Pricing Multiple Products with the Multinomial Logit and Nested Logit Models: Concavity and Implications

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We consider the problem of pricing multiple differentiated products with the Nested Logit model and, as a special case, the Multinomial Logit model. We prove that concavity of the total profit function with respect to market share holds even when price sensitivity may vary with products. We use this result to analytically compare the optimal monopoly solution to oligopolistic equilibrium solutions. To demonstrate further applications of the concavity result, we consider several multi-period dynamic models that incorporate the pricing of multiple products in the context of inventory control and revenue management, and establish structural results of the optimal policies.

Keywords: *Multinomial Logit Model, Nested Logit Model, Consumer Choice, Multi-Product Pricing, Price Competition with Differentiated Products, Quantity Competition with Differentiated Products*

1 Introduction and Literature Review

Since Luce (1959) first proposed the logit model for consumer choice half a century ago, it has received much attention from researchers in several fields including marketing, economics and operations management, and it has generated a large body of literature for both theoretical models and empirical validations in a wide range of applications. In this paper, we consider the pricing problem of a firm offering a set of differentiated products, and study the total profit under the logit-based framework. One of the main contributions of this paper is to establish the joint concavity of total profit when the customer choice is modeled using the Nested Logit model, which includes the standard Multinomial Logit (MNL) model as a special case. Our contribution also includes the characterization of the difference between monopoly and oligopoly outcomes. We further show how the concavity result may be applied to efficiently solve dynamic problems in inventory control and revenue management.

The Standard MNL Model and the Nested Logit Model with Pricing. To motivate the Nested Logit model, we first discuss the more commonly known standard MNL model. The MNL model describes the decision process of a customer who makes a selection from an array of products. This model has been used in marketing and econometric applications for predicting consumer choices (McFadden, 1986), for analyzing aggregate market shares (Berry, 1994), and for estimating price sensitivities in several industries (Guadagni and Little, 1983; Train et al., 1987; Greene, 1991). The often-cited book by Anderson et al. (1992) provides a detailed presentation of the MNL model based on consumer choice theory and demonstrates its desirable properties.

In the MNL model with $M \geq 2$ products, the representative customer associates a utility u_i with product $i \in \{1, \ldots, M\}$ defined by $u_i = \alpha_i + \epsilon_i$, where α_i relates to a measure of attractiveness for product i based on known attributes such as quality and price, and ϵ_i is a random term representing unobserved utility. Under the condition that the random terms $\{\epsilon_i\}$ are IID with Gumbel distribution, it is well known (Luce, 1959; McFadden, 1974) that the customer selects product i with probability

$$\frac{\exp \alpha_i}{1 + \sum_{j=1}^{M} \exp \alpha_j} \quad . \tag{1}$$

(The probability that the customer decides to purchase none of the products is $1/[1 + \sum_{j=1}^{M} \exp \alpha_j]$.) By aggregating customer decisions, we obtain (1) as the proportion of total potential demand captured by product i. That is, it is the expected sales quantity of product i normalized by the total number of potential customers, including those who purchase nothing. For brevity, we refer to it as the *market share* of product i in the remainder of the paper.

One of the criticisms for the standard MNL model, which serves as a motivation for the nested model, is the *Independence-from-Irrelevant-Alternatives* (IIA) property – that the ratio of the purchase probabilities for any two alternatives is independent of the presence of other alternatives. This is often explained with the famous "red bus/blue bus" example (see Train (2003) for a detailed explanation). The IIA property not only limits empirical applications of the standard MNL model, but also weakens the conclusions and insights of model-based papers obtained under the premise of standard MNL. For example, Chen and Hausman (2000) recognize that their analysis of the product line selection and pricing problem cannot be applied if the IIA property does not hold. To address this drawback, McFadden (1978, 1980) have generalized the MNL model of Luce (1959) to a Generalized Extreme Value (GEV) model with a nested choice structure, which is also known as the *Nested Logit model*. Under this model, a consumer first chooses a group of products among several possible groups, and then limits her subsequent selection within the chosen group.The Nested logit model is found empirically to be more widely applicable than a standard MNL model (see for example, Dubin (1986), Kannan and Wright (1991), Bhat (1995), and Goldberg (1995)). However, the use of the Nested Logit model has primarily been limited to the context of descriptive representation rather than prescriptive optimization.

In the pricing and revenue management literature, the Nested Logit model with pricing as a lever has not been used, to our knowledge. Instead, the price-dependent MNL model has been one of the standard methods for modeling consumer behavior and optimizing prices for a firm offering multiple products (for example, Dong et al. (2009) and Song and Xue (2007)). While our paper fits into the larger context of the multi-product pricing and revenue management literature, we deviate from prior methodological treatments (for example, Gallego and van Ryzin (1997), and Maglaras and Meissner (2006)), which typically focus on mathematical reduction or heuristics. Instead we focus on pricing solutions for a particular family of choice models, the Nested Logit models, and explore their theoretical properties in detail. We note one particular paper that studies multi-product pricing using a nested structure in the demand model, which has characteristics similar to the Nested Logit model. Bitran et al. (2010) consider a demand model where the customers first make a decision among "subfamily" groups which are differentiated by quality (i.e., a vertically differentiated assortment), and then choose at the product level, which is a horizontally differentiated assortment. Although the second level customer decision is modeled as a MNL model, the top level customer decision is, however, more aligned with the "reservation price" interpretation: The customers have fixed valuations for the top-level product groups, but are segmented based on budget and non-purchasing utility. They formulate the problem as a stochastic control problem, allowing consumer substitution based on both price and inventory availability, and show that the optimal prices follow the same ordering as the utility levels of the product choices.

Profit Maximization Objective. The objective function of our interest is the total profit, which is given by the difference between revenue and cost. The most common way of incorporating prices into the MNL model in this literature stream and previous empirical applications of the MNL model is to adopt a demand model in which the attractiveness is a linear function of price. Assuming a known cost function, the total profit is then a function of the price vector. It turns out that this profit function is not concave in prices. The results of this paper apply equally well to revenue management settings in which marginal costs are negligible relative to fixed costs, and the objective is maximization of expected revenue (In the literature, such profit maximization has not been considered in the Nested Logit framework, and we are aware of only those in the standard MNL setting). Hanson and Martin (1996) are the first to show that the logit profit function is not concave in prices. They identify a path-following approach for finding a globally optimal solution. This method was the first viable approach for circumventing the difficulty of non-concavity. Other approaches presented by subsequent papers in the literature consider the revenue maximization problem as part of a more complex problem such as product line selection, inventory control or revenue management. Chen and Hausman (2000) study a product line selection problem under the MNL model, restricting price to one of several discrete price points. They formulate an integer program and show that the linear program relaxation has a quasi-concave revenue function. When the prices are not restricted, Aydin and Porteus (2008) establish that the optimal price vector satisfying the first-order condition is unique. Several other papers use the first-order condition to find optimal prices including Aydin and Ryan (2000), Hopp and Xu (2005), Akçay et al. (2010), and Maddah and Bish (2007). While the objective function is not concave in the price vector, it turns out that it is concave with respect to the market share vector, which is a one-to-one transformation of the price vector. This result is established by Dong et al. (2009) and Song and Xue (2007), for the standard MNL model. In this paper, we show the concavity property in more generalized settings, characterized by the Nested Logit model.

Note first that incorporating the *nested* model into the optimization framework has been perceived to be important but difficult:

> The environments where product dissimilarity is asymmetric . . . can be modeled by a nested MNL model. However, the nested MNL model introduces substantially complex expressions and obscures the fundamental underlying driving forces of the optimal price behavior . . . (Dong et al., 2009)

However, showing the concavity of the total profit is one of the key contributions of this paper. Furthermore, we derive a single dimension search solution for the optimal prices and market shares under the Nested Logit model (Theorem 1 and Theorem 2).

Second, we remark that our result provides an important generalization even in the non-nested setting. The concavity result of Dong et al. (2009) and Song and Xue (2007) was established for the standard MNL model under the condition that the price sensitivity parameter is uniform across all products; in all of the papers mentioned above, with the exception of Hanson and Martin (1996) and Aydin and Porteus (2008), the price-dependent MNL models assume *identical price sensitivity parameters*, i.e., $b_i = b$ for all i's, and that in most cases the value of b is restricted to 1. Fixing the value of this parameter at 1 limits the applicability of the models. Empirical fittings of the MNL have shown that the value of b varies widely (see Berry et al. (1995) and Nevo (2001)). Furthermore, identical price sensitivity implies that the impact of each price on the market share vector is the same across all products. Several empirical papers recognize the importance of allowing different price sensitivity parameters in the MNL model (for example, Erdem et al. (2002) and Luo et al. (2007)).

In this paper, we use a different proof approach (Theorem 1) from Dong et al. (2009) and Song and Xue (2007) and generalize the concavity result to the nested choice structure, as well as to general asymmetric price sensitivity parameters, broadening the applicability of the choice model in pricing problems.

Monopoly and Oligopoly Solutions. While our preceding discussion was on the maximization of total profit as a monopolist's problem, there are a number of papers in the literature taking a game-theoretical decentralized approach. Many pricing papers on product line selection consider the assortment decisions under competition (Anderson and de Palma, 1992; Besanko et al., 1998, 2003; Hopp and Xu, 2008; Cachon and Kok, 2007; Cachon et al., 2008). The theoretical focus in these papers is often to establish the uniqueness of the Nash equilibrium and study the impact of competition on equilibrium assortment decisions including price. For example, Anderson and de Palma (1992) show that when all products have equal quality, the equilibrium prices are an equal mark-up over the product cost. Cachon and Kok (2007) study retail assortment competition using a Nested Logit demand model and show how prices and variety levels are affected under various duopoly competition scenarios. Cachon et al. (2008) consider how the consumer search cost affects assortment competition using a standard MNL model.

