Product Design under Multinomial Logit Choices: Optimization of Quality and Prices in an Evolving Product line

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Abstract

Problem Definition We study a product-line design problem in which customer choice among multiple products is given by a multinomial logit (MNL) model. A firm determines product quality and prices in an evolving product line to maximize profit. In particular, given the prices and quality of products that already exist in a product line, the firm optimizes prices and/or quality of the new products.

Academic/Practical Relevance We extend the literature on discrete choice models to include the interaction between product quality and product price and consider two variations of the problem, each mirroring a relevant decision setting found in practice: Variation (i) is a price-optimization problem in which the firm determines prices of the new products given the quality. Variation (ii) is a joint price and quality optimization of the new products.

Methodology We apply convex optimization techniques and analyze properties of optimal solutions.

Results We establish concavity of the profit function under price optimization and present tractable solution approaches for the joint quality-price optimization. For each problem variation, we characterize the optimal solution and develop efficient algorithms. We show that the interaction of price and quality is central not only to reconciling the divergence of the existing literature's equal-markup price prediction from differentiated markups observed in practice, but also for explaining differentiated quality measures across products; this empirically observed strategy can now be quantified and optimized with the model developed in this paper. In addition, we show that the presence of existing products tends to drive the firm to offer new products with both higher quality and prices due to the price-quality interaction.

Managerial Implications Findings of this paper offer not only managerial guidelines, but also tools for decision support due to the wide empirical applicability of the MNL model. An important managerial implication is that the lack of realism in the linear utility of the MNL model can be addressed by including price-quality interaction, which is central to understanding the quality and price decision in product line design. The interaction rationalizes the matching of high markup with high quality and justifies differentiated offering of new products in the presence of existing products.

Keywords: *Pricing, Revenue Management, Multinomial Logit, Product Line Design*

1 Introduction

Firms frequently introduce new products and retire old products to renew customers' interests in purchasing and consuming new products. For example, restaurants regularly add new items to its menu to spur new interests. Hotels introduce new room choices (e.g., free breakfast combo, executive package, and so on) from time to time to attract more customers. Manufacturers of home appliances periodically launch new models with improved features and efficiency driven by new trends in customer preference and life style. This phenomenon is even more pronounced in the high-tech industry due to a faster industry clockspeed. At Intel, for example, new microprocessor products are released on a quarterly basis. Such a product cadence leads to constantly evolving product lines and, at each change epoch, a firm has to determine what products to add to an existing product line and at what price points. That is, given the quality and prices of the existing products, the firm optimizes the quality and/or prices of the new products to maximize the total profit from the product line. In this paper, we interpret "quality" as a general term referring to any dimension of product attributes that can be vertically differentiated. Such business decisions are plagued by multiple complications. Higher quality and lower prices increase product appeal and attract more customers, but higher quality is costly and lower prices decrease unit revenue, both of which contribute to lower margins. In addition, the attractiveness of new products affects the market share of existing products and may cannibalize the profit of existing products. Thus, the firm needs to strike a balance between these competing forces.

This decision problem arises in a variety of industries but the nature of the problem is similar across industries. Consider a resort hotel that is adding new room choices to its existing offerings. Management would like to offer value packages that include basic room service plus resort credit to be used for ancillary services such as restaurants, gift shop, spa and entertainment. Table 1 provides information on existing products and a plausible set of new product offerings. The hotel's decision problem is to set the proper resort credit level and price for each new offer. In another example, a

Table 1: Room Offerings at a Resort Hotel.

Existing	Room	Resort Credit	Price	New	Room	Resort Credit	Price
1	One King	0	259	4	One King	50	289
2	Two Queen	0	299	5	Two Queen	80	359
3	King Suite	0	359	6	King Suite	100	429

smart phone manufacturer introduces a new model (M2) of phones to its product line, which will be sold concurrently with an existing phone model (M1). The manufacturer offers several storage-size variations of each model and needs to determine the storage size and price of each variation of the new model (Table 2). In both examples, the attribute to be optimized (i.e., quality) is a dimension

Table 2: Existing and New Phone Models for a Smartphone Manufacturer

Existing	Model	Storage	Price	New	Model	Storage	Price
1	M1	16	149	4	M2	64	299
2	M1	32	199	5	M2	128	349
3	M1	64	269	6	M2	256	499

that vertically differentiates the products and the firm optimizes the new products in the presence of existing products.

In this paper, we explore the quality and price decisions for a product line. Specifically, we consider two variations of the problem that are relevant in practice. In variation (i), the quality levels of all products are given and the firm optimizes the prices of the new products. This often arises when quality is pre-determined during the design and engineering stage and price is decided close to product launch. In variation (ii), the firm optimizes both price and quality of the new products. The joint decision on price and quality typically occurs during strategic planning when a firm plans the next generation of product offerings. In practice, variations (i) and (ii) may be adopted by the same firm for different decision scopes and contexts. For instance, a firm may solve a joint quality-price optimization problem during strategic planning but may re-optimize prices later by solving a variation (i) problem as a tactical adjustment.

We model demand using a widely-adopted choice model for customers facing multiple product options – the multinomial logit (MNL) model, which can be parameterized with consumer choice or sales data and is realistic for decision support. A new feature we include in the MNL model is the interaction between product quality and product price, which allows the marginal utility of quality to depend on the price level of the product, and likewise, marginal utility of price to depend on the quality level of the product. This captures the commonly observed phenomenon that customers are less sensitive to price changes in high-quality product than low-quality product (Tversky and Kahneman, 1981). Historically, two schools of thought dominate the literature on price quality relations. One proposes that consumers use price as a cue for product quality and thus price elasticities are reduced for high quality products (e.g., Monroe and Krishnan 1985); the other argues that customers' marginal quality valuation changes with willingness-to-pay of customers, hence effectively modifies price elasticity (e.g., Spence 1975). As a result, an interaction term is often used in empirical studies to test how quality (resp. price) moderates the effect of price (resp. quality) on customer utility or demand (e.g., Hagerty 1978, Siderelis et al. 2000). A few studies have included the price-quality interactions in logit choice models. Carpenter and Lehmann (1985) point to earlier studies for the fact that price and quality may interact to affect each product's utility ratings and conduct empirical studies including interactions of price with product quality in the MNL model. Crawford (2012) and Crawford et al. (2015) present logit models with a price-quality interaction term and provide interpretations for the coefficient of the interaction. From a theoretical perspective, Keeney (1968) extends the von Neumann and Morgenstern (1947) expected utility theory to accommodate multiple attributes and shows that with two additional axioms governing individual preferences among alternatives with two attributes (e.g., quality x and price p), an individual's utility has the following quasi-separable form: $u(x, p) = u_1(x) + u_2(p) + ku_1(x)u_2(p)$ where k is a real number. Our utility model of a product with attribute x and price p is consistent with this bivariate utility function form.

The ease of use and interpretation makes the interaction term a simple but powerful tool for incorporating the moderating effect of price and quality on how they each affect customer utility or demand. We build on this addition and adopt the interaction term in the MNL model in a normative decision setting. To our knowledge, we are the first to apply this modification to a firm's product line decision under MNL demand. We show that this interaction rationalizes the matching of high markup with high quality which supplements the equal-markup pricing result in the literature. The equal-markup result derives from the orthogonality of price and quality in their effect on customer utility and is not always consistent with observations in practice (Berry et al., 1995). For example, Morris and Bronson (1969) study quality-price rank correlation for 48 commodity data sets and find that 38 out of the 48 studies exhibit positive correlations. Gerstner (1985) considers a data set of 145 products and finds that price and quality are positively correlated for most durable items. Berry et al. (1995) and Feenstra and Levinsohn (1995) empirically estimate markups in the automobile industry and find evidence that markup is generally increasing in price. The practice of charging higher markups for high quality is ubiquitous in today's market as well. For example, Apple currently sells two models of iPhone 7 at $549 and $649 for 32GB and 128GB capacity respectively, while the two models of iPhone 8 with 64GB and 256GB capacity are selling for $699 and $849 respectively (Apple Corporation Website,, 2018b). Intel has a long history of selling its top-bin products (those with higher speed performance) at a much higher margin than the lower-bin products (Intel Corporation Website,, 2018). Therefore, our result presents a more realistic characterization of the pricing decision. More significantly, it has important implications for decisions involving product quality. We show that the sequence of optimal quality values among the newly offered products matches the sequence of prices, controlling for other parameters. Our analysis indicates that the interaction term plays a central role in justifying differentiated offering of new products. With the interaction effect of price and quality, the optimal quality and markup vary across products even under identical price sensitivity and cost function, which is a commonly-adopted strategy that can now be quantified and optimized with the model developed in this paper.

Our theoretical contributions are threefold: First, ours is the first in the literature to solve joint pricing and quality decisions under the MNL model allowing continuous quality values. Existing literature that addresses joint pricing and quality decisions examines a given assortment of products, i.e., the quality values are a finite set of pre-selected discrete values and thus the insights are limited to assortment selection. Our paper, in contrast, yields strategic insights on how firm should design its product line and optimally set product quality on a continuum in conjunction with prices, providing decision support with a new dimension. We establish concavity of the profit functions under price optimization and with considerations of existing products in the product line and we identify a sufficient condition for unique optimal solution for the joint price-quality optimization as well as providing a tractable solution method when the condition does not hold. Second, we characterize the optimal price and quality and develop efficient algorithms for each problem variation. Third, we are the first to include price-quality interaction in the optimization which helps reconcile the divergence of existing literature's equal-markup price prediction from empirical practices and uncovers new insights on product quality decisions as the product line evolves.

The remainder of the paper is organized as follows. In Section 2, we review relevant literature and further elaborate on our contributions. In Section 3, we present the model formulations, the solution methods, structural properties as well as numerical examples. We conclude with a discussion in Section 4.

2 Literature Review

Understanding customers' behavior when they make choices has been an interesting topic of study and has captured attention from scholars in economics and marketing. Thurstone (1927) develops a utility model to estimate the choice probability which is later refined by Marschak et al. (1959) and Luce (1959) to a Random Utility Maximization (RUM) model. A commonly adopted RUM model is the multinomial logit (MNL) model which describes customer choices among multiple alternatives (e.g., McFadden et al. 1973; McFadden 1980, 1986; Anderson et al. 1992; Berry 1994). The MNL model has been applied to predict and understand behavior in different markets, including transportation (McFadden, 1974), housing (McFadden, 1978), grocery (Guadagni and Little, 1983), and telephone services (Train et al., 1987). A key advantage of the MNL model is attributed to its flexibility in incorporating customer preference and sensitivity toward various product attributes and price. This allows a firm to take advantage of both past sales data and customer preference information to make informed decisions.

