and $k = 0,1,\ldots,w$ takes $O(W^2)$ time. For each link (u,v), computing p_{uv}^k and $(1-p_{uv})^k$ for all $k \in \{0,\ldots,W\}$ takes $O(W^2)$ time. For each node y, computing q_y^k and $(1-q_y)^k$ for all $k \in \{0,\ldots,W\}$ takes $O(W^2)$ time. Since π has $H(\pi)$ links and $H(\pi)-1$ intermediate nodes, the above computations for all links and nodes on π takes $O(H(\pi)W^2)$ time. The computation in eqn. (1) takes $O(W)$ time for all k. The computation in eqn. (3) takes $O(W^2)$ time for all k. This proves the asymptotic bound on the worst-case time complexity for computing $\mathbf{p}^{\pi,\mathcal{O}}(s,t)$. The example illustrated in Fig. 3 shows that $EXT^{\pi}(\mathcal{O})$ is a non-constant function of $\mathcal{O}$. ■

So far we have shown that **both the path metric vector $\mathbf{p}^{\pi,\mathcal{O}}(s,t)$ and the expected throughput $EXT^{\pi}(\mathcal{O})$ of path π depends on not only the path itself, but also the order in which the entanglements are swapped**. It is natural to define the *optimal swapping order* of an s-t path π by

$$\mathcal{O}_{opt}^{\pi} = \arg\max_{\mathcal{O}\in\mathcal{O}(\pi)} EXT^{\pi}(\mathcal{O}), \tag{15}$$

where $\mathcal{O}(\pi)$ denotes the set of all swapping orders of path π.

We say two swapping orders are *equivalent* if they correspond to the same association order. For an n-hop path, there are C_{n-1} *non-equivalent* swapping orders, where $C_n = \frac{(2n)!}{(n+1)!n!}$ is the n-th Catalan number [5]. C_n is the number of distinct ways to associate n applications of the binary operator $\otimes$. The fast growth of this sequence makes it prohibitive to enumerate all swapping orders to find an optimal order. However, it is a theoretical concept that is worthy of discussion.

The following two swapping orders have been studied in the literature [4, 14]. The *sequential order*, denoted by $\mathcal{O}_{seq}$, assumes the binary operator $\oplus$ ($\otimes$, respectively) to be *left associative*. Under this swapping order, we have

$$\mathbf{e}(a,b)\oplus\mathbf{e}(b,c)\oplus\mathbf{e}(c,d) \stackrel{\text{def}}{=} (\mathbf{e}(a,b)\oplus\mathbf{e}(b,c))\oplus\mathbf{e}(c,d), \tag{16}$$

$$\mathbf{p}(a,b)\otimes\mathbf{p}(b,c)\otimes\mathbf{p}(c,d) \stackrel{\text{def}}{=} (\mathbf{p}(a,b)\otimes\mathbf{p}(b,c))\otimes\mathbf{p}(c,d). \tag{17}$$

For the n-hop path $\pi = v_0$-v_1-v_2-$\cdots$-v_n, we first perform swapping of $\mathbf{e}(v_0,v_1)$ and $\mathbf{e}(v_1,v_2)$ at v_1, resulting in $\mathbf{e}(v_0,v_2)$. We then perform swapping of $\mathbf{e}(v_0,v_2)$ and $\mathbf{e}(v_2,v_3)$ at v_2, resulting in $\mathbf{e}(v_0,v_3)$. We continue this process for $n-1$ iterations to get $\mathbf{e}(v_0,v_n)$. In our example in Fig. 3, $\mathcal{O}_1$ is the sequential order.

The *parallel order*, denoted by $\mathcal{O}_{par}$, is designed for paths whose hop-count is a power of 2. Let $\pi = v_0$-v_1-v_2-$\cdots$-v_n where $n = 2^k$. We perform $\log n$ iterations of swapping. In the first iteration, we perform $\frac{n}{2}$ swappings in parallel: for $i = 1,2,\ldots,\frac{n}{2}$, swapping $\mathbf{e}(v_{2i-2},v_{2i-1})$ and $\mathbf{e}(v_{2i-1},v_{2i})$ at node v_{2i-1} to get $\mathbf{e}(v_{2i-2},v_{2i})$. In the next iteration, we perform $\frac{n}{4}$ swappings in parallel: for $i = 1,2,\ldots,\frac{n}{4}$, swapping $\mathbf{e}(v_{4i-4},v_{4i-2})$ and $\mathbf{e}(v_{4i-2},v_{4i})$ at node v_{4i-2} to get $\mathbf{e}(v_{4i-4},v_{4i})$. We continue this process for $\log n$ iterations to get $\mathbf{e}(v_0,v_n)$. In our example in Fig. 3, $\mathcal{O}_2$ is the parallel order.