In this paper, in addition to solving a centralized optimal pricing problem, we derive oligopoly equilibrium prices and market shares when the products are owned by separate revenue-maximizing firms. We consider both quantity competition (Cournot) and price competition (Bertrand) equilibria. In the quantity competition, each firm determines its own production quantities, and equilibrium prices are given by the inverse demand function of the Nested Logit model. In the price competition, each firm determines its own price(s) and demand is given by the Nested Logit model. Farahat and Perakis (2008) consider both price-competition and quantity-competition for multiple differentiated products where each product is sold by a separate company. They study the non-nested MNL model. The usefulness of studying these equilibrium outcomes, as Farahat and Perakis (2008) point out, is that the Cournot and Bertrand outcomes are equivalent to the equilibrium solutions of more prevalent two-stage scenarios where firms make price and quantity decisions sequentially (Friedman, 1988): either a make-to-stock situation where the production decisions precede the pricing decisions (Cournot), or a make-to-order situation where the pricing decisions precede the production decisions (Bertrand). In comparison with Farahat and Perakis (2008), we consider oligopolies under the more general Nested logit model and obtain comparisons of the centralized optimal pricing solution with oligopoly equilibria.

To our knowledge, we are the first to characterize the quantity competition equilibrium price and market share solutions in a "closed" form in the Nested Logit model (Theorem 3). This is achieved by employing the Lambert W function. Further, for the price-competition model, we characterize the equilibrium price and market share solutions by defining a modified version of the Lambert W function. We compare these two oligopoly solutions with the centralized optimal pricing solution (which we also refer to as the "monopoly" solution since all products are owned by a single firm), and show how the price and the market share for each product change from the centralized solution to the oligopoly solutions (Corollary 2 and Corollary 3). We find that, although competition drives up the total market share (in the sense that more consumers choose to buy one of the products instead of not purchasing) and drives down the prices, the market share and revenue for a particular product may increase or decrease due to competition. Specifically, the direction of change has a predictable pattern based on an ordering of product-specific parameters, namely the product quality and the price-sensitivity parameter. The findings of this paper provide a general framework for studying the optimal pricing and assortment decisions either with or without competition even when a nested structure exists in the customer choice model.

Applications of the Concavity Result. Furthermore, we explore the implications of the concavity result in a number of multi-period dynamic models arising in inventory control and revenue management. For assemble-to-order products with a single critical component and a fixed ordering cost, we show that the optimal replenishment policy for the component in this joint pricing-inventory model is the well-known (s, S) policy. The (s, S) policy has proven to be optimal in several single-product inventory control settings (e.g., Scarf (1959) and Chen and Simchi-Levi (2004)), and this paper generalizes the applicability of the (s, S) policy with multiple products under the logit models. For an inventory model that maintains the inventory of finished products, Song and Xue (2007) have shown structural results of the optimal policy when the fixed ordering cost is absent, under a number of demand models including the standard MNL model with identical price sensitivity parameters. We show that the Song and Xue (2007) results are applicable to the general MNL model (with asymmetric price sensitivity parameters) and the Nested Logit choice model. In addition, Dong et al. (2009) have studied dynamic pricing of multiple products without replenishment and have shown the property of equal mark-up across all products in any given period. We find that the equal mark-up property no longer holds in more general settings (such as the Nested Logit model), and the optimal mark-up may depend on the price sensitivity of products and dissimilarity indices of product groups. These examples show that the concavity result allows tractable analysis and optimization of dynamic systems.

The remainder of this paper is organized as follows. In Section 2, we study the profit maximization problem under the Nested Logit model, and present the concavity property of profit, a computationally efficient method for finding the optimal solution, and a comparison of the optimal solution and the equilibrium outcome. The applications of the concavity result to multi-period inventory control and revenue management problems are shown in Section 3, and we conclude in Section 4.

2 Analysis: Nested Logit Model

We focus the analysis of this section to the more general Nested Logit model. All the results in this section are also applicable to the special case of the standard MNL model, allowing for non-identical price sensitivity parameters.

We adopt the general Nested Logit model defined in Greene (2003). Suppose that there are a total of M alternatives (or products), which can be divided into $K \geq 1$ groups (also known as *branches*). Let M_k denote the number of alternatives that belong to branch k.

Thus, $\sum_{k=1}^{K} M_k = M$. Let Q_k denote the probability that the customers select branch k among K possible branches, and let $q_{j|k}$ denote the probability for product j conditional on branch k. We use $a_{jk} - b_k p_{jk}$ to represent the *attractiveness* of product j in group k, which is a decreasing function of the price p_{jk}. Here, the price-independent component (quality) is specific to each product and given by a_{jk} while the price-sensitive component is dependent on the group to which the product belongs and is denoted by b_k. This is a reasonable modeling assumption since the set of products that belong to the same group are likely to have similar attributes. Let p_{jk} and q_{jk} denote the price and market share of product j within group k, respectively. We define the conditional market share of product j within group k, and the market share of group k as the following:

$$q_{j|k} \;=\; \frac{e^{a_{jk}-b_k p_{jk}}}{\sum_{\ell=1}^{M_k} e^{a_{\ell k}-b_k p_{\ell k}}} \qquad \text{and} \qquad Q_k \;=\; \frac{e^{\tau_k I_k}}{1+\sum_{\ell=1}^{K} e^{\tau_\ell I_\ell}}\ , \tag{2}$$

where $I_k = \log \sum_{j=1}^{M_k} e^{a_{jk}-b_k p_{jk}}$ represents the aggregate attractiveness of branch k, and $\tau_k \in [0,1]$ represents an index for the degree of *dissimilarity* among products in branch k. Then,

$$q_{jk} \;=\; q_{j|k} \cdot Q_k \qquad \text{and} \qquad \sum_{j=1}^{M_k} q_{jk} = Q_k\ . \tag{3}$$

We remark that this model is a generalization of the *standard* MNL model, which corresponds to the case where each group contains exactly a single alternative, i.e., $M_k = 1$ and $\tau_k = 1$ for all k's. (When referring to the standard MNL model, we suppress the first subscript provided that there is no ambiguity.)

In this section, we first show that the problem of maximizing the total profit is concave with respect to the vector of market shares which can be thought of as a one-to-one transformation of the price vector (Section 2.1). Furthermore, we present a computationally efficient method for finding the optimal solution for the profit maximization problem, which can be accomplished by searching a single dimensional space regardless of the number of products (Section 2.2). We then compare the optimal solution to the outcome of oligopoly models in which each product branch is managed by a separate firm or manager, and conclude that competition decreases the price of each product and increases the total demand (Section 2.3).

2.1 Concavity of Total Profit

The objective of Section 2.1 is to show that the total profit, i.e., the difference between the total revenue and the total cost, is a concave function of the market share vector. We will first express the total revenue in terms of the market share vector $\mathbf{q} = (q_{11}, q_{21}, \ldots, q_{M_1 1}, \ldots, q_{1K}, q_{2K}, \ldots, q_{M_K K})$, where q_{jk} represents the market share of product j in group k. We model the cost based on c_{jk}, the per-unit cost of product j in group k. Then, we prove the concavity of total profit in terms of $\mathbf{q}$.

Given a price vector $\mathbf{p} = (p_{11}, p_{21}, \ldots, p_{M_1 1}, \ldots, p_{1K}, p_{2K}, \ldots, p_{M_K K})$, the Nested Logit model determines market share vector $\mathbf{q}$ according to (2) and (3). Conversely, we can construct a mapping from market share vector $\mathbf{q}$ to price vector $\mathbf{p}(\mathbf{q})$, where

$$p_{jk}(\mathbf{q}) = \frac{a_{jk}}{b_k} + \frac{1}{b_k}\left[\log(1 - \sum_{\ell=1}^{K} Q_\ell) - \log q_{jk}\right] + \frac{1}{b_k}\frac{1-\tau_k}{\tau_k}\left[\log(1 - \sum_{\ell=1}^{K} Q_\ell) - \log Q_k\right], \quad (4)$$

and $Q_\ell = \sum_{j=1}^{M_\ell} q_{j\ell}$. (For the proof of (4), see the appendix.)

Now, we can express the total revenue also in terms of $\mathbf{q}$:

$$R(\mathbf{q}) = \sum_{k=1}^{K}\sum_{j=1}^{M_k} p_{jk}(\mathbf{q}) q_{jk} . \quad (5)$$

Furthermore, the total profit is given by $\Gamma(\mathbf{q}) = R(\mathbf{q}) - C(\mathbf{q})$, where the total cost is $C(\mathbf{q}) = \sum_{k=1}^{K}\sum_{j=1}^{M_k} c_{jk} q_{jk}$. Establishing the concavity property of total revenue $R(\mathbf{q})$ and total profit $\Gamma(\mathbf{q})$ with respect to $\mathbf{q}$ (Theorem 1) is one of the main results of this section.

One standard approach to show the concavity of $R(\mathbf{q})$ under the *standard* MNL model is to compute the Hessian matrix and show its negative semi-definiteness. This is the approach taken by both Song and Xue (2007) and Dong et al. (2009). However, their results are limited to the case of identical price sensitivity parameters, which we further explain below. Under the standard MNL model (i.e., $M_k = 1$ and $\tau_k = 1$ for each k), Song and Xue (2007) rewrite the total revenue function in (5) as

$$R(\mathbf{q}) = \sum_{i=1}^{M} \frac{a_i}{b_i} \cdot q_i - \left(1 - \sum_{i=1}^{M} \frac{q_i}{b_i}\right) \log\left[1 - \sum_{j=1}^{M} q_j\right] + \log\left[1 - \sum_{j=1}^{M} q_j\right] - \sum_{i=1}^{M} \frac{q_i}{b_i} \cdot \log q_i ,$$

and consider each of the four terms on the right-hand side separately. It is straightforward to show the concavity of these terms except for the second term. Suppose $b_i = b$ for all i's.