One of the widely studied applications of customer choice is pricing, in which companies set prices of multiple substitutable products considering customers' choice preferences. There exists a large body of pricing literature based on the MNL model. One early study by Greene (1991) introduces an automobile pricing problem considering the fuel efficiency of each product. Further studies of product line pricing include Chen and Hausman (2000), Aydin and Ryan (2000), and Luo et al. (2007) among others. For theoretical developments, Hanson and Martin (1996) show that profit is not concave in price. They introduce a heuristic method based on a path-following approach to set prices. Many researchers have studied joint pricing and inventory decisions under the MNL demand (e.g., Aydin and Porteus 2008; Hopp and Xu 2005; Maddah and Bish 2007; Akçay et al. 2010). Song and Xue (2007) and Dong et al. (2009) show that the problem is not concave in price, but is concave in the choice probability. Dong et al. (2009) identify an interesting "equal-markup" property, i.e., the optimal markups of all products are the same. Such a property diverges from differentiated markups commonly observed in practice, which drives researchers to search for more complex models that do not need to conform to this property (Alptekinoğlu and Semple, 2016). Li and Huh (2011) and Gallego and Wang (2014) consider pricing under the nested logit model in which the choice alternatives are grouped into "nests". Li and Huh (2011) generalize the concavity property to the nested model and Gallego and Wang (2014) identify an "adjusted equal-markup" property. In this paper, we show that, due to the interaction of price and quality in customer utility, not only does the optimal markup vary with product quality in price optimization, but both the optimal markup and optimal quality vary across products in the joint optimization of price and quality. While the mixed MNL (MMNL) may also result in non-equal markups in optimal pricing decision, both the underlying driving mechanism and insights differ from that accomplished with the interaction. The MMNL model addresses heterogeneity in customer price sensitivity that is independent of quality. For example, Li et al. (2018) model a discrete MMNL model in which the customer population consists of multiple segments each with segment-specific price sensitivity. The optimal prices, therefore, have to balance the marginal utilities of different customer segments in addition to the standard tradeoffs captured in a MNL model to maximize the total profit. This often leads to markups that do not follow the sequence of product quality such as the steak versus tofu scenario illustrated in Li et al. (2018). Although many other generalizations of the MNL model address certain limitations of the MNL model, they do not capture the price-quality interactions. For example, the nested Logit (NL) model overcomes the Independence of Irrelevant Alternatives (IIA) limitation by grouping the choice alternatives into nests, which in effect allows correlated choice alternatives (e.g., McFadden 1978, 1980). The cross nested logit (CNL) further extends it by allowing fractional group membership (i.e., each choice alternative can be included in multiple nests) thus enabling more complex correlation structures (e.g., Small 1987, Vovsha 1997). Like the MNL, when price sensitivities are symmetric across products, both NL and CNL maintain the equal markup property (see a detailed discussion in Li and Webster (2017)). In other words, correlation among choice alternatives itself does not drive differentiated markups. Naturally, the analysis and solution approaches developed in this paper for including the price-quality interaction in the MNL model will also pave a foundation for extending these more complex logit models with price-quality interaction.

Assortment decisions under the MNL model is also well studied (e.g., Cachon and Kök 2007; Maddah and Bish 2007; Kök and Fisher 2007; Cachon et al. 2008). In these problems, there is a finite set of pre-determined product alternatives and the firm decides which alternatives to include in its assortment. Such an assortment optimization is a combinatorial problem and a well-known result of the optimal assortment is the "revenue-ordered" structure which includes a subset of products with higher revenues than the rest (Talluri and van Ryzin, 2004). Some consider joint assortment and pricing problems under MNL (e.g., Cachon et al. 2008; Wang 2012; Wang and Sahin 2017) and related models (e.g., Kök and Xu 2011; Li et al. 2015). With joint pricing and assortment optimization, the optimal decision is to include all available products in the assortment (Wang, 2012). In contrast to this literature, we examine a problem in which the number of products is given and we search for optimal quality values that are continuously adjustable. Further, we analytically characterize the relationships between the optimal quality or price and the model primitives to shed light on the optimal product line design.

Our work also relates broadly to the literature on product line design. There are two lines of research in this literature – those that seek analytical properties of the optimal solution on a continuum and those that seek practical solutions for optimal product selections among discrete options (see Dobson and Kalish 1993).

The former often focuses on product quality and/or price decisions under diversity of customer preferences (e.g., Mussa and Rosen 1978; Moorthy 1984; Oren et al. 1984; Villas-Boas 1998; Kim and

Chhajed 2002; Netessine and Taylor 2007; Pan and Honhon 2012). A key tradeoff identified in these papers is that, when facing a heterogenous customer population, selling to low-valuation customers may create negative externality on high-value customers and thus it is important that firms set quality and/or prices such that customers self select. Common in this stream of work is that the price schedule is dictated by a pre-assumed willingness-to-pay schedule, and thus price increases with quality when differentiation is possible. This is consistent with our result of higher markup for higher quality. However, these stylized models are difficult to parameterize with real data and do not accommodate practical complexities of a large product line, i.e., a multitude of product offerings that differentiate across multiple dimensions including product features and prices. They are useful for conceptual illustration but not designed to identify optimal price/quality decisions for a discrete product line in a practical business setting. In comparison, rigorous econometric methods have long been established for the multinomial logit (MNL) model to parameterize customer choices, to fit and predict demand of multiple differentiated products (Greene, 2003), providing the pertinent input to subsequent price/quality optimizations. Lancaster (1990) summarizes the structural properties identified for the Chamberlinian and Hoteling models and their analogs or extensions in this context; specifically, the optimal level of product variety is affected by (i) scale economy (ii) customer sensitivity to product differentiation and (iii) market competitiveness. Papers in this line of research adopt highly stylized demand models that yield strategic guidance but are not ideal for decision support. Compared to this line of work, we consider the product quality and price decisions in the MNL model, which has empirical support (McFadden, 1974, 1978; Guadagni and Little, 1983; Train et al., 1987) and is efficient to parameterize (Greene, 2003; Train, 2003). In addition, we explicitly model the quality and price interaction. We provide not only structural properties, but also efficient solution methods to joint price and quality optimizations on a continuum.

The latter line of work focuses on discrete decision of product selection and the focus is on heuristic algorithms that achieve efficiency. This line of work further diverges into two branches: (1) one that selects products from a predefined set (i.e., optimizing over the product scope, e.g., McBride and Zufryden 1988; Green and Krieger 1985, 1993; Bertsimas and Mišić 2016) which is essentially the assortment optimization problem, and (2) one that selects discrete attribute levels of the products (i.e, optimizing over the attribute space, see, e.g., Kohli and Sukumar 1990; Wang et al. 2009). In addition, some papers consider assortment in conjunction with price and typically optimize over the product scope but allow price to be continuous (e.g., Dobson and Kalish 1988, 1993; Kraus and Yano 2003; Schön 2010b) or discrete (e.g., Schön 2010a; Chen and Hausman 2000). Belloni et al. (2008) identify a method to find a guaranteed optimal solution and use it to evaluate heuristic but speedier methods. Several of the above papers, Chen and Hausman (2000) and Schön (2010a,b) use the MNL demand model. The primary goal of these papers is to seek efficient solution methods for the discrete optimization, but they do not offer insights on structural properties of the solution due to the combinatorial nature of the problem formulation.

In contrast with existing work, our research contributes to the literature by addressing three novel issues. First, we consider the situation when a company is given an existing product line and determines the prices of the new products, which is a realistic setting, yet has not been addressed. Secondly, we develop a continuous quality optimization model with joint pricing decisions under the MNL model which is new to the product line design literature, as well as to the pricing and revenue management literature. We present both efficient solution methods and structural property characterizations for two realistic problem variations. Thirdly, we allow for a nonlinear interaction among price and quality in consumer utility, which enables us to more accurately model how consumer utility changes over multiple dimensions (e.g., price and quality) and derive useful insights for the quality and pricing decisions.

3 Model

A customer makes a selection of one of n product choices and a no-purchase alternative. The product purchase probabilities are given by the MNL model. Let the utility of product $i, i = 1, 2, \ldots, n$ be

$$u_i = x_i - b_i p_i + \beta_i x_i p_i + a_i + \epsilon_i$$

where x_i is the quality, p_i is the price of product i, a_i represents an observable utility term that is independent of x_i and p_i, and ϵ_i is a noise term which is a Gumbel random variable that represents unobserved utility. We remark that a_i refers to utility from attributes that are exogenously determined and orthogonal to x_i. For example, for hotel rooms, a_i may capture utility associated with the type of room such as King, Queen and Suite, whereas x_i reflects utility of ancillary services such as packages that include resort credit, event activities and meals. For smart phone products, a_i may be associated with the model type (e.g., iPhone 7, iPhone 7plus), and x_i may be associated with the storage size (e.g., 64GB, 256GB), similar to the example in Table 2. That is, the quality measure x is a linear utility scale transformed from the nominal scale of a certain attribute such as storage size or resort credit. Without loss of generality, we assume $x_i \in [0, x_i^+]$ where $x_i = 0$ and $x_i = x_i^+$ align with the lowest and highest possible quality level respectively. For example, the quality scale for smart phones could be the logarithm transformation of the nominal storage size. Then, adjusting for a minimum required size of 16GB (i.e., align $x_i = 0$ with 16GB), nominal values of 64GB, 128GB and 256GB correspond to x values of 0.6, 0.9, and 1.2 respectively, while the maximum x_i^+ may correspond to the scaled value of some practical upper limit of storage size. Parameters $b_i > 0$ and $\beta_i \geq 0$ are the coefficients for price sensitivity where b_i is quality-independent sensitivity and β_i is the coefficient for the interaction term and captures the heterogeneity in customer sensitivity towards the product price at different quality levels. We assume that $b_i - \beta_i x_i$ is always positive for all $x_i \in [0, x_i^+]$, i.e., customers always experience a disutility toward higher prices.

Interaction terms in regression models are used to capture how the marginal effect of one explanatory variable on the dependent variable is modified by another explanatory variable and are prevalent in statistics and econometric applications (Allison, 1977; Rajan and Zingales, 1998; Greene, 2003). An interaction term is typically modeled as the product of two variables in the regression equation

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{12} X_1 X_2 + \epsilon \ .$$

The same form of interaction terms is also commonly adopted in logit and probit models (Nagler, 1994; Ai and Norton, 2003). The interaction term in the utility function of the logit model allows the marginal utility of quality x to depend on price p and equivalently, the marginal disutility of price p to depend on x. Specifically, rewrite the utility function in two alternative forms

$$u_i = x_i - (b_i - \beta_i x_i)p_i + a_i + \epsilon_i \quad \text{and} \tag{1}$$

$$u_i = (1 + \beta_i p_i)x_i - b_i p_i + a_i + \epsilon_i \ . \tag{2}$$

The marginal disutility of p_i is given by $(b_i - \beta_i x_i)$ which decreases with attribute; the marginal utility of x_i is $(1 + \beta_i p_i)$ which increases with price. As discussed in the introduction, this in effect models the empirical observation that customers are less sensitive to price change at high quality, or equivalently, customers are more sensitive to quality change at high price (i.e., $-\frac{\partial}{\partial x_i}\left(\frac{\partial(-u_i)}{\partial p_i}\right) = \frac{\partial}{\partial p}\left(\frac{\partial u_i}{\partial x_i}\right) \geq 0$ where $\frac{\partial(-u_i)}{\partial p_i}$ is the marginal disutility of price).

Let J be the set of existing products and I be the set of new products. Assume the no-purchase option has a utility of zero. For product $j \in J$, its price, quality, and cost values are fixed at $\bar{p}_j$, $\bar{x}_j$, and $\bar{c}_j$ respectively. Let $\boldsymbol{x} = (x_i)_{i \in I}$ be the vector of quality of the new products. For the ease of notation, we also define $\bar{u}_j = \bar{x}_j - b_j \bar{p}_j + \beta_j \bar{x}_j \bar{p}_j + a_j$, $\bar{m}_j = \bar{p}_j - \bar{c}_j$ and $\bar{\pi}_J = \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}}$. Note that $\bar{\pi}_J$ is the firm's expected profit prior to the addition of the new products.

The purchase probability of a new product $i \in I$ is

$$q_i = \frac{e^{x_i - b_i p_i + \beta_i x_i p_i + a_i}}{1 + \sum_{j \in J} e^{\bar{u}_j} + \sum_{i' \in I} e^{x_{i'} - b_{i'} p_{i'} + \beta_{i'} x_{i'} p_{i'} + a_{i'}}} , \tag{3}$$

the purchase probability of an existing product $j \in J$ is

$$q_j = \frac{e^{\bar{u}_j}}{1 + \sum_{j' \in J} e^{\bar{u}_{j'}} + \sum_{i \in I} e^{x_i - b_i p_i + \beta_i x_i p_i + a_i}} \quad \text{and} \tag{4}$$

$$q_0 = \frac{1}{1 + \sum_{j \in J} e^{\bar{u}_j} + \sum_{i \in I} e^{x_i - b_i p_i + \beta_i x_i p_i + a_i}}$$

is the no-purchase probability. Therefore,

$$q_i = q_0 e^{x_i - b_i p_i + \beta_i x_i p_i + a_i} \quad \text{and} \tag{5}$$

$$q_j = q_0 e^{\bar{u}_j} . \tag{6}$$

3.1 Price Optimization

Price optimization arises when the firm sets prices of the new products to maximize the total profit from the product line. For instance, in Table 1, the resort hotel decides the prices of the new room offers based on the information of existing room offers and the planned service packages for each room type in the new offers; similarly in Table 2, the smartphone manufacturer decides the prices of the new phones given all other information. Price optimization under the MNL model is well-established and we provide incremental contributions to this problem through two features: (1) consideration of existing products in the product line, and (2) consideration of quality-price interaction. In the presence of existing products, the pricing decision of the new products affects not only the relative market share of the new products, but also those of the existing products. Hence, it does not merely imply an enlarged no-purchase utility with the additional constant term $\sum_{j \in J} e^{\bar{u}_j}$ as equation (3) might have suggested. Incorporating the interaction of quality and price enables us to characterize how quality differences translate to price differences across products, thereby reconciling the counterintuitive equal-markup solution in the literature.