Among $\mathcal{O}_{seq}$, $\mathcal{O}_{par}$ and $\mathcal{O}_{opt}$, $\mathcal{O}_{seq}$ is the most useful in path finding algorithms, because when we extend a partial path by one hop, the contribution of the new hop to the metric vector of the extended path is known due to the left association assumption. In contrast, under either $\mathcal{O}_{par}$ or $\mathcal{O}_{opt}$, the contribution of the new hop to the metric vector of the extended path is not known until the whole path is computed. With the aid of Fig. 4, we present an example to show that under any of the swapping orders, the problem of computing an s-t path with maximum expected throughput does not have the sub-path optimality property.

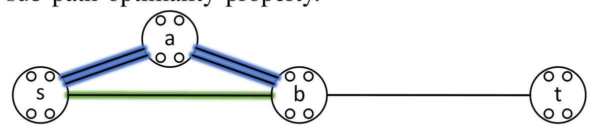

Fig. 4: $W = 2$, $p_{uv} = p = 0.5$ for all links, $q_v = q = 0.84$ for all nodes. Links (s,a) and (a,b) each have 2 available channels. Links (s,b) and (b,t) each have 1 available channel. Each node has 4 qubits.

In this example, there are two s-t paths. The path $\pi_1 = s$-a-b-t has an expected throughput of 0.202 under the optimal swapping order $<a,b>$ (the expected throughput is 0.198 under the swapping order $<b,a>$). The path $\pi_2 = s$-b-t has an expected throughput of 0.210 (there is only one swapping order). Therefore π_2 is the unique optimal s-t path. There are two s-b paths. The path $\pi_3 = s$-a-b has an expected throughput of 0.525 (there is only one swapping order). The path $\pi_4 = s$-b has an expected throughput of 0.500 (no swapping is needed). Therefore π_3 is the unique optimal s-b path. This example shows that the s-b path π_4 (which is a sub-path of the optimal s-t path π_2) is not an optimal s-b path. Hence we have proven the following theorem.

Theorem 5: The problem of computing an s-t path with maximum expected throughput under any entanglement swapping order does not have the sub-path optimality property. □

The lack of sub-path optimality property indicates that an optimal algorithm might not be a trivial generalization of Dijkstra's algorithm. Realistic path metrics that ensure the sub-path optimality property can make entanglement routing easier.

5. Simulation Results

To validate our theoretical findings, we wrote a simulator and performed extensive experiments. The *simulation of entanglement generation* is as follows. For each available channel with adequate qubits on a link (u,v), we generate a random number $r \in (0,1)$. If $r \leq p_{uv}$, we consider the attempt for entanglement generation successful. If $r > p_{uv}$, we consider the attempt unsuccessful.

The *simulation of entanglement swapping* is as follows. Suppose $\mathbf{e}(x,y)$ contains L entanglements, $\mathbf{e}(y,z)$ contains R entanglements, and we attempt to perform entanglement swapping at node y. We simulate $m = \min\{L,R\}$ attempts of entanglement swapping at node y. For each attempt, we

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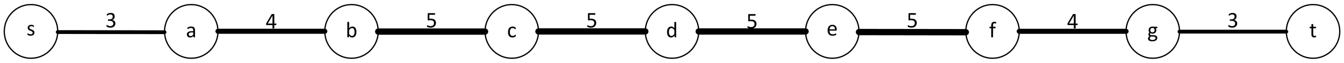

Fig. 5: The number on each link is the number of available channels. Each node has 10 qubits. $W = 5$.

Swapping order: $\mathcal{O}_{seq} = <a, b, c, d, e, f, g>$				Swapping order: $\mathcal{O}_{par} = <a, c, e, g, b, f, d>$				Swapping order: $\mathcal{O}_{opt} = <c, b, e, f, d, g, a>$			
Experimental result		Theoretical result		Experimental result		Theoretical result		Experimental result		Theoretical result	
f_0 :	5734	p_0 :	0.058	f_0 :	2602	p_0 :	0.026	f_0 :	2100	p_0 :	0.021
f_1 :	30184	p_1 :	0.301	f_1 :	23285	p_1 :	0.233	f_1 :	20061	p_1 :	0.200
f_2 :	48367	p_2 :	0.484	f_2 :	54205	p_2 :	0.539	f_2 :	53747	p_2 :	0.536
f_3 :	15715	p_3 :	0.157	f_3 :	19908	p_3 :	0.202	f_3 :	24092	p_3 :	0.244
$total$:	174063	**ext** :	**1.741**	$total$:	191419	**ext** :	**1.917**	$total$:	199831	**ext** :	**2.003**

TABLE I: Result over 100000 experiments for each swapping order: For each swapping order $\mathcal{O}$, f_k is the number of times when exactly k entanglements are formed between s and t; p_k denotes $p_k^{\pi,\mathcal{O}}(s,t)$; $total = \sum_{k=1}^{W} k \times f_k$; $ext = \sum_{k=1}^{W} k \times p_k$.

generate a random number $r \in (0, 1)$. If $r \leq q_y$, we consider the attempt successful, resulting in an entanglement connecting x and z. If $r > q_y$, we consider the attempt unsuccessful.