Then, the first derivative of the second term is given by

$$\frac{\partial}{\partial q_j}\left\{-\left(1-\sum_{i=1}^{M}\frac{q_i}{b}\right)\log\left[1-\sum_{i=1}^{M}q_i\right]\right\} \;=\; \frac{1-\sum_{i=1}^{M}q_i/b}{1-\sum_{i=1}^{M}q_i}+\frac{1}{b}\log\left[1-\sum_{i=1}^{M}q_i\right]\;,$$

and the Hessian matrix of the second term is given by

$$\left\{\frac{-1}{1-\sum_{\ell=1}^{M}q_\ell}\right\}\cdot\left(\frac{2}{b}-\frac{1-\sum_{\ell=1}^{M}q_\ell/b}{1-\sum_{\ell=1}^{M}q_\ell}\right)\cdot[\mathbf{1}]$$

where $[\mathbf{1}]$ is the unit matrix with all elements equal to 1. Since the first scalar factor (in the curly bracket) is non-positive and $[\mathbf{1}]$ is positive semi-definite, the negative semi-definiteness of the Hessian matrix depends on the sign of the second scalar factor (within the round bracket). In Song and Xue (2007) where $b = 1$, this factor becomes 1 and thus the Hessian matrix is negative semi-definite. This result can be extended if $0 < b \leq 1$; however, the above Hessian matrix is not negative semi-definite if $b \geq 2$. Thus their approach cannot be generalized for all values of b's, let alone the non-identical price sensitivity parameter case. Dong et al. (2009) consider the $(M + 1)$-dimensional function where an additional dimension is introduced by $q_0 = 1 - \sum_{i=1}^{M} q_i$ and verify the negative semi-definiteness of the Hessian matrix using the first principle. They consider the case of identical price sensitivity parameters only and it is not straightforward how one could generalize the proof given in their paper.

We now discuss other approaches based on the first-order condition on prices. Akçay et al. (2010) use the Lambert W function to solve the first-order-condition price in the MNL model with identical price sensitivity parameters and show that the solution is unique and that the revenue function is unimodal. Their approach does not extend to the Nested Logit model or the MNL model with product-specific price sensitivity, because as we illustrate in our paper, the optimal price is not easily reduced to a form of the Lambert W function and the line of proof in Akçay et al. (2010) does not carry through. In Farahat and Perakis (2008), each firm maximizes its own revenue in the oligopoly, and thus their analysis can be limited to the first order derivative of each firm's revenue with respect to its *own* price and they show that the first order derivative yields a unique Nash equilibrium. This does not easily generalize to the concavity of *total* monopoly revenue or profit with respect to the *market share vector*.

In Theorem 1 below, we present a different approach to establish the concavity of $R(\mathbf{q})$ and $\Gamma(\mathbf{q})$ which we believe is simpler than those found in the literature.

Theorem 1. *In the Nested Logit model, $R(\mathbf{q})$ and $\Gamma(\mathbf{q})$ are jointly concave in $\mathbf{q}$.*

Proof. From (4) and (5),

$$\begin{aligned} R(\mathbf{q}) &= \sum_{k=1}^{K}\sum_{j=1}^{M_k} \frac{a_{jk}}{b_k} \cdot q_{jk} + \sum_{k=1}^{K}\sum_{j=1}^{M_k} \frac{1}{b_k} \cdot q_{jk} \left[\log(1 - \sum_{\ell} Q_\ell) - \log q_{jk}\right] \\ &\quad + \sum_{k=1}^{K}\sum_{j=1}^{M_k} \frac{1}{b_k}\frac{1-\tau_k}{\tau_k} \cdot q_{jk} \left[\log(1 - \sum_{\ell} Q_\ell) - \log Q_k\right] . \end{aligned}$$

Consider the right side of the above expression. The first term is linear in $\mathbf{q}$. The concavity of the second term can easily be established (using Lemma 2 in the appendix and the preservation of concavity in Boyd and Vandenberghe (2004)). Finally, the third term can be expressed as

$$\sum_{k=1}^{K} \frac{1}{b_k}\frac{1-\tau_k}{\tau_k} \cdot Q_k \left[\log(1 - \sum_{\ell} Q_\ell) - \log Q_k\right] ,$$

which is also concave in $\mathbf{q}$ by a similar argument.

The concavity of $\Gamma(\mathbf{q}) = R(\mathbf{q}) - C(\mathbf{q})$ now follows from the linearity of $C(\mathbf{q})$. □

We remark that the price sensitivity parameter b_k is based on the group, not on each product in a group. Without this assumption – if the price sensitivity parameter depends on each product within a group – then showing the concavity of $R(\mathbf{q})$ would have required to prove the concavity of $q_{jk}\left[\log(1 - \sum_\ell Q_\ell) - \log Q_k\right]$ given in the proof of Theorem 1. However, it can be verified that this is not concave. Hence, our modeling assumption on the price sensitivity parameter is necessary for the proof of Theorem 1. Since multiple products belong to the same group based on similarity, it is reasonable to assume that all the products within a given group have the same price sensitivity parameter.

2.2 Maximization of Total Profit

An immediate implication of Theorem 1 is that we can use the first-order condition to identify the optimal solution of the profit maximization problem. Finding the optimal solution for a problem such as this one typically requires performing descent steps in M-dimensional space, where M is the number of products. However, we show in this section how to find the optimal solution and the optimal profit based on a *single* dimensional search, regardless of the number of products.

Before doing this, it is convenient to define the following quantity which represents an aggregate measure of branch k's overall quality, which depends on $(a_{1k}, \ldots, a_{M_k k})$:

$$A_k \;=\; e^{-1}\left[\sum_{j=1}^{M_k} e^{a_{jk}}\right]^{\tau_k} . \tag{6}$$

We comment that the *aggregate quality* A_k is higher if the quality of individual products in the group denoted by a_{jk} is strong, and firm k offers a wide range of products (high dissimilarity index τ_k). In addition, we note that, compared to the *aggregate attractiveness* I_k, A_k is independent of prices. We also define a *cost-adjusted aggregate quality* $\overline{A}_k$ which signifies the attractiveness of branch k if the customers are only charged the cost of a product (without any mark-up):

$$\overline{A}_k \;=\; e^{-1}\left[\sum_{j=1}^{M_k} e^{a_{jk}-b_k c_{jk}}\right]^{\tau_k} . \tag{7}$$

Note that $\overline{A}_k$ is the same as A_k if each cost c_{jk} is zero for any j and k. This quantity will prove useful in stating and proving the analytical results of this paper.

Let $\mathbf{q}^*$ and $\mathbf{p}^*$ denote the optimal market share vector and the optimal price vector, respectively. Let ρ^* denote the optimal expected profit, which we recall is the profit after normalizing the total potential demand to 1. (In other words, ρ^* is the expected profit scaled down by a constant that corresponds to total potential demand.)

Theorem 2. *In the Nested Logit model, the optimal expected profit ρ^* is the unique value of ρ satisfying*

$$\rho \;=\; \sum_{k=1}^{K} \frac{\overline{A}_k e^{-b_k\tau_k\rho}}{b_k\tau_k} . \tag{8}$$

Furthermore,

$$p^*_{jk} \;=\; \rho^* + \frac{1}{b_k\tau_k} + c_{jk} \qquad \textit{and} \qquad q^*_{jk} \;=\; \frac{\overline{A}_k e^{-b_k\tau_k\rho^*}}{1+\sum_{\ell=1}^{K}\overline{A}_\ell e^{-b_\ell\tau_\ell\rho^*}} \cdot \frac{e^{a_{jk}-b_k c_{jk}}}{\sum_{i=1}^{M_k} e^{a_{ik}-b_k c_{ik}}} .$$

Theorem 2 is useful in computationally finding the optimal solution because it reduces the multi-product pricing problem to a single-variable root-finding problem. Equation (8) has a single unknown, ρ, and its left-hand side is increasing and its right-hand side is decreasing in r. Thus, it is straightforward to find ρ^*, for example by a bisection algorithm. Once the optimal profit ρ^* is found, one can readily obtain the optimal price vector and the optimal market share vector.

When the price sensitivity parameters and the dissimilarity indices are identical for all products, all the prices are marked up by the same amount – this observation has already been made in the literature by Aydin and Ryan (2000) and Hopp and Xu (2005) in the case of the standard MNL model. However, when the b_k values are not identical, Theorem 2 characterizes how the optimal price vector depends on b_k's, and shows that the price mark-up for a product with a high price sensitivity should be set low. From Theorem 2, we also note that the optimal price for a product of group k decreases not only in its own price sensitivity b_k, but also in the price sensitivities of products in other groups, b_j, where $j \neq k$; furthermore, it increases in $\overline{A}_k$ and $\overline{A}_j, j \neq k$. (The proof of this statement is straightforward and available from the authors upon request).

In the special case of the standard MNL model with identical price sensitivity parameters, we can express the optimal profit (and thus the optimal price and market share vectors) in a closed-form expression involving the Lambert W function (Corless et al., 1996). For any nonnegative z, $W(z)$ is the solution w satisfying

$$z = we^w . \tag{9}$$

The W function is positive, increasing and concave in the interval of our interest, $[0, \infty]$. The use of the Lambert W function to characterize the optimal price under the standard MNL model can be found in Li (2007), Li and Graves (2010), and Akçay et al. (2010).

Corollary 1. *In the standard MNL model, if* $b_k = b$ *for all* $k \in \{1, \ldots, K\}$, *then* $\rho^* = W\left(\sum_{k=1}^K \exp(a_k - b_k c_k - 1)\right)/b$.

Proof. Suppose $M_k = 1$ and $\tau_k = 1$ for each k. If $b_k = b$, equation (8) implies that $\rho^* = \sum_{k=1}^K e^{a_k - 1 - b_k c_k - 1 - b\rho^*}/b$. Then,

$$b\rho^* = \sum_{k=1}^K \exp(a_k - b_k c_k - 1 - b\rho^*) = \exp(-b\rho^*) \sum_{k=1}^K \exp(a_k - b_k c_k - 1) .$$

Thus, it follows that $b\rho^* = W\left(\sum_k \exp(a_k - b_k c_k - 1)\right)$, which implies the required result. $\square$

2.3 Comparison to the Oligopoly Equilibrium

We have computed the optimal monopoly solution for the problem of maximizing total profit. Below, we compare it to the outcome of a corresponding oligopolistic setting where multiple firms compete with one another. The oligopolistic competition with the MNL model has been extensively studied in the literature. The existence of the unique equilibrium has been established (for example, Gallego et al. (2006), Bernstein and Federgruen (2004) and Allon et al. (2010)), and the coefficients for the models have been estimated in various settings (for example, Berry (1994), Berry et al. (2004) and Goldberg (1995)). However, most of the analytical results are for the standard MNL model. In this section, we study the more general Nested Logit model. Our focus is to develop closed-form expressions for the equilibrium, and then to make a comparison with the optimal solution. Such a comparison between the oligopoly equilibrium solution and the optimal monopoly solution has been made in the literature for other types of demand (for example, see Farahat and Perakis (2010) for the nonnegative affine demand), but not in the context of standard MNL or Nested Logit models, to our knowledge.