Let $\boldsymbol{p} = (p_i)_{i \in I}$ be the vector of prices of the new products. The firm's price optimization problem is

$$\max_{\boldsymbol{p}} \pi(\boldsymbol{p}) = \sum_{i \in I} (p_i - c_i) q_i(\boldsymbol{p}) + \sum_{j \in J} \bar{m}_j q_j(\boldsymbol{p}) ,$$

which is not a concave or a quasiconcave maximization even for the special case of $J = \emptyset$ (Dong et al., 2009). We rewrite profit as a function of choice probabilities of the new products, $\boldsymbol{q} = (q_i)_{i \in I}$, and show that this function is concave. From (5) and (6),

$$\begin{aligned} p_i &= (\bar{x}_i + a_i + \log q_0 - \log q_i)/(b_i - \beta_i \bar{x}_i) \quad \text{and} \\ q_0 &= 1 - \sum_{i \in I} q_i - \sum_{j \in J} q_j = 1 - \sum_{i \in I} q_i - \left(\sum_{j \in J} e^{\bar{u}_j} \right) q_0 \ , \end{aligned} \tag{7}$$

the latter of which is equivalent to

$$q_0 = \frac{1 - \sum_{i \in I} q_i}{1 + \sum_{j \in J} e^{\bar{u}_j}} \ . \tag{8}$$

Therefore, the price optimization problem can be restated as

$$\max_{\boldsymbol{q}} \hat{\pi}(\boldsymbol{q}) = \sum_{i \in I} (p_i(\boldsymbol{q}) - c_i) q_i + \sum_{j \in J} \bar{m}_j q_j(\boldsymbol{q}), \tag{9}$$

where

$$\begin{aligned} p_i(\boldsymbol{q}) &= \left[\bar{x}_i + a_i + \log \left(\frac{1 - \sum_{i \in I} q_i}{1 + \sum_{j \in J} e^{\bar{u}_j}} \right) - \log q_i \right] / (b_i - \beta_i \bar{x}_i) \quad \text{and} \\ q_j(\boldsymbol{q}) &= \frac{1 - \sum_{i \in I} q_i}{1 + \sum_{j \in J} e^{\bar{u}_j}} e^{\bar{u}_j} \ . \end{aligned}$$

Theorem 1. *$\hat{\pi}(\boldsymbol{q})$ is concave in $\boldsymbol{q}$.*

Dong et al. (2009) and Song and Xue (2007) have shown that the profit is concave in the choice probability vector for a clean-slate problem in which $J = \emptyset$. Theorem 1 extends the state-of-art literature and establishes a unique optimal solution in the presence of existing products.

Theorem 2. *The optimal prices of the new products and the firm's optimal total profit are*

$$\begin{aligned} p_i^* &= c_i + \frac{1}{b_i - \beta_i \bar{x}_i} + \theta \\ \pi^* &= \theta \end{aligned} \tag{10}$$

where θ solves the single-variable equation

$$\theta = \bar{\pi}_J + \frac{\sum_{i \in I} e^{\bar{x}_i + a_i - 1 - (b_i - \beta_i \bar{x}_i)(c_i + \theta)} / (b_i - \beta_i \bar{x}_i)}{1 + \sum_{j \in J} e^{\bar{u}_j}} \ . \tag{11}$$

From Theorem 2, all else equal, the optimal markup is higher for higher-quality product, and for product with lower b_i and higher β_i values, as stated in the following corollary.

Corollary 1. *At optimality, the following holds for $i, i' \in I, i \neq i'$: (i) Let $b_i = b_{i'}$ and $\beta_i = \beta_{i'}$. Then $p_i^* - c_i > p_{i'}^* - c_{i'}$ if and only if $\bar{x}_i > \bar{x}_{i'}$. (ii) Let $\beta_i = \beta_{i'}$ and $\bar{x}_i = \bar{x}_{i'}$. Then $p_i^* - c_i > p_{i'}^* - c_{i'}$ if and only if $b_i < b_{i'}$. (iii) Let $b_i = b_{i'}$ and $\bar{x}_i = \bar{x}_{i'}$. Then $p_i^* - c_i > p_{i'}^* - c_{i'}$ if and only if $\beta_i > \beta_{i'}$.*

Hence, controlling other parameters, a hotel room with a higher resort credit package or a smartphone with larger storage size should command a higher markup than its peer products. The next corollary implies that the well-known equal mark-up property holds if $\beta_i = 0$ and $b_i = b$ for all $i \in I$.

Corollary 2. *If $b_i = b$ and $\beta_i = 0$ for all $i \in I$, then the optimal prices become*

$$p_i^* \quad = \quad c_i + \bar{\pi}_J + \frac{1}{b}\left[1 + W\left(\frac{\sum_{i \in I} e^{\bar{x}_i + a_i - bc_i - 1 - b\bar{\pi}_J}}{1 + \sum_{j \in J} e^{\bar{u}_j}}\right)\right]$$

where $W(\cdot)$ is the Lambert W function.

We remark that, treating $b_i - \beta_i \bar{x}_i$ as the effective price sensitivity, the relationship in (10) reproduces the more general equal "adjusted mark-up" property identified in Gallego and Topaloglu (2014) but specializes it in terms of quality-price interaction.

Theorem 2 also leads to the following bounds for π^*.

Corollary 3. $\bar{\pi}_J \leq \pi^* \leq \bar{\pi}_J + \frac{\sum_{i \in I} e^{\bar{x}_i + a_i - 1 - (b_i - \beta_i \bar{x}_i)(c_i + \bar{\pi}_J)}/(b_i - \beta_i \bar{x}_i)}{1 + \sum_{j \in J} e^{\bar{u}_j}}$.

These bounds, along with equation (11), lead to an efficient bisection search algorithm for solving the optimal profit and prices.

Algorithm 1. *(Price Optimization)*

1. *Let $\theta^- = \bar{\pi}_J$ and $\theta^+ = \bar{\pi}_J + \frac{\sum_{i \in I} e^{\bar{x}_i + a_i - 1 - (b_i - \beta_i \bar{x}_i)(c_i + \bar{\pi}_J)}/(b_i - \beta_i \bar{x}_i)}{1 + \sum_{j \in J} e^{\bar{u}_j}}$*
2. *Let $\theta = (\theta^- + \theta^+)/2$.*
3. *Compute $f = \bar{\pi}_J + \frac{\sum_{i \in I} e^{\bar{x}_i + a_i - 1 - (b_i - \beta_i \bar{x}_i)(c_i + \theta)}/(b_i - \beta_i \bar{x}_i)}{1 + \sum_{j \in J} e^{\bar{u}_j}}$.*
4. *If $f > \theta$, let $\theta^- = \theta$; if $f < \theta$, let $\theta^+ = \theta$.*
5. *Repeat Steps 2-4 until $f = \theta$.*
6. *Compute optimal prices according to equation* (10).

Examples

Consider a manufacturer with a product cost function $c(a, x) = 0.5a + x^2$. Suppose the manufacturer currently offers three products with a_j, $\bar{x}_j$ and $\bar{p}_j$ values shown in Table 3. The manufacturer plans to add three new products, products 4-6, with quality given in Table 3, while still keeping products 1-3 in its portfolio and maintaining their current prices. The price coefficients are $b_i = b_j = 1$ and $\beta_i = \beta_j = 0.2$ for all $i \in I$ and $j \in J$. Algorithm 1 is applied to obtain the optimal prices for the new products. We observe that the optimal markups vary across products despite that all products

Table 3: Price Optimization.

Initial Products ($j \in J$)	a_j	$\bar{x}_j$	$\bar{p}_j$	$\bar{m}_j$	New Products ($i \in I$)	a_i	$\bar{x}_i$	p_i^*	m_i^*
1	0.0	0.5	2	1.75	4	0.0	0.8	3.39	2.75
2	1.0	0.8	3	1.86	5	1.0	1.0	4.31	2.81
3	2.0	1.0	4	2.00	6	2.0	1.2	5.32	2.86

have the same b and β values. This is more inline with practice than the equal-markup solution.

Table 4 presents a comparison of problem instances and sheds light on how optimal prices are affected by the magnitude of b, β and the quality of the new products. In these examples, the quality and prices for three existing products are $\{a\}_{j\in J} = [0, 1, 2]$, $\{\bar{x}\}_{j\in J} = [0.5, 0.8, 1.0]$ and $\{\bar{p}\}_{j\in J} = [2, 3, 4]$. The cost function is the same as in the example of Table 3. The comparison

Table 4: Optimal Quality Vary with Prices.

			New Quality			Optimal Prices			
Instance	b	β	$\bar{x}_4$	$\bar{x}_5$	$\bar{x}_6$	p_4^*	p_5^*	p_6^*	profit
1	2.00	0.05	0.8	1.0	1.2	1.39	2.25	3.19	0.24
2	1.00	0.05	0.8	1.0	1.2	2.95	3.83	4.78	1.27
3	0.50	0.05	0.8	1.0	1.2	5.09	6.00	6.99	2.27
4	1.00	0.10	0.8	1.0	1.2	3.09	3.98	4.94	1.36
5	1.00	0.20	0.8	1.0	1.2	3.39	4.31	5.32	1.56
6	1.00	0.05	0.3	0.5	0.7	2.42	3.09	3.85	1.32
7	1.00	0.05	0.2	0.4	0.6	2.36	3.00	3.71	1.31
8	1.00	0.05	0.2	0.5	0.8	2.36	3.09	3.99	1.31
9	1.00	0.05	0.5	1.0	1.5	2.51	3.79	5.57	1.24

of instances 1-3 demonstrates the effect of the parameter b while the comparison of instances 2, 4 and 5 demonstrates the effect of the parameter β – the optimal prices decrease in b and increase in β. Instances 2, 6, and 7 suggest that higher quality levels lead to higher prices while instances 7-9 show that a larger quality gap between products (i.e., larger $\bar{x}_i - \bar{x}_{i'}$ value) results in a larger price gap (i.e., larger $p_i^* - p_{i'}^*$ value).

3.2 Quality and Price Optimization

In this section, we study the joint optimization of product quality and price. For example, the resort hotel considers optimizing both the service package value and the price of each new room offer to maximize the total profit. Let $\boldsymbol{x} = (x_i)_{i\in I}$ denote the vector of quality levels for the new products and define $\Omega = \{\boldsymbol{x} | 0 \le x_i \le x_i^+, i \in I\}$.

$$\max_{\boldsymbol{x}\in\Omega,\boldsymbol{p}} \pi(\boldsymbol{x},\boldsymbol{p}) = \sum_{i\in I}(p_i - c_i(x_i))q_i(\boldsymbol{x},\boldsymbol{p}) + \sum_{j\in J}\bar{m}_j q_j(\boldsymbol{x},\boldsymbol{p}).$$

Recall that in the hotel example, a_i represents customers' utility for a particular room type (i.e., King, Queen or Suite) and x_i represents customers' utility for a particular service package (e.g., \$50, \$80, or \$100 resort credit). In general, the value of a_i reflects the composite utility of all attributes of product i that are not part of the design decision (i.e., features of the product that are not to be changed), whereas quality x_i is the utility from the attribute to be optimized. For example, a_i may reflect a certain product type, for example, room type in a hotel, type of seats in an airplane (main cabin, business class, first class), or size of a rental car (compact, mid-size, full-size) and x_i may represent add-on services. The joint quality-price optimization model helps us optimally set both the level of add-on services and the price for the product to be offered in each type.