In Fig. 5, we have an 8-hop path $\pi = s$-a-b-c-d-e-f-g-t. Links (s, a) and (g, t) each have 3 available channels. Links (a, b) and (f, g) each have 4 available channels. All other links each have 5 available channels. We assume $p_{uv} = p = 0.9$ for all links and $q_v = q = 0.95$ for all nodes.

We study 3 swapping orders: $\mathcal{O}_{seq}$, $\mathcal{O}_{par}$, and $\mathcal{O}_{opt}$, where $\mathcal{O}_{opt}$ is obtained by enumerating all 429 swapping orders. For each swapping order, we perform 100000 experiments. We use f_k to record the number of times when exactly k entanglements connecting s and t are formed, $k = 0, 1, 2, 3$. We compute $total = \sum_{k=1}^{W} k \times f_k$. We also (theoretically) compute the probability of having exactly k entanglements connecting s and t, for each $\mathcal{O} \in \{\mathcal{O}_{seq}, \mathcal{O}_{par}, \mathcal{O}_{opt}\}$, and denote it as p_k $(= p_k^{\pi,\mathcal{O}}(s,t))$. The expected throughput is denoted by ext. These results are reported in Table I.

We observe that the expected throughput for $\mathcal{O}_{opt}$ is 2.003, which is larger than that for both $\mathcal{O}_{seq}$ and $\mathcal{O}_{par}$. In all cases, $\frac{f_k}{100000}$ is very close to p_k, and $\frac{total}{100000}$ is very close to ext. This validates the correctness of our theoretical derivations from the perspective of experimental probability.

6. Conclusions and Future Research

In this paper, we have studied the impact of swapping order on the resulting expected throughput of a path in a quantum network. We show how to efficiently compute the metric vector and expected throughput of a path for any given swapping order. We prove that the binary operator corresponding to entanglement swapping is not associative. We further show that the problem of finding an s-t path maximizing the expected throughput for a given swapping order lacks the sub-path optimality property. Extensive simulations validate our theoretical result. It is of interest to know whether two different swapping orders of an entanglement path π can lead to significantly different expected throughput values. Designing polynomial time algorithms for computing an s-t path and its corresponding swapping order leading to maximum expected throughput is also challenging.

References

[1] G. Apostolopoulos, D. Williams, S. Kama, R. Guerin, A. Orda, and T. Przygienda. QoS routing mechanisms and OSPF extensions. IETF RFC 2676, 1999.

[2] E. Dijkstra. A note on two problems in connexion with graphs. *Numerische Mathematik*, 1(1):269–271, 1959.

[3] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller. Long-distance quantum communication with atomic ensembles and linear optics. *Nature*, 414(6862):413–418, 2001.

[4] W. Dür, H.-J. Briegel, J. I. Cirac, and P. Zoller. Quantum repeaters based on entanglement purification. *Physical Review A*, 59(1):169, 1999.

[5] https://en.wikipedia.org/wiki/Catalan_number.

[6] C. Li, T. Li, Y.-X. Liu, and P. Cappellaro. Effective routing design for remote entanglement generation on quantum networks. *NPJ Quantum Information*, 7(1):1–12, 2021.

[7] W. J. Munro, K. Azuma, K. Tamaki, and K. Nemoto. Inside quantum repeaters. *IEEE Journal of Selected Topics in Quantum Electronics*, 21(3):78–90, 2015.

[8] S. Muralidharan, L. Li, J. Kim, N. Lütkenhaus, M. D. Lukin, and L. Jiang. Optimal architectures for long distance quantum communication. *Scientific Reports*, 6(1):1–10, 2016.

[9] M. Pant, H. Krovi, D. Towsley, L. Tassiulas, L. Jiang, P. Basu, D. Englund, and S. Guha. Routing entanglement in the quantum internet. *NPJ Quantum Information*, 5(1):1–9, 2019.

[10] N. Sangouard, C. Simon, H. De Riedmatten, and N. Gisin. Quantum repeaters based on atomic ensembles and linear optics. *Reviews of Modern Physics*, 83(1):33, 2011.

[11] A. Sen, K. S. Candan, A. Ferreira, B. Beauquier, and S. Perennes. On shortest path problems with "non-markovian" link contribution to path lengths. In *Proceedings of IFIP Networking*, pages 859–870, 2000.

[12] S. Shi and C. Qian. Concurrent entanglement routing for quantum networks: Model and designs. In *Proc. ACM SIGCOMM*, pages 62–75, 2020.

[13] R. Van Meter. *Quantum Networking*. John Wiley & Sons, 2014.

[14] R. Van Meter, T. Satoh, T. D. Ladd, W. J. Munro, and K. Nemoto. Path selection for quantum repeater networks. *Networking Science*, 3(1-4):82–95, 2013.

[15] S. Wehner, D. Elkouss, and R. Hanson. Quantum internet: A vision for the road ahead. *Science*, 362(6412), 2018.

[16] Y. Zhao and C. Qiao. Redundant entanglement provisioning and selection for throughput maximization in quantum networks. In *Proc. IEEE INFOCOM*, 2021.

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