We now consider an oligopoly model using the Nested Logit model, where each branch corresponds to a set of products offered by one of several competing firms. In this section, we consider both quantity competition and price competition.

Quantity Competition

The market share of a particular product j of firm k is given by q_{jk}. For a company indexed by k, Q_k given in (2) represents the total market share for firm k. The objective of firm k is to maximize its profit, given by

$$\Gamma_k(\mathbf{q}) \;=\; R_k(\mathbf{q}) - C_k(\mathbf{q}) \;, \quad \text{where} \quad R_k(\mathbf{q})=\sum_{j=1}^{M_k} p_{jk}(\mathbf{q}) \cdot q_{jk} \quad \text{and} \quad C_k(\mathbf{q}) = \sum_{j=1}^{M_k} c_k \cdot q_{jk} \;.$$

Above, firm k's decision is the set of market-shares for its products, $(q_{1k}, \ldots, q_{M_k k})$. Let $\hat{\mathbf{q}}$ denote the equilibrium market share vector under quantity competition, and let $\hat{\mathbf{p}}$ denote the price vector associated with the equilibrium. Recall that $\mathbf{p}^*$ and $\mathbf{q}^*$ are the optimal monopoly price and market share vectors.

In the next theorem, we present expressions for quantity-competition equilibrium solutions. Recall that the Lambert W function is a mapping from z to w such that $z = we^w$.

Theorem 3. *In the quantity competition oligopoly with the Nested Logit model,*

$$\hat{p}_{jk} = \frac{1+W(\overline{A}_k)}{b_k\tau_k} + c_{jk}\ , \qquad \text{and}$$

$$\hat{q}_{jk} = \hat{Q}_k \cdot \frac{e^{a_{jk}-b_k c_{jk}}}{\sum_{\ell=1}^{M_k} e^{a_{\ell k}-b_k c_{\ell k}}} \qquad \text{where} \qquad \hat{Q}_k = \frac{W(\overline{A}_k)}{1+\sum_{\ell=1}^{K} W(\overline{A}_\ell)}\ .$$

We make a few remarks on Theorem 3. First, both the equilibrium prices and market shares are expressed as closed-form expressions in terms of the W function, and therefore they are easy to compute. Even for the special case of the standard MNL model, Farahat and Perakis (2008) present the equilibrium solution as a solution to a system of several equations. Second, the mark-up $(\hat{p}_{jk} - c_{jk})$ does not depend on product index j, and firm k charges the same mark-up for all of its products. This occurs since we have assumed that the price sensitivity parameter b_k is the same for all products of firm k, and this observation is consistent with the existing literature. However, the market share $\hat{q}_{jk}$ depends on each product j within the firm k – in a manner that is proportional to the exponent of its *cost-adjusted* quality $(a_{jk}-b_k c_{jk})$. In comparison to the optimal monopoly price, which depends on the quality and price-sensitivity of all product groups, the quantity-competition equilibrium price for each product depends only on the characteristics of its own product group.

Third, we comment on the impact of the dissimilarity index τ_k for products offered by firm k. Under a technical condition (that the market share of firm k is bigger than that of the "no-purchase" option when firm k charges only the cost for its products), firm k's equilibrium market share $\hat{Q}_k$ always increases in its own dissimilarity index τ_k and decreases in other firm's dissimilarity index $\tau_\ell, \ell \neq k$. In addition, if the dissimilarity index increases for firm k, then it can be shown that the equilibrium price for firm k decreases unless $\overline{A}_k$ is very large; but the price for other firms remain unchanged. To see why the price may change in different directions with respect to τ_k as $\overline{A}_k$ becomes very large, let us also interpret $b_k\tau_k$ as the "aggregate price sensitivity" measure for each firm. We note that the equilibrium price is influenced by both the aggregate price sensitivity $b_k\tau_k$ and the aggregate quality $\overline{A}_k$, both of which are affected by τ_k. Clearly, an increased τ_k increases the aggregate price sensitivity and can cause price to fall. But it also leads to a higher overall quality. When $\overline{A}_k$ is very large, the impact of τ_k is primarily reflected in an increase of $\overline{A}_k$ and thus causes the equilibrium price to increase. Managerially, as a company's product offerings become more diverse (larger τ_k), the overall "quality" of its product portfolio increases, which allows it to obtain a larger market share and also *may* cause price to increase. However, as the product offering becomes more diverse, customers have more distinctive choices within the same company, and thus become *effectively* more price sensitive, which may result in a decrease in the equilibrium price.

Price Competition

We now consider the price competition oligopoly, where each firm k's decision is the set of prices for its products $(p_{1k}, p_{2k}, \ldots, p_{M_k k})$. The objective of firm k is to maximize its profit, which is now given by

$$\Gamma_k(\mathbf{p}) \;=\; R_k(\mathbf{p}) - C_k(\mathbf{p}) \;, \quad \text{where} \quad R_k(\mathbf{p}) = \sum_{j=1}^{M_k} p_{jk} \cdot q_{jk}(\mathbf{p}) \;, \qquad C_k(\mathbf{p}) = \sum_{j=1}^{M_k} c_k \cdot q_{jk}(\mathbf{p}) \;,$$

and $q_{jk}(\mathbf{p})$ denotes the market share associated with product j of firm k as a function of price vector $\mathbf{p}$ (according to equations (2) and (3)). Let $\tilde{\mathbf{p}}$ denote the equilibrium prices under price competition, and let $\tilde{\mathbf{q}}$ denote the vector of market shares at the equilibrium.

Recall that the Lambert W function has been very useful in expressing the monopoly optimal solution and the quantity-price equilibrium solution. While this function is no longer directly useful for analyzing the price competition oligopoly, it turns out that a variant of the W function, which we call the *Modified Lambert W* function, is applicable. Let V be a mapping from $(0, \infty)$ to $(0, 1)$ such that, for any $x \in (0, \infty)$, $V(x)$ is the unique solution v in $(0, 1)$ satisfying

$$v \cdot \exp\left(\frac{v}{1-v}\right) \;=\; x \;.$$

(By comparison, recall that the Lambert W function satisfies $W(x) \cdot \exp(W(x)) = x$; see (9).) It is easy to verify, for example using calculus, that V is a strictly increasing and concave function. Figure 1 shows the shapes of the W and V functions.

We first prove a property of how the V function compares to the Lambert W function.

Lemma 1. *Let $x \in (0, \infty)$. If $\lambda \geq W(x)$, then*

$$V(x) \;<\; W(x) \;\leq\; (1+\lambda) \cdot V\left(\frac{x}{1+\lambda}\right) \;\leq\; x \;.$$

The next theorem gives a closed-form expression for the price-competition equilibrium solution in terms of the V function. Let $\tilde{Q}_0$ be the unique solution to the single-variable

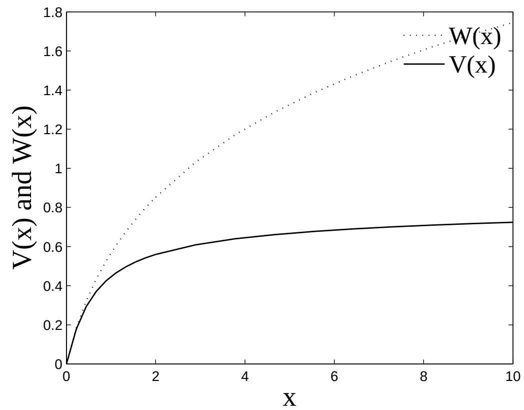


Figure 1: The Lambert W Function and the Modified Lambert W function, denoted by V.

equation

$$Q_0 + \sum_{k=1}^{K} V(\overline{A}_k Q_0) \ = \ 1 \ . \tag{10}$$

The choice of $\tilde{Q}_0$ is unique since the V function is strictly increasing from 0 to 1.

Theorem 4. *In the price competition oligopoly with the Nested Logit model, the proportion of non-purchasers in the total potential market is $\tilde{Q}_0$ as given by equation (10). The equilibrium price and market share for each product are given by*

$$\begin{aligned} \tilde{p}_{jk} &= \frac{1}{b_k \tau_k} \cdot \frac{1}{1 - V(\overline{A}_k \tilde{Q}_0)} \ + \ c_{jk} \qquad \textit{and} \\ \tilde{q}_{jk} &= \tilde{Q}_k \cdot \frac{e^{a_{jk} - b_k c_{jk}}}{\sum_{\ell=1}^{M_k} e^{a_{\ell k} - b_k c_{\ell k}}} \qquad \textit{where} \qquad \tilde{Q}_k \ = \ V(\overline{A}_k \tilde{Q}_0) \ . \end{aligned}$$

Since $\tilde{p}_{jk}$ is independent of j, we can write it as $\tilde{p}_k$.

Comparison of Oligopoly Equilibrium Outcomes and Monopoly Optimal Solution

Next, we draw a comparison between the oligopoly equilibrium solutions and the monopoly optimal solution. As before, we use hat and tilde to indicate the quantity competition and price competition, respectively, and we use star to denote the monopoly optimal solution.