The choice of quality affects product cost. Let $c_i(x_i)$ be the unit cost of product $i \in I$ at quality x_i, which is assumed to be nonnegative. The product cost function may differ across products, reflecting differences in fixed and variable attributes, i.e., $c_i(x_i) = c(a_i, x_i)$. For brevity, we denote the cost function with $c_i(x_i)$ but we emphasize that it is also a function of a_i. To ensure that the cost function $c_i(x_i)$ is well-behaved, we make the following assumption.

Assumption 1. *The cost function $c_i(x_i), i \in I$ is twice differentiable, increasing and convex in x_i for all $x_i \in [0, x_i^+]$.*

We remark that convexity of $c_i(x_i)$ does not necessitate convexity of the nominal cost curve of a product attribute. Note that x_i is a linear utility measure of quality that can be different from its natural or nominal measure. In the smart phone example, suppose that the cost of memory increases linearly with size. Since the x values are generated by taking logarithm, the cost function in terms of x becomes exponential which is convexly increasing. A similar argument holds for the hotel example. Suppose customer utility does not grow linearly with the resort credit amount, but at a lower order of growth (e.g., the rate of a square root function) while the cost of the resort credit grows linearly with the amount. Then the cost function in terms of x becomes convex (e.g., quadratic). In other words, Assumption 1 is satisfied if the cost of the focal attribute is convexly increasing in its linear utility contribution x, or equivalently, the utility of the focal attribute exhibits diminishing return on cost. For the remainder of the paper we assume that Assumption 1 holds.

Let $\boldsymbol{m} = (m_i)_{i \in I}$ where $m_i = p_i - c_i(x_i)$. From (5), $q_i = q_0 e^{x_i + a_i - b_i p_i + \beta_i x_i p_i} = q_0 e^{x_i + a_i - (b_i - \beta_i x_i) p_i}$ $= q_0 e^{x_i + a_i - (b_i - \beta_i x_i) c_i - (b_i - \beta_i x_i) m_i}$ and thus solve m_i as a function of $\boldsymbol{x}$ and $\boldsymbol{q}$:

$$m_i = \left[x_i + a_i - (b_i - \beta_i x_i) c_i(x_i) - \log q_i + \log q_0 \right] / (b_i - \beta_i x_i) \ .$$

We can express the total profit as

$$\hat{\pi}(\boldsymbol{x},\boldsymbol{q}) = \sum_{i\in I} \frac{q_i}{b_i - \beta_i x_i} \left[x_i + a_i - (b_i - \beta_i x_i)c_i(x_i) - \log q_i + \log q_0(\boldsymbol{q})\right] + \left(\sum_{j\in J} \bar{m}_j e^{\bar{u}_j}\right) q_0(\boldsymbol{q}) \ .$$

Theorem 3. *Given $\boldsymbol{x}$, $\hat{\pi}(\boldsymbol{x},\boldsymbol{q})$ is concave in the choice probability vector $\boldsymbol{q}$. The optimal markup is given by*

$$m_i^*(\boldsymbol{x}) = \frac{1}{b_i - \beta_i x_i} + \theta(\boldsymbol{x})$$

where $\theta(\boldsymbol{x})$ solves

$$\theta = \frac{\sum_{i\in I} e^{x_i + a_i - (b_i - \beta_i x_i)c_i(x_i) - 1 - (b_i - \beta_i x_i)\theta} / (b_i - \beta_i x_i)}{1 + \sum_{j\in J} e^{\bar{u}_j}} + \frac{\sum_{j\in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j\in J} e^{\bar{u}_j}} \ . \tag{12}$$

Let $\tilde{\pi}(\boldsymbol{x}) = \max_q \hat{\pi}(\boldsymbol{x},\boldsymbol{q})$. We can establish the following relationship.

Lemma 1. *$\tilde{\pi}(\boldsymbol{x}) = \theta(\boldsymbol{x})$.*

Therefore, to maximize $\tilde{\pi}(\boldsymbol{x})$, we only need to maximize $\theta(\boldsymbol{x})$.

In the special case when $\beta_i = 0$, the optimal solution is unique and given in the following theorem.

Theorem 4. *Let $\boldsymbol{x}^* = (x_i^*)_{i\in I}$ and $\boldsymbol{p}^* = (p_i^*)_{i\in I}$ be the optimal solution to the joint quality and price optimization problem. Suppose $\beta_i = 0$. Then the optimal solution is given by*

$$x_i^* = \begin{cases} c_i'^{-1}\left(\frac{1}{b_i}\right), & \text{if } c_i'^{-1}\left(\frac{1}{b_i}\right) \in [0, x_i^+] \\ 0, & \text{if } c_i'^{-1}\left(\frac{1}{b_i}\right) < 0 \\ x_i^+, & \text{if } c_i'^{-1}\left(\frac{1}{b_i}\right) > x_i^+ \ , \end{cases} \qquad i \in I. \tag{13}$$

$$p_i^* = c_i(x_i^*) + \frac{1}{b_i} + \theta^* \tag{14}$$

where θ^* solves

$$\theta = \frac{\sum_{i\in I} e^{x_i^* + a_i - b_i c_i(x_i^*) - 1 - b_i\theta} / b_i}{1 + \sum_{j\in J} e^{\bar{u}_j}} + \bar{\pi}_J \ .$$

Theorem 4 describes the optimal solution of the joint quality and price optimization in the absence of interaction. Consider a cost function $c_i(\cdot)$ that is additively separable in a_i and x_i. Also, consider a case with symmetric b_i, i.e., $b_i = b$ for all $i \in I$. From equations (13) and (14), it must be that $x_i^* = x_{i'}^*$ and $p_i^* - c_i(x_i^*) = p_{i'}^* - c_i(x_{i'}^*)$ for all $i, i' \in I$. That is, the optimal prices and quality are such that all new products have equal markup and equal quality, as summarized in the following corollary.

Corollary 4. *Suppose that the cost function is additively separable in a_i and x_i, i.e., $c_i(x_i) = c_a(a_i) + c_x(x_i)$ and that $b_i = b$ and $\beta_i = 0$ for all $i \in I$. Then at optimality, $x_i^* = x_{i'}^*$ and $m_i^* = m_{i'}^*$ for any $i, i' \in I$.*

The result in Corollary 4 lacks realism and is an oversimplification of the effect of quality and price on customers' utility. However, it serves as a benchmark case for understanding the impact of interaction. Next, we illustrate how the inclusion of a simple quality and price interaction term leads to a different conclusion by capturing a more realistic relationship between customer preference and product quality/price.

In general, $\beta_i > 0$ and $\theta(\boldsymbol{x})$ is defined by the implicit function (12). Taking derivatives with respect to x_i, and with algebraic transformation, we obtain

$$\frac{\partial \theta(\boldsymbol{x})}{\partial x_i} = \frac{q_i}{b_i - \beta_i x_i}\left[1 - (b_i - \beta_i x_i)c_i'(x_i) + \beta_i c_i(x_i) + \beta_i \theta + \frac{\beta_i}{b_i - \beta_i x_i}\right] . \tag{15}$$

From (15), a necessary condition for an internal optimal solution is

$$h_i(\boldsymbol{x}) := -1 - \beta_i\left(\theta(\boldsymbol{x}) + c_i(x_i)\right) + (b_i - \beta_i x_i)c_i'(x_i) - \frac{\beta_i}{b_i - \beta_i x_i} = 0 \quad \text{for all } i \in I \ , \tag{16}$$

which can be rewritten as

$$(b_i - \beta_i x_i)c_i'(x_i) - \beta_i\left(c_i(x_i) + \frac{1}{b_i - \beta_i x_i}\right) = 1 + \beta_i \theta \quad \text{for all } i \in I \tag{17}$$

If for any given θ, there exists a unique x_i such that the above is satisfied, then the joint quality and price optimization is reduced to a single-variable fixed point solution. If, in addition, the Jacobian of $h(\boldsymbol{x}) = (h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), \ldots, h_n(\boldsymbol{x}))$ evaluated at $\boldsymbol{x}^*$

$$J(\boldsymbol{x}^*) = \begin{bmatrix} \frac{\partial h_1(\boldsymbol{x}^*)}{\partial x_1} & \cdots & \frac{\partial h_1(\boldsymbol{x}^*)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n(\boldsymbol{x}^*)}{\partial x_1} & \cdots & \frac{\partial h_n(\boldsymbol{x}^*)}{\partial x_n} \end{bmatrix}$$

is positive semidefinite for any $\boldsymbol{x}^*$ satisfying (16), then $\boldsymbol{x}^*$ is a global maximum.

In the following theorem, we identify a sufficient condition for positive semidefinite $J(\boldsymbol{x}^*)$ that uses a lower bound on the value $\frac{c_i''(x)}{c_i'(x)}$ which is a measure of the normalized convexity of the cost function $c_i(\cdot)$. The value of $\frac{c_i''(x)}{c_i'(x)}$ is generally not difficult to evaluate. For example, for polynomial cost functions of the form $c_i(x) = a + bx^n$ where $n > 1$, $\frac{c_i''(x)}{c_i'(x)} = \frac{n-1}{x}$; for exponential cost functions of the form $c_i(x) = a + be^{\alpha x}$ where $\alpha > 0$, $\frac{c_i''(x)}{c_i'(x)} = \alpha$.

Assumption 2. *For any $x_i \in \left[0,\ x_i^+\right]$, the cost function $c_i(\cdot)$ satisfies*

$$\frac{c_i''(x_i)}{c_i'(x_i)} > \frac{3\beta_i}{b_i - \beta_i x_i} . \tag{18}$$

Assumption 2 ensures that for a given θ, the left side of (17) can only cross $1 + \beta_i\theta$ from below. Therefore, if a solution to (17) exists, it must be unique. Under Assumption 2, the Jacobian matrix $J(\boldsymbol{x}^*)$ is a diagonal matrix with nonnegative diagonal elements (note that $\frac{\partial h_i(\boldsymbol{x})}{\partial x_i} = (b_i - \beta_i x_i^*)c_i''(x_i^*) - 2\beta_i c_i'(x_i^*) - \left(\frac{\beta_i}{b_i-\beta_i x_i^*}\right)^2 > 0$ according to Lemma 2 in the appendix and $\frac{\partial h_i(\boldsymbol{x})}{\partial x_j} = 0$), which is positive semi-definite. This implies global optimality. Later in the paper we also consider the setting where Assumption 2 does not hold and identify an optimization procedure that accommodates multiple stationary points.

Assumption 2 requires that the cost function be "sufficiently" convex. In most realistic scenarios, the interaction effect is small relative to the main effect of price and we expect the fraction $\frac{\beta_i}{b_i-\beta_i x_i}$ to be small. Thus the condition is not as restrictive as it might appear. For polynomial cost functions of the form $c_i(x) = a + bx^n$, it can be shown that the condition reduces to $\frac{n-1}{3} > \frac{\beta_i x_i^+}{b_i-\beta_i x_i^+}$; for exponential cost functions of the form $c_i(x) = a + be^{\alpha x}$, condition (18) reduces to $\frac{\alpha}{3} > \frac{\beta_i}{b_i-\beta_i x_i^+}$.

Theorem 5. *If Assumption 2 holds, then the optimal profit θ^* to the joint quality and price optimization problem is the fixed-point solution to*

$$\theta = \frac{\sum_{i\in I} e^{x_i(\theta)+a_i-(b_i-\beta_i x_i(\theta))c_i(x_i(\theta))-1-(b_i-\beta_i x_i(\theta))\theta}/(b_i-\beta_i x_i(\theta))}{1+\sum_{j\in J} e^{\bar{u}_j}} + \bar{\pi}_J \ . \tag{19}$$

where $x_i(\theta)$ is the unique solution of (17) for any given θ if a solution to (17) exists and $x_i(\theta) = 0$ or x_i^+ otherwise (specifically, if $1 - b_i c_i'(0) + \beta_i c_i(0) + \beta_i\theta + \frac{\beta_i}{b_i} < 0$, then $x_i(\theta) = 0$; if $1 - (b_i - \beta_i x_i^+)c_i'(x_i^+) + \beta_i c_i(x_i^+) + \beta_i\theta + \frac{\beta_i}{b_i-\beta_i x_i^+} > 0$, then $x_i(\theta) = x_i^+$). In addition, the optimal quality and price values are given by

$$x_i^* = x_i(\theta^*) \tag{20}$$

$$p_i^* = c_i(x_i^*) + \frac{1}{b_i - \beta_i x_i^*} + \theta^* \ . \tag{21}$$

We remark that, given additively separable cost functions and symmetric b and β, the optimal quality is not identical across products but varies based on a_i values, which can be derived from equation (17). As a result, the optimal markup must also differ across products with different a_i values due to equation (21). The following corollary provides a key insight into the implication of the interaction term.

Corollary 5. *Suppose Assumption 2 holds. In addition, assume that the cost function is additively separable in a_i and x_i, i.e., $c_i(x_i) = c_a(a_i) + c_x(x_i)$ where $c_a(\cdot)$ is a non-decreasing function, and that $b_i = b$ and $\beta_i = \beta$ for all $i \in I$. Then $x_i^* \geq x_{i'}^*$ and $m_i^* \geq m_{i'}^*$ if and only if $a_i \geq a_{i'}$ for any $i, i' \in I$.*

Contrasting this with Corollary 4, the optimal quality levels and markups now differ by a_i values and products with a larger a_i value is matched with a higher quality as well as a higher markup. In a practical setting, this implies, for example, that the smart phone manufacturer shall design its product line such that a premium model (which corresponds to a high a_i value) is matched with a premium storage size as well as a premium price – a commonly-adopted strategy which can now be quantified and optimized with the model developed in this paper.

From (21), we also observe that the properties of the optimal prices identified in Corollary 1 for the price optimization problem continue to hold for the joint optimization problem. When price and quality can be determined jointly, lower price sensitivity of a product allows the firm to charge a higher price for the product, and subsequently to also set a higher quality value. Thus the relative magnitude of x_i versus other products depends on both β_i and b_i, as shown in the following corollary.

Corollary 6. *Suppose Assumption 2 holds. For any $i, i' \in I$ and $i \neq i'$,*

(i) let $b_i = b_{i'}, \beta_i = \beta_{i'}$ and $c_i(\cdot) = c_{i'}(\cdot)$, then $x_i^* > x_{i'}^*$ if and only if $p_i^* > p_{i'}^*$.

(ii) if $\beta_i = \beta_{i'}$ and $c_i(\cdot) = c_{i'}(\cdot)$, then $x_i^* > x_{i'}^*$ if and only if $b_i < b_{i'}$.

In addition, we derive the following bounds for θ^*.

Corollary 7. *Under Assumption 2,* $\bar{\pi}_J \leq \theta^* \leq \bar{\pi}_J + \frac{\sum_{i \in I} e^{\max\left\{a_i + \frac{b_i}{\beta_i} - \frac{b_i^2}{\beta_i} c_i'(0),\ a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)\right\}} / b_i}{1 + \sum_{j \in J} e^{\bar{u}_j}}$.

Theorem 5 identifies the fixed-point equation for the optimal solution but does not establish uniqueness of the solution or identify an efficient solution algorithm. Next, we show that the fixed-point solution to (19) is unique and can be obtained with a bisection search. Define

$$g(\theta) := \frac{\sum_{i \in I} e^{x_i(\theta) + a_i - (b_i - \beta_i x_i(\theta)) c_i(x_i(\theta)) - 1 - (b_i - \beta_i x_i(\theta))\theta} / (b_i - \beta_i x_i(\theta))}{1 + \sum_{j \in J} e^{\bar{u}_j}}.$$

Theorem 6. *$g(\theta)$ monotonically decreases in θ and equation* (19) *has a unique fixed-point solution.*

As a result of Theorem 6, the solution of equation (19) can be obtained through an efficient bisection search algorithm.

Algorithm 2. *(Quality and Price Optimization)*

1. *Let $\theta^- = \bar{\pi}_J$ and $\theta^+ = \bar{\pi}_J + \frac{\sum_{i \in I} e^{\max\left\{a_i + \frac{b_i}{\beta_i} - \frac{b_i^2}{\beta_i} c_i'(0),\ a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)\right\}} / b_i}{1 + \sum_{j \in J} e^{\bar{u}_j}}$.*
2. *Let $\theta = (\theta^- + \theta^+)/2$ and solve* (17) *for $x_i(\theta), i \in I$ following Steps (a)-(e):*

(a) *Let* $y^- = 0$ *and* $y^+ = x_i^+$.

(b) *Let* $y = (y^- + y^+)/2$.

(c) *Compute* $z = (b_i - \beta_i x_i)c_i'(x_i) - \beta_i\left(c_i(x_i) + \frac{1}{b_i - \beta_i x_i}\right)$.

(d) *If* $z > 1 + \beta_i\theta$, *let* $y^- = y$; *if* $z < 1 + \beta_i\theta$, *let* $y^+ = y$.

(e) *Repeat Steps (a)-(e) until* $z = \theta$ *or* $y^+ = y^-$. *Then* $x_i(\theta) = y$.

3. *If, for the given* θ *value, Steps (a)-(e) do not converge, then let* $\theta^+ = \theta$ *and repeat Step 2. If no solution is found for* θ *at its lowest value, then set* $x_i^* = x_i^+$.

4. *Let* $c_i(x_i) = c_i(x_i(\theta)), i \in I$ *and compute* $f = \frac{\sum_{i\in I} e^{x_i + a_i - (b_i - \beta_i x_i)c_i(x_i) - 1 - (b_i - \beta_i x_i)\theta}/(b_i - \beta_i x_i)}{1 + \sum_{j\in J} e^{\bar{u}_j}} + \bar{\pi}_J$.

5. *If* $f > \theta$, *let* $\theta^- = \theta$; *if* $f < \theta$, *let* $\theta^+ = \theta$.

6. *Repeat Steps 2-4 until* $f = \theta$.

If, for a given cost function, Assumption 2 is not satisfied, then we cannot apply Algorithm 2. Instead, we note a special characteristic of the optimality condition in (17). That is, given the value of θ, the left side of the equation does not depend on $x_j, j \neq i$. This implies that, given θ, we can use a single-dimension search to find all stationary x_i values for all $i \in I$. For any given θ, let $x_i^k(\theta), k = 1, \ldots, K$ where $K > 1$ be the multiple paths of solutions to equation (17). For any given path (defined by some selection rule when picking a solution to (17) from potentially multiple possibilities), we can show that $g(\theta)$ is decreasing in θ by applying Theorem 6 for this generalization. In other words, we have multiple g functions, i.e., $g^1(\theta), \ldots, g^K(\theta)$ which are decreasing in θ. The function that yields the largest fixed-point solution to (19) yields the global maximum. Denote this function with $g^*(\theta)$ and the global maximum with θ^*.

Since each $g^k(\theta)$ function decreases monotonically in θ, it is easy to see that the global maximum must satisfy $g^*(\theta^*) \geq g^k(\theta^*)$ for all $k = 1, \ldots, K$. Consequently, to locate the global maximum, it suffices to locate the fixed-point solution of $\theta = g_{\max}(\theta) + \bar{\pi}_J$ where $g_{\max}(\theta) := \max_k g^k(\theta)$. Since $g_{\max}(\theta)$ must also be decreasing in θ, we can apply bisection search to obtain the optimal value of θ. Therefore, we propose the following algorithm for obtaining the optimal solution when Assumption 2 does not hold or cannot be verified.

Algorithm 3. *(Quality and Price Optimization without Assumption 2)*

1. *Let* $\theta^- = \bar{\pi}_J$ *and* $\theta^+ = \bar{\pi}_J + \frac{\sum_{i\in I} e^{x_i^+ + a_i - (b_i - \beta_i x_i^+)c_i(0) - 1 - (b_i - \beta_i x_i^+)\theta^-}/(b_i - \beta_i x_i^+)}{1 + \sum_{j\in J} e^{\bar{u}_j}}$.

2. *Let* $\theta = (\theta^- + \theta^+)/2$.

(a) *Search in the range of* $[0, x_i^+]$ *for all values of* x_i *that satisfy* (17) *and place them in set* $\mathcal{X}_i$. *In addition, place* 0 *and* x_i^+ *in set* $\mathcal{X}_i$ *if they are not already included.*

(b) *For each* $\boldsymbol{x} = (x_i)_{i\in I}$ *where* $x_i \in \mathcal{X}_i$, *compute*

$$f(\boldsymbol{x}) = \frac{\sum_{i\in I} e^{x_i + a_i - (b_i - \beta_i x_i)c_i(x_i) - 1 - (b_i - \beta_i x_i)\theta}/(b_i - \beta_i x_i)}{1 + \sum_{j\in J} e^{\bar{u}_j}} + \bar{\pi}_J \ .$$

(c) Let $f_{\max} = \max_{x_i \in \mathcal{X}_i, i \in I} f(\boldsymbol{x})$.

3. *If $f_{\max} > \theta$, then let $\theta^- = \theta$; if $f_{\max} < \theta$, let $\theta^+ = \theta$.*
4. *Repeat Steps 2-3 until $f_{\max} = \theta$.*

We remark that the upper bound from Corollary 7 holds true only under Assumption 2. Therefore in Algorithm 3 we construct an upper bound based solely on $x_i \in [0, x_i^+]$.

Examples

Consider a manufacturer with the product cost function $c_i(a_i, x_i) = 0.1a_i + 0.1e^{1.5x_i}$ where $x_i \in [0, 4]$. The manufacturer has an existing set of products with $\{a_j\}_{j \in J} = [0, 5, 10]$, $\{\bar{x}_j\}_{j \in J} = [0.5, 0.8, 1]$, $\{\bar{p}_j\}_{j \in J} = [2, 3, 4]$. It introduces three new products with $\{a_i\}_{i \in I} = [0, 5, 10]$ and jointly optimizes prices and quality of the new products in the expanded product line. The values of b and β are the same across products and are given in Table 5. It can be verified that Assumptions 2 holds for all parameter combinations presented in the table. We apply Algorithm 2 to optimize both prices and quality (i.e., $p_i, x_i, i \in I$) of the new products and the results are recorded in Table 5.

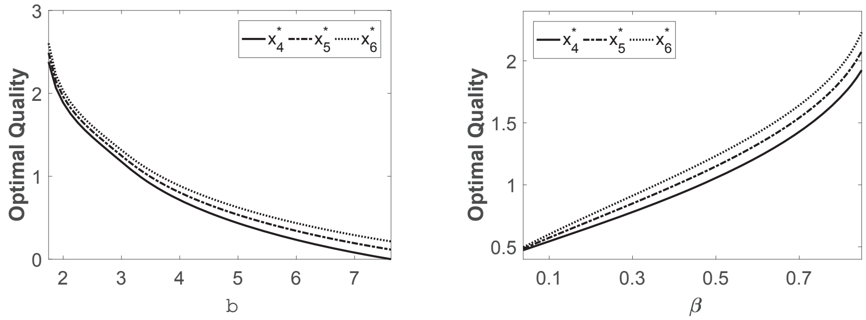
Table 5: Jointly Optimization of Quality and Prices.