Corollary 2. *In the Nested Logit model, $Q_0^* \geq \hat{Q}_0 \geq \tilde{Q}_0$ and $p_{jk}^* \geq \hat{p}_{jk} \geq \tilde{p}_{jk}$.*

Corollary 2 qualitatively establishes the relationship between the optimal monopoly price vector and the equilibrium price vector. Competition decreases the price of all the products, and the equilibrium price under the price competition oligopoly is the lowest among all three cases. These findings are consistent with insights obtained from standard Cournot and Bertrand competition models (see, for example, Mas-Colell et al. (1995), Propositions 12.C.1 and 12.C.2). In addition, competition also increases the total market share across all M products, and from this we can deduce that at least one product has an increase in its market share due to competition; however, it is possible that the market share of each individual product may increase or decrease. Farahat and Perakis (2008) compare the two types of oligopoly equilibria, namely quantity competition and price competition, under the standard MNL model and show that price competition yields lower prices and higher total market shares, which is a special case of Corollary 2. In addition, they suggest that the relationship between the equilibrium quantities of an individual product under the two types of competitions is more complex. In what follows, we explore this relationship under the more general Nested Logit model.

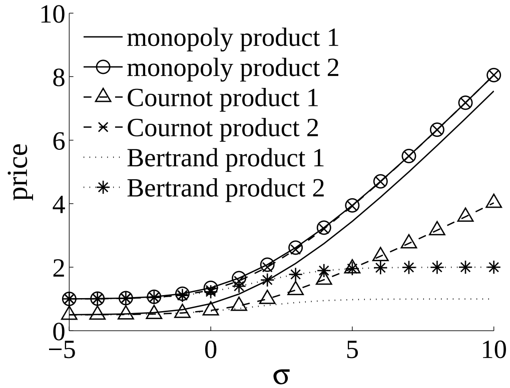
Recall that $\overline{A}_k$ is the *cost-adjusted* "aggregate quality" measure for each firm k that produces multiple differentiated products (see equation (6)), and that $b_k\tau_k$ is the "aggregate price sensitivity" measure for each firm. In the statement of Corollary 3 below, we assume that it is possible to order the firms based on the dominance of the cost-adjusted "aggregate quality" $\overline{A}_k$ and the "aggregate price sensitivity" $b_k\tau_k$. This corollary shows that, under perfect coordination between the firms, brands with weaker market power (i.e., the firm with smaller $\overline{A}_k$ value and higher $b_k\tau_k$ value) would receive smaller market shares than under oligopolistic competition. This can be explained by the fact that the firm of the lowest brand power does not need to be concerned with the effect of cannibalization (which is an issue for the monopolist managing multiple product groups.) By comparison, the market shares of products belonging to the firm with strong brand power may decrease due to competition.

Corollary 3. *In the Nested Logit model, suppose $\overline{A}_1 \leq \cdots \leq \overline{A}_K$ and $b_1\tau_1 \geq \cdots \geq b_K\tau_K$. Then, (a) there exist $\hat{k}_Q^* \in \{1, 2, \cdots, K\}$ such that*

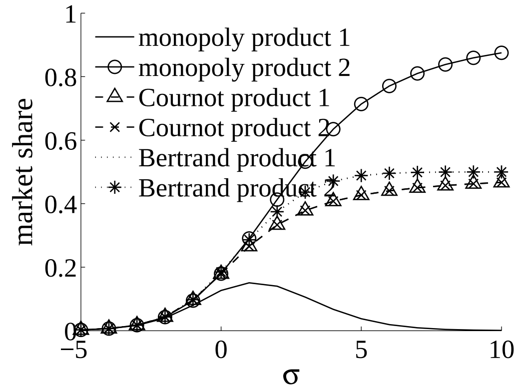
$$\begin{cases} q_{jk}^* \leq \hat{q}_{jk} & \text{for any } j \in \{1, \dots, M_k\} \quad \text{if } k \in \{1, \dots, \hat{k}_Q^*\} \\ q_{jk}^* \geq \hat{q}_{jk} & \text{for any } j \in \{1, \dots, M_k\} \quad \text{if } k \in \{\hat{k}_Q^* + 1, \dots, K\} \end{cases} .$$

(b) Furthermore, there exist $\hat{\hat{k}}_Q \in \{0, 1, 2, \cdots, K\}$ such that

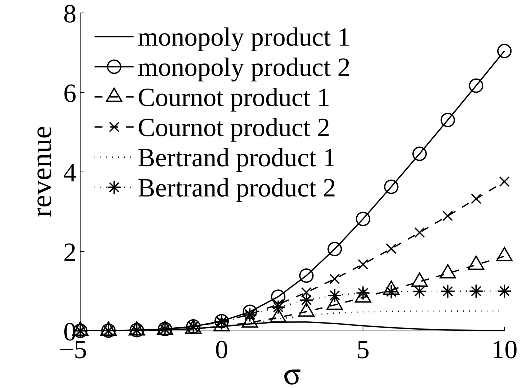
$$\begin{cases} \hat{q}_{jk} \geq \tilde{q}_{jk} & \text{for any } j \in \{1, \dots, M_k\} \quad \text{if } k \in \{1, \dots, \hat{\hat{k}}_Q\} \\ \hat{q}_{jk} \leq \tilde{q}_{jk} & \text{for any } j \in \{1, \dots, M_k\} \quad \text{if } k \in \{\hat{\hat{k}}_Q + 1, \dots, K\} \end{cases} .$$



(a) Prices

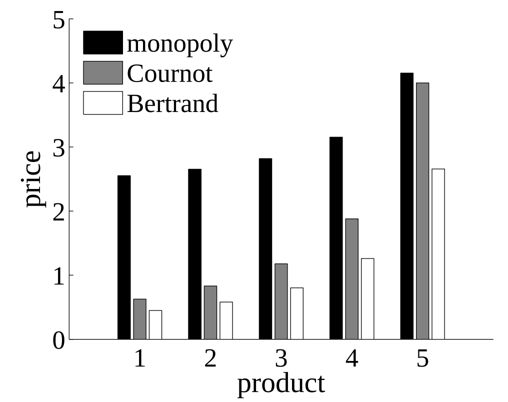


(b) Market Shares

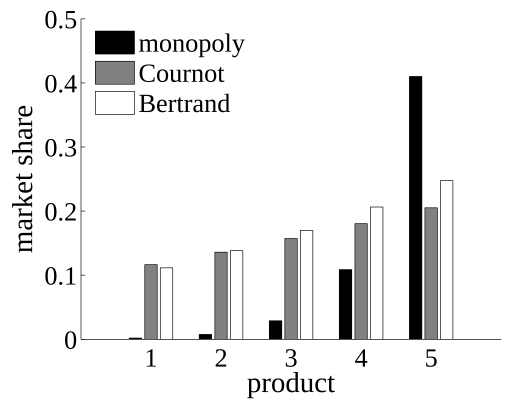


(c) Revenues

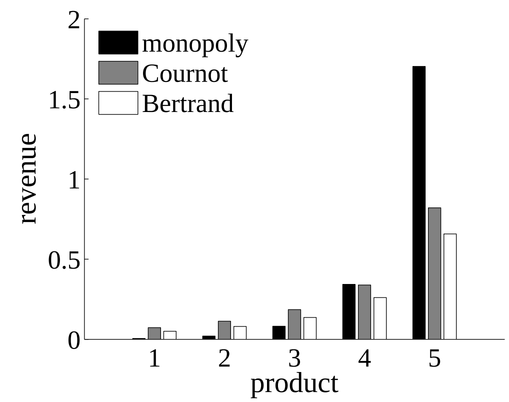
Figure 2: Comparison of the Optimal Centralized Pricing and the Oligopoly Equilibriums as Product Quality σ increases ($b_1 = 2$, $b_2 = 1$, and $c_1 = c_2 = 0$)



(a) Prices



(b) Market Shares



(c) Revenues

Figure 3: Illustration of Corollaries 2 and 3 – Comparison of Monopoly Solution and Oligopoly Equilibrium Solutions with 5 distinct products where $(a_1, b_1) = (1.0, 2.5), (a_2, b_2) = (1.25, 2.0), \ldots, (a_5, b_5) = (2.0, 0.5)$, and $c_1 = c_2 = \cdots = c_5 = 0$

Before we close this section, we show by an example that it is indeed possible that competition may decrease the market share of a product, and also increase the profit associated with a product. For simplicity, we let each branch contain a single product, i.e., we use the standard MNL model. Suppose that there are two products, i.e., $M = 2$. We consider a family of problems parameterized by σ. Suppose $a_1 = a_2 = \sigma$, and let $b_1 > b_2$. We assume that all costs are zero; thus, revenue and profit are the same. We compare the prices, market shares and revenues under the monopoly setting and under the duopoly setting as we vary the value of σ. In Figure 2(a), we see that the prices under the monopoly are always higher than prices under the quantity-competition (Cournot) duopoly, which is higher than prices under the price-competition (Bertrand) duopoly. In Figure 2(b), we observe that with increasing quality σ, the market share becomes dominated by the low price-sensitivity product (product 2) under the monopoly but shared evenly between the two products under the duopolies. Further, the monopolist generates most of its revenue from the low price-sensitivity product (product 2) and the other product (product 1) plays little role in the market; but under both duopoly settings, the firm producing the high price-sensitivity product (product 1) prices more aggressively to obtain a more significant market share and revenue (Figure 2(c)). This is consistent with Corollary 3 that the market share of the product with higher price sensitivity is more likely to increase as a result of competition.

This observation becomes even more evident when the number of products increases. Figure 3 illustrates a 5-product example that satisfies the conditions in Corollary 3. In addition to the behavior predicted by the corollary, we notice that competition increases product variety; compared to the oligopoly equilibrium, the revenue-maximizing monopolist effectively prices out the less attractive products and increases the price for the more attractive ones. A common finding in the literature of product line selection problems has been that when limited by the number of products within an assortment, firms should choose an assortment based on the ordering of product "attractiveness"(Cachon and Kok, 2007; Aydin and Ryan, 2000; Hopp and Xu, 2008). Although we do not limit the number of products in an assortment, the results shown in Figure 3 seem to provide the same insight that firms should exclude products that rank low on attractiveness (i.e., with lower a_i and higher b_i values).

Revenue Implications for Joint versus Decentralized Pricing

Another interesting interpretation of the oligopoly results is to think of each product as a separate business unit within a firm that acts autonomously and hence makes its own pricing decisions. With this interpretation, the difference between the total revenue of the oligopoly context and the monopoly context can shed light on the benefits of coordinate/joint pricing for product assortments. Using the same two-product example in Figure 2, we show in Figure 4 the differences in total revenue as product quality increases.