			Optimal Quality			Optimal Prices			
Instance	b	β	x_4^*	x_5^*	x_6^*	p_4^*	p_5^*	p_6^*	profit
1	3.0	0.35	1.08	1.15	1.22	2.85	3.41	3.97	1.96
2	3.0	0.40	1.17	1.25	1.32	3.03	3.60	4.17	2.05
3	3.0	0.45	1.27	1.35	1.42	3.22	3.81	4.40	2.14
4	3.5	0.35	0.84	0.92	0.99	2.01	2.56	3.11	1.35
5	3.5	0.40	0.91	0.99	1.07	2.11	2.66	3.22	1.40
6	3.5	0.45	0.98	1.07	1.15	2.22	2.78	3.35	1.45
7	4.0	0.35	0.66	0.75	0.82	1.52	2.05	2.59	0.98
8	4.0	0.40	0.71	0.80	0.89	1.57	2.11	2.66	1.01
9	4.0	0.45	0.77	0.86	0.95	1.62	2.18	2.73	1.03

Note that products 4-6 are differentiated only by their a_i value prior to the optimization. The optimized prices and quality follow a sequence matching that of a_i's, as Corollary 5 implies.

Table 5 and Figure 1 illustrate that the optimal quality decreases with price sensitivity b: as customers become more price sensitive (equivalently, less willing to pay), the manufacturer lowers prices, which, due to the interaction of quality and price, leads to lower marginal utility of quality (equation (2)); this, consequently, drives the manufacturer to reduce quality. Managerially, with more price-sensitive customers, it is optimal for the firm to play the low-end strategy (i.e., setting lower quality and lower price).

Now consider the effect of β. Larger β implies higher marginal utility of x, creating incentive to increase quality; the increased quality in turn, reduces customers' marginal disutility of price and



(a) $\beta = 0.4$ (b) $b = 3.5$

Figure 1: Optimal Quality Values vs. b and β

thus drives up optimal prices, which in turn further increases the marginal utility of x. Due to such a reinforcement effect on the marginal utility, increasing β leads to both higher optimal quality and higher optimal prices. This discussion reveals a central role that the interaction effect plays in the joint optimization of quality and prices, i.e., it causes the optimal prices and quality to move in tandem, and as the interaction intensifies, both move upward. Recall that the interaction effect has been adopted to model the phenomenon that customers often use price as a cue for quality. Thus according to our findings, strong price-cue effect drives a firm more toward the high-end strategy (i.e., offering products with premium quality and high prices).

Joint optimization dominates price optimization alone and leads to higher profit. We illustrate such improvement with numerical experiments that consider multiple b and β parameter combinations. For each parameter combination, we generate 100 random problem instances by drawing the quality values of the new products from a uniform distribution on $[0, 2]$ and perform price optimization for each instance. We then compute the percentage profit improvement of joint optimization over price optimization for each instance and average over these 100 instances; we also present the average profit under price optimization and that under joint optimization. See results in Table 6. As is shown, profit improvement with joint optimization can be substantial.

3.3 Effect of Existing Products on Optimal Decision

In this section, we explore the impact of existing products on the optimal quality and price decisions of new products, as well as how the impact is modified by the price-quality interaction.

Recall that I is the set of new products and J is the set of existing products. Let $\pi^*_{I \cup J}$ denote the total profit of the product line given the set of existing product J and that the price and quality of the new products in I are optimally determined via methods in Section 3.2. Let π^*_I denote the

Table 6: Profit Improvement (Joint vs. Price Optimization).

Combination	b	β	Average Profit for Price Optimization	Profit for Joint Optimization	Average Improvement
1	3.5	0.35	1.23	1.35	11.1%
2	3.5	0.40	1.28	1.40	9.8%
3	3.5	0.45	1.34	1.45	8.9%
4	4.5	0.35	0.60	0.74	35.8%
5	4.5	0.40	0.62	0.76	31.8%
6	4.5	0.45	0.65	0.77	28.2%
7	5.5	0.35	0.31	0.43	91.0%
8	5.5	0.40	0.32	0.44	81.6%
9	5.5	0.45	0.33	0.45	73.2%

optimal profit with no existing product. Similarly, let $\left(p^*_{i|I\cup J},\ x^*_{i|I\cup J}\right)_{i\in I}$ and $\left(p^*_{i|I},\ x^*_{i|I}\right)_{i\in I}$ denote the optimal decisions with and without the existing products.

The following corollaries show the effect of existing products on quality/price decisions with and without price-quality interaction. Corollary 8 follows directly from Theorem 4. Corollary 9 follows directly from Theorem 5 and the fact that the left-hand side of equation (17) is increasing in x_i under Assumption 2.

Corollary 8. *Suppose $\beta_i = 0$ for all $i \in I$. (i) The optimal quality of the new product $x^*_i, i \in I$ is independent of price and quality of any existing product. (ii) If $\pi^*_{I\cup J} > \pi^*_I$, then the presence of existing products J causes the prices of new products to increase, i.e., $p^*_{i|I\cup J} > p^*_{i|I}$ for all $i \in I$; otherwise, the opposite holds true.*

Corollary 9. *Suppose $\beta_i > 0$ for all $i \in I$ and Assumption 2 holds. If $\pi^*_{I\cup J} > \pi^*_I$, then the presence of existing products J causes both the quality and prices of new products to increase, i.e., $x^*_{i|I\cup J} > x^*_{i|I}$ and $p^*_{i|I\cup J} > p^*_{i|I}$ for all $i \in I$; otherwise, the opposite holds true.*

The effect of existing products on the optimal price/quality decisions for new products is determined by the relationship between $\pi^*_{I\cup J}$ and π^*_I. From a practical perspective, the existing products J that remain in the offer set at the time new products are introduced are such that $\pi^*_{I\cup J} > \pi^*_I$, i.e., if $\pi^*_{I\cup J} < \pi^*_I$, then one or more of the existing products should be dropped from the product line.

Without interaction, assuming that the firm has made the right decision for including products $j \in J$, the presence of existing products drives up the optimal prices of the new products, but does not affect the optimal quality of the new products. The independence of optimal quality from existing products is a consequence of zero price-quality interaction and is arguably unrealistic in most industry contexts. With interaction, the presence of existing products (assuming that inclusion is a good decision) ought to drive the firm to offer new products positioned higher in both quality and price. This resonates with practical observations. For example, the latest iPhone model X has been introduced in the presence of existing iPhone7 and iPhone8 products, priced at hefty \$999, \$1149, and \$1349 for 64GB, 256GB, and 512GB storage size respectively and a slew of other fancy high-end features (Apple Corporation Website,, 2018a). Had iPhone7 and iPhone8 not been included, Apple would probably not have aimed its new products at such extreme high-end target position.

4 Conclusion and Discussion

Constantly evolving product lines create challenges for product design. In this paper, we address this complex problem by formulating the pricing and quality decision problem using a MNL model with quality-price interaction. We consider two practical variations of the problem: (i) optimize prices of the new products in the presence of existing products in the product line and pre-determined product quality of the new products, and (ii) optimize both prices and quality of the new products in the presence of existing products. We characterize the profit function and the optimal solution, in particular, how the optimal prices and/or optimal quality vary across products and with the parameters. Our analysis yields efficient solution algorithms for each problem variation.

An important message that this paper brings forth is that the lack of realism in the linear utility of the MNL model and the resulting equal markup and equal quality properties can be addressed with an interaction term. This interaction term is a simple but powerful extension that is central to understanding the quality and price decision in product line design. With the interaction effect, the optimal quality and markup vary across products even under identical price sensitivity and cost function. Illustrative examples add further insights on how the optimal solution is affected by coefficients of the utility model and how the joint optimization improves the firm's profit beyond what is accomplished by price optimization alone.

This paper considers price and quality decisions when a firm adds to an existing product line but we note that the "clean-slate" version of the problem, i.e., when the firm can decide prices and/or quality of all products to be offered, is a special case of the model in this paper. This includes the case where the firm is starting a new product line (i.e., without any existing product), as well as the case in which the firm decides to re-optimize the price and quality of both existing and new products. In practice, a mixed optimization in which price and/or quality optimization may be performed on different sets of products is also likely. For example, in some industries, the older version products are discounted when the new versions are introduced. In this case, the firm may decide to optimize prices of both new and old products but only optimize quality of the new products. This scenario is not modeled in the current two variations. However, it is straightforward to extend our model to accommodate such a scenario. Furthermore, in practice, new product introduction often involves the decision on the number of new products to introduce as well as product features. Cost as a function of the number of new products is generally complex and nonlinear (e.g., incorporating costs related to production setup and product roll-out, and diseconomy of scope). Our methods allow management to compare profits with the associated costs, thereby supporting a decision on the number of new products to introduce. In sum, the model and methods presented in this paper apply broadly to practical decision scenarios.

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A Appendix

A.1 Proof of Theorem 1

Proof. Substitute (6) and (7) into (9) to obtain

$$\begin{aligned}\pi(\boldsymbol{q}) &= \sum_{i\in I}\left(\frac{\bar{x}_i + a_i + \log q_0(\boldsymbol{q}) - \log q_i}{b_i - \beta_i \bar{x}_i} - c_i\right) q_i + \left(\sum_{j\in J} \bar{m}_j e^{\bar{u}_j}\right) q_0(\boldsymbol{q}) \\ &= \sum_{i\in I}\left(\frac{\bar{x}_i + a_i - \log(1 + \sum_{j\in J} e^{\bar{u}_j})}{b_i - \beta_i \bar{x}_i} - c_i\right) q_i + \left(\sum_{j\in J} \bar{m}_j e^{\bar{u}_j}\right) q_0(\boldsymbol{q}) \\ &\quad - \sum_{i\in I} \frac{q_i\left[\log q_i - \log(1 - \sum_{i\in I} q_i)\right]}{b_i - \beta_i \bar{x}_i}\end{aligned}$$

where the second equation follows from equation (8). The first and third terms are both linear in $\boldsymbol{q}$ (see equation (8)), thus if the term $q_i\left[\log q_i - \log(1 - \sum_{i\in I} q_i)\right]$ is convex in $\boldsymbol{q}$, then the total profit is concave in $\boldsymbol{q}$. From Lemma 2 in Li and Huh (2011), $q_i\left[\log q_i - \log(1 - \sum_{i\in I} q_i)\right]$ is jointly convex in q_i and $\sum_{i\in I} q_i$. Since $\sum_{i\in I} q_i$ is an affine transformation of $\boldsymbol{q}$, convexity is preserved. Thus the term $q_i\left[\log q_i - \log(1 - \sum_{i\in I} q_i)\right]$ is convex in $\boldsymbol{q}$, which implies $\pi(\boldsymbol{q})$ is concave in $\boldsymbol{q}$. □

A.2 Proof of Theorem 2

Proof. Take the first order derivative of $\hat{\pi}(\boldsymbol{q})$ in (9) with respect to q_i to obtain

$$\begin{aligned}\frac{\partial \hat{\pi}(\boldsymbol{q})}{\partial q_i} &= p_i - c_i + \sum_{i'\in I} \frac{\partial p_{i'}(\boldsymbol{q})}{\partial q_i} q_{i'} + \sum_{j\in J} \bar{m}_j \frac{\partial q_j(\boldsymbol{q})}{\partial q_i} \\ &= p_i - c_i + \frac{1}{b_i - \beta_i x_i} - \frac{\sum_{i'\in I} \frac{q_{i'}/q_0}{b_{i'} - \beta_{i'} x_{i'}} + \sum_{j\in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j\in J} e^{\bar{u}_j}}\end{aligned}$$

where the second equality follows from (7) and (8). Thus the first order condition becomes

$$p_i - c_i - \frac{1}{b_i - \beta_i x_i} = \frac{\sum_{i'\in I} \frac{q_{i'}/q_0}{b_{i'} - \beta_{i'} x_{i'}} + \sum_{j\in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j\in J} e^{\bar{u}_j}} .$$

Since the right side of the above is independent of the subscript i, rewrite the above as

$$p_i - c_i - \frac{1}{b_i - \beta_i x_i} = \theta \quad \forall i \in I \tag{22}$$

where

$$\theta = \frac{\sum_{i'\in I} \frac{q_{i'}/q_0}{b_{i'} - \beta_{i'} x_{i'}} + \sum_{j\in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j\in J} e^{\bar{u}_j}} . \tag{23}$$