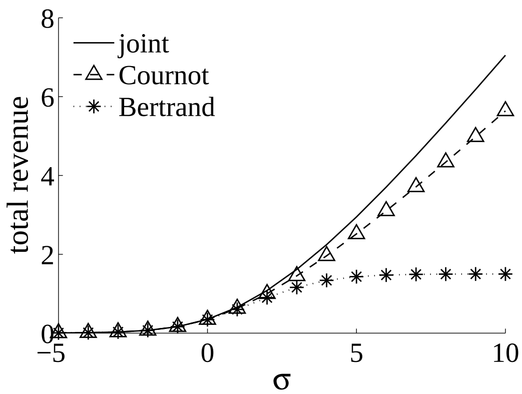


Figure 4: Total Revenue of All Products as Quality Increases ($b_1 = 2$, $b_2 = 1$, and $c_1 = c_2 = 0$).

In both oligopolies, as the quality increases, the total revenue deviates more from the optimal joint-pricing revenue, suggesting that the loss due to decentralized decision making increases with quality. It is clear that the price-competition (Bertrand) oligopoly results in the worst total revenue. Moreover, as the quality exceeds a certain level, improved quality does not bring any increase in the total revenue under price competition, all suggesting that price competition is a more fierce competition. We also observe this from Figures 2(b) and 2(c), which indicate that the equilibrium price and market share both plateau as quality σ improves above a certain level. Therefore, if product lines that belong to the same customer choice structure are managed by different business units, production decisions and pricing decisions should be made jointly, especially in a make-to-order scenario (which corresponds to the price competition oligopoly as explained earlier).

3 Applications

3.1 Joint Inventory-Pricing Control: Make-To-Assemble

In this section, we suppose that the set of differentiated products shares a critical common component in a make-to-assemble setting. We ignore all other components. We consider a periodic-review inventory control of the common component when demands are stochastic and based on the logit models. While the results of this section are valid for both the standard MNL model and the Nested Logit model, we use the notation of the standard MNL model for expositional simplicity. The manager decides the inventory level of the critical component and the vector of prices for the set of differentiated products.

In each period t, the following sequence of events occur. (1) The manager observes the inventory level of the common component denoted by x^t, where the negative value of x^t represents the backlog. (2) The manager orders the component. We use y^t to denote the inventory level after ordering, where $y^t \geq x^t$. If any positive quantity is ordered, i.e., $y^t > x^t$, then the fixed cost of $K \geq 0$ is charged. Replenishment is instantaneous. (3) The manager sets prices for all M products, $\mathbf{p}^t = (p_1^t, \ldots, p_M^t)$, or equivalently $\mathbf{q}^t = (q_1^t, \ldots, q_M^t)$. (4) Stochastic demand vector for period t, $\mathbf{D}^t(\mathbf{q}) = (D_1^t(\mathbf{q}), \ldots, D_M^t(\mathbf{q}))$, is realized. We suppose that $\mathbf{D}^t$ is given by

$$\mathbf{D}^t(\mathbf{q}^t) \quad = \quad \mathbf{q}^t \Lambda^t + \mathbf{L}(\mathbf{q}^t) \varepsilon^t$$

where Λ^t represents the customer arrival rate, $\mathbf{L}(\mathbf{q})$ is an M-by-M matrix where each entry is a linear function of $\mathbf{q}$, and both $\{\Lambda^t\}$ and $\{\varepsilon^t = (\varepsilon_1^t, \ldots, \varepsilon_M^t)\}$ are random and IID. This demand model is adopted by Song and Xue (2007) and encompasses many commonly used stochastic demand models in the literature. (5) Demands are satisfied to the extent possible. Any excess demand is backlogged at the cost of b per unit, and any excess inventory is carried over to the next period at the cost of h per unit. Thus, $x^{t+1} = y^t - \sum_i D_k^t(\mathbf{q}^t)$.

In the literature that addresses joint price-inventory control with the MNL model, Song and Xue (2007) are the closest to our model in this section, and indeed our demand model is taken from their paper. While we consider the stocking of a single common component, they consider the problem of managing multiple stockpiles of inventory, one for each product. Since their problem is more complex, they can show only a partial structural result, even when the fixed order cost $K = 0$. In this section, we allow positive K, and characterize the optimal inventory policy of the common component. Related to this section, Dong et al. (2009) also consider a dynamic system, but they do not allow inventory replenishment.

Denote the expected single-period profit by

$$\pi(y,\mathbf{q}) \quad = \quad R(\mathbf{q}) - h\cdot E[y - \sum_i D_i(\mathbf{q})]^+ - b\cdot E[\sum_i D_i(\mathbf{q}) - y]^+ \ .$$

(Because of the IID assumption, we suppress the superscript t.) We consider the infinite-horizon discounted cost criteria, i.e., minimizing $\sum_{t=1}^{\infty}\beta^t\pi(y^t,\mathbf{q}^t)$. For this problem, we show that the structure of the optimal inventory policy is a simple two-threshold policy.

Theorem 5. *For the joint inventory-pricing control problem of Section 3.1, an (s,S) policy is optimal. Furthermore, if $K=0$, then a base-stock policy is optimal, i.e., $s=S$.*

Proof. This proof is based on Huh and Janakiraman (2008), who provide a sufficient condition for the optimality of an (s,S) policy. In our context, their condition is satisfied if we could verify two sufficient conditions. The first condition is the concavity of π, which follows from the concavity of R (Theorem 1). Let $(y^*,\mathbf{q}^*)$ be the maximizer of π. The second condition is that, for any y^A and y^B satisfying $y^*\le y^A<y^B$ and $\mathbf{q}^B$, there exists $\mathbf{q}^A$ such that (a) $\pi(y^A,\mathbf{q}^A)\ge\pi(y^B,\mathbf{q}^B)$, and (b) $y^A-\sum_i D_i^t(\mathbf{q}^{A,t})\le\max\{y^B-\sum_i D_i^t(\mathbf{q}^{B,t}),y^*\}$ for any realization of Λ and ε. Since y^A is a convex combination of y^* and y^B, we choose $\mathbf{q}^A$ such that the vector $(y^A,\mathbf{q}^A)$ is a convex combination of $(y^*,\mathbf{q}^*)$ and $(y^B,\mathbf{q}^B)$. Since π is concave and is maximized at $(y^*,\mathbf{q}^*)$, we obtain that $\pi(y^A,\mathbf{q}^A)$ is bounded below by the convex combination of $\pi(y^*,\mathbf{q}^*)$ and $\pi(y^B,\mathbf{q}^B)$; thus, (a) holds. Also, since $\mathbf{D}(\mathbf{q})$ is a linear function of $\mathbf{q}$, it follows that $y^A-\sum_i D_i^t(\mathbf{q}^{A,t})$ is a convex combination of $y^*-\sum_i D_i^t(\mathbf{q}^*)$ and $y^B-\sum_i D_i^t(\mathbf{q}^{B,t})$, implying (b). This completes the verification of the sufficient conditions. □

We note that the exact form of the MNL demand model is not used in the proof of Theorem 5 except through the concavity of $R(\mathbf{q})$ in $\mathbf{q}$. Thus, it can also be shown that the optimality of the (s,S) policy remains valid if we adopt a more general demand model described in Song and Xue (2007) since their demand model satisfies the concavity requirement.

3.2 Joint Inventory-Pricing Control: Make-To-Stock

In the joint inventory-pricing model of Song and Xue (2007), the inventory is stocked not at the common component level but at each individual finished product level. Furthermore, the fixed cost of ordering K is negligible. The sequence of events is similar to Section 3.1, except that the before-ordering and after-ordering inventory levels are now specified for each product which we can denote by $\mathbf{x}^t = (x_1^t, \ldots, x_M^t)$ and $\mathbf{y}^t = (y_1^t, \ldots, y_M^t)$.

For this problem, Song and Xue (2007) have shown the concavity of the dynamic programming value function, and established that the structure of the optimal ordering policy is an order-up-to policy where the order-up-to level depends on a subset of products that can be identified by an algorithm. The requirement for their result to hold is the concavity of the single-period revenue function with respect to the market-share vector, which they have shown to hold in the case of the standard MNL model where all price sensitivity parameters are one, i.e., $b_i = 1$ for all i's. The concavity results of this paper (Theorem 1) imply that all the results of Song and Xue (2007) also hold for the standard MNL model with general price sensitivity parameters as well as for the Nested Logit model.

3.3 Dynamic Pricing Model with Non-Replenishable Inventory

Now, consider a multi-period price control model with a finite horizon without inventory replenishment. The model is similar to the model of Section 3.2 except that inventory cannot be replenished. Furthermore, at most one customer arrives in each period, and λ denotes the probability that a customer shows up. Dong et al. (2009) have studied this setting when the price sensitivity parameters are identical and shown that the optimal prices in each period have identical profit margins. In this section, we extend their model to include the standard MNL model with non-identical price sensitivity parameters and the Nested Logit model.

Let T be the planning horizon, and let $t \in \{1, \ldots T\}$ index time periods in a forward manner. Let $\mathbf{x}^t = (x_1^t, \ldots, x_M^t) \geq \mathbf{0}$ denote the vector of initial inventory levels. We present a dynamic programming formulation for the optimal pricing problem, assuming that the standard MNL model is used. (The extension to the Nested Logit model is similar.) It can be written as

$$V^t(\mathbf{x}^t) \;=\; \max_{\mathbf{q}^t} \quad \left\{ J^t(\mathbf{x}^t, \mathbf{q}^t) \mid \mathbf{q}^t \in [0,1]^M,\; q_1^t + \cdots + q_M^t \le 1 \right\}$$
$$J^t(\mathbf{x}^t, \mathbf{q}^t) \;=\; \lambda \sum_{i=1}^{M} q_i^t \cdot \{p_i(\mathbf{q}^t) - \Delta_i V^{t+1}(\mathbf{x}^t)\} \;+\; V^{t+1}(\mathbf{x}^t) \tag{11}$$

where

$$\Delta_i V^{t+1}(\mathbf{x}^t) \;=\; V^{t+1}(\mathbf{x}^t) - V^{t+1}(\mathbf{x}^t - \mathbf{e}_i) \qquad \text{if } x_i^t \ge 1 \;,$$

and $\Delta_i V^{t+1}(\mathbf{x}^t) = \infty$ otherwise (which ensures that the inventory remains nonnegative). Above, $\mathbf{e}_i$ is an all-zeros vector except the entry of 1 corresponding to index i. We assume the zero salvage value by setting $V^T(\mathbf{x}) = 0$, for the simplicity of exposition.