From equations (5) and (22),

$$q_{i'}/q_0 = e^{x_{i'}+a_{i'}-(b_{i'}-\beta_{i'}x_{i'})p_{i'}} = e^{x_{i'}+a_{i'}-1-(b_{i'}-\beta_{i'}x_{i'})(c_i+\theta)}.$$

Substitute this expression into (23) to obtain

$$\theta = \frac{\sum_{i'\in I}\frac{e^{x_{i'}+a_{i'}-1-(b_{i'}-\beta_{i'}x_{i'})(c_i+\theta)}}{b_{i'}-\beta_{i'}x_{i'}} + \sum_{j\in J}\bar{m}_j e^{\bar{u}_j}}{1+\sum_{j\in J}e^{\bar{u}_j}}.$$

In addition, from (9),

$$\begin{aligned}
\pi^* &= \sum_{i\in I}(\theta + \frac{1}{b_i-\beta_i x_i})q_i + \sum_{j\in J}\bar{m}_j q_j = \sum_{i\in I}(\theta + \frac{1}{b_i-\beta_i x_i})q_i + \sum_{j\in J}\bar{m}_j q_0 e^{\bar{u}_j} \\
&= \theta - \left(1-\sum_{i\in I}q_i\right)\theta + q_0\left(\sum_{i\in I}\frac{1}{b_i-\beta_i x_i}q_i/q_0 + \sum_{j\in J}\bar{m}_j e^{\bar{u}_j}\right) \\
&= \theta - \left(1-\sum_{i\in I}q_i\right)\theta + q_0\left(1+\sum_{j\in J}e^{\bar{u}_j}\right)\theta \\
&= \theta - \left(1-\sum_{i\in I}q_i\right)\theta + \frac{1-\sum_{i\in I}q_i}{1+\sum_{j\in J}e^{\bar{u}_j}}\left(1+\sum_{j\in J}e^{\bar{u}_j}\right)\theta = \theta
\end{aligned}$$

where the second equality follows from (6), the fourth equality follows from (23), and the second-to-last equality follows from (8). □

A.3 Proof of Corollary 2

Proof. Since $b_i = b$ and $\beta_i = 0$, equation (11) reduces to $\theta = \bar{\pi}_J + \frac{\left(\sum_{i\in I}e^{\bar{x}_i+a_i-1-bc_i}/b\right)e^{-b\theta}}{1+\sum_{j\in J}e^{\bar{u}_j}}$, which is equivalent to $b(\theta-\bar{\pi}_J)e^{b(\theta-\bar{\pi}_J)} = \frac{\sum_{i\in I}e^{\bar{x}_i+a_i-1-b(c_i+\bar{\pi}_J)}}{1+\sum_{j\in J}e^{\bar{u}_j}}$. By the definition of the Lambert W function, we have $b(\theta-\bar{\pi}_J) = W\left(\frac{\sum_{i\in I}e^{\bar{x}_i+a_i-1-b(c_i+\bar{\pi}_J)}}{1+\sum_{j\in J}e^{\bar{u}_j}}\right)$, hence $\theta^* = \bar{\pi}_J + \frac{1}{b}W\left(\frac{\sum_{i\in I}e^{\bar{x}_i+a_i-1-b(c_i+\bar{\pi}_J)}}{1+\sum_{j\in J}e^{\bar{u}_j}}\right)$. From equation (10), $p_i^* = c_i + \bar{\pi}_J + \frac{1}{b}\left(1+W\left(\frac{\sum_{i\in I}e^{\bar{x}_i+a_i-1-b(c_i+\bar{\pi}_J)}}{1+\sum_{j\in J}e^{\bar{u}_j}}\right)\right)$. □

A.4 Proof of Theorem 3

Proof. The proof of concavity follows the same approach as in Theorem 1 and we omit the details here. We can derive the first order condition

$$m_i - \frac{1}{b_i-\beta_i x_i} = \left(\sum_{i'\in I}\frac{q_{i'}}{b_{i'}-\beta_{i'}x_{i'}}\right)\frac{1}{1-\sum_{i'\in I}q_{i'}} + \left(\sum_{j\in J}\bar{m}_j e^{\bar{u}_j}\right)\frac{1}{1+\sum_{j\in J}e^{\bar{u}_j}}$$

with the right side independent of i. Let $m_i - \frac{1}{b_i - \beta_i x_i} = \theta$. Then at optimality,

$$\left(1 - \sum_{i \in I} q_i\right) \theta = \left(\sum_{i \in I} \frac{q_i}{b_i - \beta_i x_i}\right) + \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}} \left(1 - \sum_{i \in I} q_i\right) . \tag{24}$$

Since

$$1 - \sum_{i \in I} q_i = \frac{1 + \sum_{j \in J} e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j} + \sum_{i \in I} e^{x_i + a_i - (b_i - \beta_i x_i) c_i - (b_i - \beta_i x_i) m_i}}$$

and $m_i = 1/(b_i - \beta_i x_i) + \theta$, the following holds:

$$\begin{aligned}
\theta &= \frac{\left(\sum_{i \in I} \frac{e^{x_i + a_i - (b_i - \beta_i x_i) c_i - (b_i - \beta_i x_i) m_i} / (b_i - \beta_i x_i)}{1 + \sum_{j \in J} e^{\bar{u}_j} + \sum_{i \in I} e^{x_i + a_i - (b_i - \beta_i x_i) c_i - (b_i - \beta_i x_i) m_i}}\right)}{\left(\frac{1 + \sum_{j \in J} e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j} + \sum_{i \in I} e^{x_i + a_i - (b_i - \beta_i x_i) c_i - (b_i - \beta_i x_i) m_i}}\right)} + \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}} \\
&= \frac{\sum_{i \in I} e^{x_i + a_i - (b_i - \beta_i x_i) c_i - (b_i - \beta_i x_i) m_i} / (b_i - \beta_i x_i)}{1 + \sum_{j \in J} e^{\bar{u}_j}} + \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}} \\
&= \frac{\sum_{i \in I} e^{x_i + a_i - (b_i - \beta_i x_i) c_i - 1 - (b_i - \beta_i x_i) \theta} / (b_i - \beta_i x_i)}{1 + \sum_{j \in J} e^{\bar{u}_j}} + \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}} ,
\end{aligned}$$

which is an easy-to-solve single-variable equation (i.e., the left side increases in θ and, for all feasible $x_i \in [0, x_i^+]$, the right side decreases in θ). □

A.5 Proof of Lemma 1

Proof. For brevity, we omit the argument $\boldsymbol{x}$ in the following derivation.

$$\begin{aligned}
\tilde{\pi} &= \sum_{i \in I} m_i q_i + \sum_{j \in J} \bar{m}_j q_j = \sum_{i \in I} \left(\frac{1}{b_i - \beta_i x_i} + \theta\right) q_i + \sum_{j \in J} \bar{m}_j e^{\bar{u}_j} q_0 \\
&= \sum_{i \in I} \frac{q_i}{b_i - \beta_i x_i} + \theta \sum_{i \in I} q_i + \sum_{j \in J} \bar{m}_j e^{\bar{u}_j} q_0 \\
&= \left(1 - \sum_{i \in I} q_i\right) \theta - \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}} \left(1 - \sum_{i \in I} q_i\right) + \theta \sum_{i \in I} q_i + \sum_{j \in J} \bar{m}_j e^{\bar{u}_j} q_0 \\
&= \theta - \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}} \left(1 - \sum_{i \in I} q_i\right) + \sum_{j \in J} \bar{m}_j e^{\bar{u}_j} q_0 \\
&= \theta - \sum_{j \in J} \bar{m}_j e^{\bar{u}_j} \left(\frac{1 - \sum_{i \in I} q_i}{1 + \sum_{j \in J} e^{\bar{u}_j}} - q_0\right) = \theta
\end{aligned}$$

where the second equality follows from (24) and the last equality is due to (8). □

A.6 Proof of Theorem 4

Proof. Recall that θ solves (12). Let $\boldsymbol{y} = (y_i)_{i\in I}$ and $y_i = x_i^* - b_i c_i(x_i^*)$. Then (12) becomes:

$$\theta = \frac{\sum_{i\in I} e^{y_i+a_i-1-b_i\theta}/b_i}{1+\sum_{j\in J} e^{\bar{u}_j}} + \frac{\sum_{j\in J} \bar{m}_j e^{\bar{u}_j}}{1+\sum_{j\in J} e^{\bar{u}_j}} \ . \tag{25}$$

We implicitly differentiate (12) with respect to y_i to obtain

$$\frac{\partial\theta(\boldsymbol{y})}{\partial y_i} = \frac{1}{1+\sum_{j\in J} e^{\bar{u}_j}} \left[\sum_{i'\in I} \frac{e^{y_{i'}+a_{i'}-1-b_{i'}\theta}}{b_{i'}}(-b_{i'})\frac{\partial\theta(\boldsymbol{y})}{\partial y_i}\right] + \frac{e^{y_i+a_i-1-b_i\theta}/b_i}{1+\sum_{j\in J} e^{\bar{u}_j}},$$

which yields

$$\frac{\partial\theta(\boldsymbol{y})}{\partial y_i} = \frac{e^{y_i+a_i-1-b_i\theta}/b_i}{1+\sum_{j\in J} e^{\bar{u}_j} + \sum_{i'\in I} e^{y_{i'}+a_{i'}-1-b_{i'}\theta}} > 0 \ .$$

Thus $\theta(\boldsymbol{y})$ strictly increases in y_i for all $i \in I$. Therefore, it suffices to maximizes y_i for all $i \in I$. Recall that $y_i = x_i - b_i c_i(x_i)$. Since $c_i(x_i)$ is convex, the optimal quality satisfies

$$c_i'(x_i) = \frac{1}{b_i} \tag{26}$$

if $c_i'^{-1}\big(\frac{1}{b_i}\big) \in [0, x_i^+]$, $x_i = 0$ if $c_i'^{-1}\big(\frac{1}{b_i}\big) < 0$, and $x_i = x_i^+$ if $c_i'^{-1}\big(\frac{1}{b_i}\big) > x_i^+$ for all $i \in I$. Thus optimal x_i^* satisfies (13) and the optimal price p_i^* satisfies (14). ☐

Lemma 2. *Suppose Assumption 2 holds. Then, for any $x_i^* \in \left[0,\ x_i^+\right]$, that satisfies* (17), *the cost function $c_i(\cdot)$ satisfies*

$$(b_i - \beta_i x_i^*)c_i''(x_i^*) > 2\beta_i c_i'(x_i^*) + \left(\frac{\beta_i}{b_i - \beta_i x_i^*}\right)^2 \ .$$

Proof. For any $x_i \in \left[0,\ x_i^+\right]$, that satisfies (17),

$$(b_i - \beta_i x_i^*)c_i'(x_i^*) = 1 + \frac{\beta_i}{b_i - \beta_i x_i^*} + \beta_i(\theta + c_i(x_i^*)) > \frac{\beta_i}{b_i - \beta_i x_i^*}$$

Thus

$$\begin{aligned} &\frac{2\beta_i}{b_i - \beta_i x_i^*} + \left(\frac{\beta_i}{b_i - \beta_i x_i^*}\right)^2 \frac{1}{(b_i - \beta_i x_i^*)c_i'(x_i^*)} \\ &< \frac{2\beta_i}{b_i - \beta_i x_i^*} + \left(\frac{\beta_i}{b_i - \beta_i x_i^*}\right)^2 \left(\frac{b_i - \beta_i x_i^*}{\beta_i}\right) = \frac{2\beta_i}{b_i - \beta_i x_i^*} + \left(\frac{\beta_i}{b_i - \beta_i x_i^*}\right) = \frac{3\beta_i}{b_i - \beta_i x_i^*} \ . \end{aligned}$$

From Assumption 2,

$$\frac{c_i''(x_i^*)}{c_i'(x_i^*)} > \frac{2\beta_i}{b_i - \beta_i x_i^*} + \left(\frac{\beta_i}{b_i - \beta_i x_i^*}\right)^2 \frac{1}{(b_i - \beta_i x_i^*)c_i'(x_i^*)}$$

Since $c_i(x_i)$ is increasing, the above is equivalent to

$$(b_i - \beta_i x_i^*)c_i''(x_i^*) > 2\beta_i c_i'(x_i^*) + \left(\frac{\beta_i}{b_i - \beta_i x_i^*}\right)^2 \ .$$

☐

A.7 Proof of Theorem 5

Proof. Define $f_i(x_i) = (b_i - \beta_i x_i)c_i'(x_i) - \beta_i \left(c_i(x_i) + \frac{1}{b_i - \beta_i x_i}\right)$. Take derivative of $f_i(x_i)$ with respect to x_i,

$$f_i'(x_i) = (b_i - \beta_i x_i)c_i''(x_i) - 2\beta_i c_i'(x_i) - \left(\frac{\beta_i}{b_i - \beta_i x_i}\right)^2 .$$

From Lemma 2, $f_i'(x_i) > 0$ for any $x_i \in \left[0,\ x_i^+\right]$ that satisfies (17). Thus $f_i(x_i)$ is monotonic in x_i. For any given θ, the solution to (17) is unique, denoted with $\boldsymbol{x}(\theta)$. It is possible that for certain θ values, equation (17) does not have a solution. Since $\theta(\boldsymbol{x})$ is unimodal in x_i (note that under Assumption 2, for all x_i that satisfies (17), $\frac{\partial^2\theta}{\partial x_i^2} < 0$.), this could occur in one of the following two cases: (i) If $1-(b_i)c_i'(0)+\beta_i c_i(0)+\beta_i\theta+\frac{\beta_i}{b_i} < 0$, then it is easy to see from (15) that $\frac{\partial\theta(\boldsymbol{x})}{\partial x_i} < 0$, hence we set x_i as small as possible, i.e., let $x_i = 0$. (ii) If $1-(b_i-\beta_i x_i^+)c_i'(x_i^+)+\beta_i c_i(x_i^+)+\beta_i\theta+\frac{\beta_i}{b_i-\beta_i x_i^+} > 0$, then $\frac{\partial\theta(\boldsymbol{x})}{\partial x_i} > 0$, in which case the optimal solution is to set x_i as large as possible, i.e., at x_i^+. From (12), the optimal profit is given by the following fixed-point condition

$$\theta = \frac{\sum_{i\in I} e^{x_i(\theta)+a_i-(b_i-\beta_i x_i(\theta))c_i(x_i(\theta))-1-(b_i-\beta_i x_i(\theta))\theta}/(b_i-\beta_i x_i(\theta))}{1+\sum_{j\in J} e^{\bar{u}_j}} + \frac{\sum_{j\in J}\bar{m}_j e^{\bar{u}_j}}{1+\sum_{j\in J} e^{\bar{u}_j}} .$$

From Theorem 4, we obtain the corresponding optimal quality and price values in (20) and (21). □

A.8 Proof of Corollary 5

Proof. Since $b_i = b$, $\beta_i = \beta$, equation (21) becomes $m_i^* = \frac{1}{b-\beta x_i^*} + \theta^*$. Hence, $m_i^* \geq m_{i'}^*$ if and only if $x_i^* \geq x_{i'}^*$. Therefore, it suffices to show that $x_i^* \geq x_{i'}^*$ if and only if $a_i \geq a_{i'}$ for any $i, i' \in I$. Since $c_i(x_i) = c_a(a_i) + c_x(x_i)$, equation (17) becomes

$$(b - \beta x_i)c_x'(x_i) - \beta\left(c_a(a_i) + c_x(x_i) + \frac{1}{b-\beta x_i}\right) = 1 + \beta\theta \quad \text{for all } i \in I,$$

which is equivalent to

$$(b - \beta x_i)c_x'(x_i) - \beta\left(c_x(x_i) + \frac{1}{b-\beta x_i}\right) = 1 + \beta\theta + \beta c_a(a_i) \quad \text{for all } i \in I\ .$$

From Assumption 2, the left side of the above is monotonically increasing in x_i, thus the solution to this equation is non-decreasing in a_i. Therefore, $x_i^* \geq x_{i'}^*$ if and only if $a_i \geq a_{i'}$ for any $i, i' \in I$. □

A.9 Proof of Corollary 6

Proof. (i) We note from equation (21) that, if $b_i = b_{i'}, \beta_i = \beta_{i'}$ and $c_i(\cdot) = c_{i'}(\cdot)$, then $x_i^* > x_{i'}^*$ if and only if $p_i^* > p_{i'}^*$. (ii) From equation (17),

$$(b_i - \beta_i x_i^*)c_i'(x_i^*) - \beta_i\left(c_i(x_i^*) + \frac{1}{b_i-\beta_i x_i^*}\right) = 1 + \beta_i\theta^* \quad \text{for all } i \in I\ .$$

The left side is increasing in x_i^* (by Assumption 2) and in b_i. Therefore, if $\beta_i = \beta_{i'}$ and $c_i(\cdot) = c_{i'}(\cdot)$, then $x_i^* > x_{i'}^*$ if and only if $b_i < b_{i'}$. $\square$

A.10 Proof of Corollary 7

Proof. From Theorem 5, $\theta^* \geq \bar{\pi}_J$ and that the optimal solution satisfies

$$\theta^* + c_i(x_i^*) = \frac{b_i - \beta_i x_i^*}{\beta_i} c_i'(x_i^*) - \left(\frac{1}{\beta_i} + \frac{1}{b_i - \beta_i x_i^*}\right) \tag{27}$$

or $x_i^* = 0$ if $\theta^* + c_i(0) < \frac{b_i}{\beta_i} c_i'(0) - \left(\frac{1}{\beta_i} + \frac{1}{b_i}\right)$, $x_i^* = x_i^+$ if $\theta^* + c_i(x_i^+) > \frac{b_i - \beta_i x_i^+}{\beta_i} c_i'(x_i^+) - \left(\frac{1}{\beta_i} + \frac{1}{b_i - \beta_i x_i^+}\right)$. Let $I_0 \in I$ be the subset of products for which $x_i^* = 0$ and $\bar{I}_0 \in I$ be the subset for which $x_i^* > 0$.

Substitute the above into (19),

$$\begin{aligned}
&\left(1 + \sum_{j \in J} e^{\bar{u}_j}\right)\left(\theta^* - \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}}\right) \\
\leq\ & \sum_{i \in \bar{I}_0} e^{x_i^* + a_i - 1 - (b_i - \beta_i x_i^*)\left[\frac{b_i - \beta_i x_i^*}{\beta_i} c_i'(x_i^*) - \left(\frac{1}{\beta_i} + \frac{1}{b_i - \beta_i x_i^*}\right)\right]} / (b_i - \beta_i x_i^*) + \sum_{i \in I_0} e^{a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)} / b_i \\
=\ & \sum_{i \in \bar{I}_0} e^{a_i + \frac{b_i}{\beta_i} - \frac{(b_i - \beta_i x_i^*)^2}{\beta_i} c_i'(x_i^*)} / (b_i - \beta_i x_i^*) + \sum_{i \in I_0} e^{a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)} / b_i \\
=\ & \sum_{i \in \bar{I}_0} \exp\left(a_i + \frac{b_i}{\beta_i}\right) \exp\left(-\log(b_i - \beta_i x_i^*) - \frac{(b_i - \beta_i x_i^*)^2}{\beta_i} c_i'(x_i^*)\right) + \sum_{i \in I_0} e^{a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)} / b_i\ .
\end{aligned}$$

It can be shown that under Assumption 2, the term $-\log(b_i - \beta_i x_i^*) - \frac{(b_i - \beta_i x_i^*)^2}{\beta_i} c_i'(x_i^*)$ is strictly decreasing in x_i^* (due to Lemma 2). Therefore,

$$\begin{aligned}
&\left(1 + \sum_{j \in J} e^{\bar{u}_j}\right)\left(\theta^* - \frac{\sum_{j \in J} \bar{m}_j e^{\bar{u}_j}}{1 + \sum_{j \in J} e^{\bar{u}_j}}\right) \\
\leq\ & \sum_{i \in \bar{I}_0} \exp\left(a_i + \frac{b_i}{\beta_i}\right) \exp\left(-\log(b_i) - \frac{b_i^2}{\beta_i} c_i'(0)\right) + \sum_{i \in I_0} e^{a_i - 1 - b_i(c_i(0) + \bar{\pi}_J} / b_i \\
=\ & \sum_{i \in \bar{I}_0} \frac{1}{b_i} \exp\left(a_i + \frac{b_i}{\beta_i} - \frac{b_i^2}{\beta_i} c_i'(0)\right) + \sum_{i \in I_0} e^{a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)} / b_i \\
\leq\ & \sum_{i \in I} \frac{1}{b_i} \exp\left(\max\left\{a_i + \frac{b_i}{\beta_i} - \frac{b_i^2}{\beta_i} c_i'(0),\ a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)\right\}\right).
\end{aligned}$$

Equivalently,

$$\theta^* \leq \frac{\sum_{i \in I} \frac{1}{b_i} \exp\left(\max\left\{a_i + \frac{b_i}{\beta_i} - \frac{b_i^2}{\beta_i} c_i'(0),\ a_i - 1 - b_i(c_i(0) + \bar{\pi}_J)\right\}\right)}{1 + \sum_{j \in J} e^{\bar{u}_j}} + \bar{\pi}_J$$

$\square$

A.11 Proof of Theorem 6

Proof.

$$
\begin{aligned}
&\left(1+\sum_{j\in J} e^{\bar{u}_j}\right) g'(\theta) \\
&= \sum_i \frac{e^{x_i(\theta)+a_i-(b_i-\beta_i x_i(\theta))c_i(x_i(\theta))-1-(b_i-\beta_i x_i(\theta))\theta}}{b_i-\beta_i x_i(\theta)} \\
&\quad \cdot \left[x_i'(\theta) - (b_i-\beta_i x_i(\theta))(1+c_i'(x_i(\theta))x_i'(\theta) + \beta_i x_i'(\theta)\left(c_i(x_i(\theta))+\theta+\frac{1}{b_i-\beta_i x_i(\theta)}\right)\right] \\
&= -\sum_i e^{x_i(\theta)+a_i-(b_i-\beta_i x_i(\theta))c_i(x_i(\theta))-1-(b_i-\beta_i x_i(\theta))\theta} + \sum_i \frac{e^{x_i(\theta)+a_i-(b_i-\beta_i x_i(\theta))c_i(x_i(\theta))-1-(b_i-\beta_i x_i(\theta))\theta}}{b_i-\beta_i x_i(\theta)} \\
&\quad \cdot x_i'(\theta)\left[1-(b_i-\beta_i x_i(\theta))c_i'(x_i(\theta)+\beta_i\left(c_i(x_i(\theta))+\theta+\frac{1}{b_i-\beta_i x_i(\theta)}\right)\right] \ .
\end{aligned}
$$

From (17), we have either

$$
(b_i-\beta_i x_i(\theta))c_i'(x_i(\theta)) - \beta_i\left(c_i(x_i(\theta))+\theta+\frac{1}{b_i-\beta_i x_i(\theta)}\right) = 1 \ .
$$

or $x_i'(\theta)$=0 (in the case $x_i(\theta)=0$ or x_i^+). Thus

$$
\left(1+\sum_{j\in J} e^{\bar{u}_j}\right) g'(\theta) = -\sum_i e^{x_i(\theta)+a_i-(b_i-\beta_i x_i(\theta))c_i(x_i(\theta))-1-(b_i-\beta_i x_i(\theta))\theta} < 0 \ .
$$

Therefore, $g(\theta)$ strictly decreases in θ. Thus, the solution to (19) is unique. $\square$