The decision in each period is to choose $\mathbf{q}^t$ that maximizes $J^t(\mathbf{x}^t, \mathbf{q}^t)$. Since $R(\mathbf{q}) = \sum_{i=1}^{M} q_i^t \cdot p_i(\mathbf{q}^t)$ is concave in $\mathbf{q}^t$ (Theorem 1), it follows easily that $J^t(\mathbf{x}^t, \mathbf{q}^t)$ is concave in $\mathbf{q}^t$. Thus, we can use the first-order conditions to identify the optimal market share vector, which we denote by $\check{\mathbf{q}}^t(\mathbf{x})$. We note that although $J^t(\mathbf{x}^t, \mathbf{q}^t)$ may not be jointly concave in both inventory and price, the optimization problem within each time period t is a concave problem since the inventory position at the beginning of the next period, x^{t+1}, is limited to a finite set given by $(\mathbf{x}^t - \mathbf{e}_i)_{i=1,\dots,M}$. It can be shown that

$$p_k(\check{\mathbf{q}}^t(\mathbf{x})) \;=\; \Delta_k V^{t+1}(\mathbf{x}) \;+\; \check{r}(\mathbf{x}, \check{\mathbf{q}}^t(\mathbf{x})) \;+\; \frac{1}{b_k} \tag{12}$$

where

$$\check{r}(\mathbf{x}, \check{\mathbf{q}}^t(\mathbf{x})) \;=\; \sum_{k=1}^{M} \left\{ p_k(\check{\mathbf{q}}^t(\mathbf{x})) - \Delta_k V^{t+1}(\mathbf{x}) \right\} \cdot \check{q}_k^t(\mathbf{x}) \;\ge\; 0 \;.$$

(See the appendix for the proof of (12).) This shows that the optimal price is higher than the future value of the unit represented by $\Delta_k V^{t+1}(\mathbf{x})$. The mark-up has two components: $\check{r}(\mathbf{x}, \check{\mathbf{q}}^t(\mathbf{x}))$ which is common across all the products, and $1/b_k$ which depends on each product k. We observe that the mark-up is higher for the product that is less price sensitive. This insight is a generalization of Dong et al. (2009) who have identified that the mark-up is the same for all products when the price sensitivity parameters are identical for all products.

With the Nested Logit model of consumer choice, the above results can be extended to

$$p_{jk}(\check{\mathbf{q}}^t(\mathbf{x})) \;=\; \Delta_{jk} V^{t+1}(\mathbf{x}) \;+\; \check{r}(\mathbf{x}, \check{\mathbf{q}}^t(\mathbf{x})) \;+\; \frac{1}{b_k \tau_k} \tag{13}$$

where

$$\check{r}(\mathbf{x}, \check{\mathbf{q}}^t(\mathbf{x})) \;=\; \sum_{k=1}^{K} \sum_{j=1}^{M_k} \left\{ p_{jk}(\check{\mathbf{q}}^t(\mathbf{x})) - \Delta_{jk} V^{t-1}(\mathbf{x}) \right\} \cdot \check{q}^t_{jk}(\mathbf{x}) \;\geq\; 0 \; .$$

Note that in the case of the Nested Logit model, the product-specific component of the mark-up also depends on the dissimilarity index τ_k. The proof for equation (13) is similar to that of equation (12) and is available from the authors upon request.

4 Concluding Remarks

We have shown the concavity of the revenue and profit functions with respect to the market share vector when a firm sells multiple differentiated products with demand given by the Nested Logit models, which include the standard MNL model as a special case. Using this property, we derive the optimal solution of the profit maximization problem for a monopolist selling multiple differentiated products, and the equilibrium solution for price-competition and quantity-competition oligopolies where each firm sells multiple differentiated products.

The comparison of the monopoly and the oligopolies yields several interesting insights for the nested choice structure. First, the price for any individual product is always the highest under the monopoly and the lowest under the price-competition (Bertrand) oligopoly. Second, while the total market share for all the products is always the lowest under the monopoly and the highest under the price-competition oligopoly, the market share for each individual product does not follow the same trend. Rather, whether a product's market share improves or worsens with competition depends on the brand power of the product group, as characterized by its aggregate quality and the customer's price sensitivity toward this product group: Products with stronger (weaker) brand power will have a smaller (larger) market share under competition than under a monopoly. Third, a comparison of the monopoly and oligopolies reveals the impact of joint versus decentralized decision making. Our results show that profit loss due to decentralized decision is greater under the price-competition (Bertrand) oligopoly than under the quantity-competition (Cournot) oligopoly. This implies that the joint pricing decision is the most critical in a make-to-order manufacturing environment, which corresponds to the price-competition oligopoly (see Friedman (1988) and Farahat and Perakis (2008)).

The application of the concavity property extends to joint inventory and pricing control problems, and provides theoretical justifications for using first-order conditions to derive the optimal pricing and market share solutions for dynamic models in which customer demand can be modeled with a Nested Logit model. For example, by extending the multiple-product dynamic pricing problem with non-replenishable inventory introduced by Dong et al. (2009), we show that the optimal mark-up may not be uniform but may include a product-specific component. For dynamic pricing problems with replenishable inventory, we extend the optimality of the (s, S) policy to a make-to-assemble scenario where a common critical component of multiple differentiated products is stocked (Huh and Janakiraman, 2008). Furthermore, we show that the structure of the optimal policy for the joint inventory-pricing problem in a make-to-order setting (a la Song and Xue (2007)) continues to hold even for a more general nested choice model.

Given the flexibility and empirical validity of the Nested Logit model, we hope the theoretical properties and applications shown in this paper prove useful for future research in operations, marketing and economics employing this family of price-dependent demand models. One interesting research direction relates to more concrete interpretations of the nested choice structure within a specific market and operations context. For example, exploiting the theoretical results derived from a general nested structure presented in this paper, we can study pricing and other decisions for product assortments that are both vertically and horizontally differentiated. Another direction to explore is the impact of demand uncertainty on pricing and market share solutions.

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Appendix

Proof of Equation (4)

To see (4), note from equation (2),

$$1-\sum_{\ell=1}^{K} Q_\ell \;=\; 1-\frac{\sum_{\ell=1}^{K} e^{\tau_\ell I_\ell}}{1+\sum_{\ell=1}^{K} e^{\tau_\ell I_\ell}} \;=\; \frac{1}{1+\sum_{\ell=1}^{K} e^{\tau_\ell I_\ell}}\;.$$

Thus, from the above equation and again from (2),

$$\frac{Q_k}{1-\sum_{\ell=1}^{K} Q_\ell} \;=\; \left(\frac{e^{\tau_k I_k}}{1+\sum_{\ell=1}^{K} e^{\tau_\ell I_\ell}}\right)\cdot\left(1+\sum_{\ell=1}^{K} e^{\tau_\ell I_\ell}\right) \;=\; e^{\tau_k I_k} \;=\; \left[\sum_{i=1}^{M_k} e^{a_{ik}-b_k p_{ik}}\right]^{\tau_k}\;,$$

where the last equality follows from the definition of $I_k = \log \sum_{j=1}^{M_k} e^{a_{jk}-b_k p_{jk}}$. Also, from the definition of $q_{j|k} = e^{a_{jk}-b_k p_{jk}} / \sum_{\ell=1}^{M_k} e^{a_{\ell k}-b_k p_{\ell k}}$ in (2),

$$\frac{q_{jk}}{Q_k} \;=\; \frac{q_{jk}}{\sum_{i=1}^{M_k} q_{ik}} \;=\; \frac{q_{j|k}}{\sum_{i=1}^{M_k} q_{i|k}} \;=\; q_{j|k} \;=\; \frac{e^{a_{jk}-b_k p_{jk}}}{\sum_{i=1}^{M_k} e^{a_{ik}-b_k p_{ik}}}\;,$$

for any product j in group k. From the two above equations, we obtain

$$e^{a_{jk}-b_k p_{jk}} = \frac{q_{jk}}{Q_k}\sum_{i=1}^{M_k} e^{a_{ik}-b_k p_{ik}} = \frac{q_{jk}}{Q_k}\left[\frac{Q_k}{1-\sum_{\ell=1}^{K} Q_\ell}\right]^{\frac{1}{\tau_k}} = \frac{q_{jk}}{1-\sum_{\ell=1}^{K} Q_\ell}\left[\frac{Q_k}{1-\sum_{\ell=1}^{K} Q_\ell}\right]^{\frac{1}{\tau_k}-1}.$$

Hence, by taking a logarithm to the above equation, we can write p_{jk} as a function of $\mathbf{q}$ as given in equation (4).

Additional Argument in the Proof of Theorem 1

Lemma 2. *Let $\phi(z_A, z_B) = z_A \cdot [\log z_A - \log(1-z_B)]$. Then, ϕ is jointly convex in (z_A, z_B) in the region $\{(z_A, z_B) \mid 0 < z_A, z_B < 1\}$.*

Proof. We can obtain that the Hessian of ϕ is $H_\phi(z_A, z_B) = \begin{bmatrix} \frac{1}{z_A} & \frac{1}{1-z_B} \\ \frac{1}{1-z_B} & \frac{z_A}{(1-z_B)^2} \end{bmatrix}$. Then, for any real numbers γ and δ, we obtain

$$\begin{pmatrix} \gamma & \delta \end{pmatrix} \cdot H_\phi(z_A, z_B) \cdot \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \frac{\gamma^2}{z_A} + \frac{\gamma \cdot \delta}{1-z_B} + \frac{\gamma \cdot \delta}{1-z_B} + \frac{\delta^2 \cdot z_A}{(1-z_B)^2} = \left(\frac{\gamma}{\sqrt{z_A}} + \frac{\delta\sqrt{z_A}}{1-z_B}\right)^2,$$

which is nonnegative for $0 < z_A, z_B < 1$. Thus, the Hessian of ϕ is positive semi-definite, and ϕ is jointly convex. □

Other Proofs

All other proofs appear in the online appendix.

Pricing Multiple Products with the Multinomial Logit and Nested Logit Models: Concavity and Implications

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A Online Supplement

A.1 Proof of Theorem 2

Proof. Taking first order partial derivative of the revenue function (5) with respect to q_{jk},

$$\frac{\partial R(\mathbf{q})}{\partial q_{jk}} = p_{jk}(\mathbf{q}) + \sum_{m=1}^{K}\sum_{\ell=1}^{M_m} q_{lm}\cdot\frac{\partial p_{lm}(\mathbf{q})}{\partial q_{jk}} .$$

Taking a partial derivative of $p_{jk}(\mathbf{q})$ given in equation (4),

$$\begin{aligned}\frac{\partial p_{lm}(\mathbf{q})}{\partial q_{jk}} = & -\frac{1}{b_m}\left[\frac{1}{1-\sum_i Q_i}\right] - \frac{1}{b_m}\frac{1-\tau_m}{\tau_m}\left[\frac{1}{1-\sum_i Q_i}\right] \\ & -\frac{1}{b_m}\frac{1}{q_{lm}}\mathbf{1}_{\{m=k \text{ and } \ell=j\}} - \frac{1}{b_m}\frac{1-\tau_m}{\tau_m}\frac{1}{Q_m}\mathbf{1}_{\{m=k\}} ,\end{aligned} \tag{14}$$

where $\mathbf{1}$ represents a binary indicator function. Thus,

$$\begin{aligned}\frac{\partial R(\mathbf{q})}{\partial q_{jk}} &= p_{jk}(\mathbf{q}) - \left[\frac{1}{1-\sum_i Q_i}\right]\sum_{m=1}^{K}\frac{\sum_{\ell=1}^{M_m} q_{lm}}{b_m} - \left[\frac{1}{1-\sum_i Q_i}\right]\sum_{m=1}^{K}\frac{1-\tau_m}{\tau_m}\frac{\sum_{\ell=1}^{M_m} q_{lm}}{b_m} \\ &\quad -\frac{1}{b_k} - \frac{1}{b_k}\frac{1-\tau_k}{\tau_k}\frac{\sum_{\ell=1}^{M_k} q_{\ell k}}{Q_k} \\ &= p_{jk}(\mathbf{q}) - \left[\frac{1}{1-\sum_i Q_i}\right]\sum_{m=1}^{K}\frac{Q_m}{b_m} - \left[\frac{1}{1-\sum_i Q_i}\right]\sum_{m=1}^{K}\frac{1-\tau_m}{\tau_m}\frac{Q_m}{b_m} - \frac{1}{b_k\tau_k} \\ &= p_{jk}(\mathbf{q}) - \left[\frac{1}{1-\sum_i Q_i}\right]\sum_{m=1}^{K}\frac{Q_m}{b_m\tau_m} - \frac{1}{b_k\tau_k} ,\end{aligned} \tag{15}$$

where the second inequality follows from $\sum_{\ell=1}^{M_m} q_{lm} = Q_m$. Also, recall $\frac{\partial C(\mathbf{q})}{\partial q_{jk}} = c_{jk}$. Since $\Gamma(\mathbf{q}) = R(\mathbf{q}) - C(\mathbf{q})$ is concave (Theorem 1), we equate the partial derivative of Γ at $\mathbf{q}^*$ to zero, and obtain

$$p_{jk}(\mathbf{q}^*) = \left[\frac{1}{1-\sum_i Q_i^*}\right]\sum_{m=1}^{K}\frac{Q_m^*}{b_m\tau_m} + \frac{1}{b_k\tau_k} + c_{jk} . \tag{16}$$

Now, substituting (16) into $R(\mathbf{q}^*) = \sum_{k=1}^{K}\sum_{j=1}^{M_k} p_{jk}(\mathbf{q}^*)q_{jk}^*$ given in (5), it follows

$$\begin{aligned}\Gamma(\mathbf{q}^*) &= \sum_{k=1}^{K}\sum_{j=1}^{M_k}\left[\frac{q_{jk}^*}{1-\sum_i Q_i^*}\right]\sum_{m=1}^{K}\frac{Q_m^*}{b_m\tau_m} + \sum_{k=1}^{K}\sum_{j=1}^{M_k}\frac{q_{jk}^*}{b_k\tau_k} + \sum_{k=1}^{K}\sum_{j=1}^{M_k} c_{jk}q_{jk}^* - \sum_{k=1}^{K}\sum_{j=1}^{M_k} c_{jk}q_{jk}^* \\ &= \left[\frac{\sum_k Q_k^*}{1-\sum_i Q_i^*}\right]\sum_{m=1}^{K}\frac{Q_m^*}{b_m\tau_m} + \sum_{k=1}^{K}\frac{Q_k^*}{b_k\tau_k} = \left[\frac{1}{1-\sum_i Q_i^*}\right]\sum_{m=1}^{K}\frac{Q_m^*}{b_m\tau_m}\ .\end{aligned}$$

Since $\rho^* = \Gamma(\mathbf{q}^*)$, we obtain

$$\rho^* = \left[\frac{1}{1-\sum_i Q_i^*}\right]\sum_{k=1}^{K}\frac{Q_k^*}{b_k\tau_k}\ . \tag{17}$$

Thus, substituting this expression into (16), we obtain $p_{jk}^* = p_{jk}(\mathbf{q}^*) = \rho^* + \frac{1}{b_k\tau_k} + c_{jk}$, which is the required expression for p_{jk}^*.

Let $I_k^* = \log\sum_{j=1}^{M_k}\exp(a_{jk} - b_k p_{jk}^*)$ represent the optimal aggregate attractiveness of branch k. Using the above expression for p_{jk}^*, it follows that $I_k^* = \log\sum_{j=1}^{M_k}\exp(a_{jk} - 1/\tau_k - b_k\rho^* - b_k c_{jk})$. Then, the optimal aggregate market share for branch k is, from (2),

$$Q_k^* = \frac{e^{\tau_k I_k}}{1+\sum_{\ell=1}^{K} e^{\tau_\ell I_\ell}} = \frac{\left[\sum_{j=1}^{M_k}\exp(a_{jk} - 1/\tau_k - b_k\rho^* - b_k c_{jk})\right]^{\tau_k}}{1+\sum_{\ell=1}^{K}\left[\sum_{j=1}^{M_\ell}\exp(a_{jl} - 1/\tau_\ell - b_\ell\rho^* - b_k c_{jk})\right]^{\tau_\ell}}\ . \tag{18}$$

Therefore, we obtain

$$1-\sum_{k=1}^{K} Q_k^* = \frac{1}{1+\sum_{\ell=1}^{K}\left[\sum_{j=1}^{M_\ell}\exp(a_{jl} - 1/\tau_\ell - b_\ell\rho^* - b_\ell c_{j\ell})\right]^{\tau_\ell}}\ ,$$

and it follows from (17) and (18),

$$\rho^* = \sum_{k=1}^{K}\frac{\left[\sum_{j=1}^{M_k}\exp(a_{jk} - 1/\tau_k - b_k\rho^* - b_k c_{jk})\right]^{\tau_k}}{b_k\tau_k} = e^{-1}\sum_{k=1}^{K}\frac{\left[\sum_{j=1}^{M_k}\exp(a_{jk} - b_k\rho^* - b_k c_{jk})\right]^{\tau_k}}{b_k\tau_k}\ ,$$

where the last equality follows from $[\exp(-1/\tau_k)]^{\tau_k} = e^{-1}$. Thus, from (7), we obtain (8). The value of ρ^* satisfying (8) is unique since the left side is increasing and the right side is decreasing in ρ^*.

Finally, let $q_{j|k}^* = e^{a_{jk} - b_k p_{jk}^*}/\sum_{\ell=1}^{M_k} e^{a_{\ell k} - b_k p_{\ell k}^*}$ as in (2). From (18), as well as $p_{jk}^* = \rho^* + \frac{1}{b_k\tau_k} + c_{jk}$,

$$\begin{aligned}q_{jk}^* &= Q_k^*\cdot q_{j|k}^* = \frac{\left[\sum_{i=1}^{M_k} e^{a_{ik} - 1/\tau_k - b_k\rho^* - b_k c_{ik}}\right]^{\tau_k}}{1+\sum_{\ell=1}^{K}\left[\sum_{i=1}^{M_\ell} e^{a_{i\ell} - 1/\tau_\ell - b_\ell\rho^* - b_\ell c_{i\ell}}\right]^{\tau_\ell}}\cdot\frac{e^{a_{jk} - 1/\tau_k - b_k\rho^* - b_k c_{jk}}}{\sum_{i=1}^{M_k} e^{a_{ik} - 1/\tau_k - b_k\rho^* - b_k c_{ik}}} \\ &= \frac{e^{-1}\left[\sum_{i=1}^{M_k} e^{a_{ik} - b_k\rho^* - b_k c_{ik}}\right]^{\tau_k}}{1+e^{-1}\sum_{\ell=1}^{K}\left[\sum_{i=1}^{M_\ell} e^{a_{i\ell} - b_\ell\rho^* - b_\ell c_{i\ell}}\right]^{\tau_\ell}}\cdot\frac{e^{a_{jk} - b_k c_{jk}}}{\sum_{i=1}^{M_k} e^{a_{ik} - b_k c_{ik}}}\ ,\end{aligned}$$

which yields the required expression for q_{jk}^*. □

A.2 Proof of Theorem 3

Proof. Taking a partial derivative of $R_k(\mathbf{q})$ with respect to q_{jk}, $\frac{\partial R_k(\mathbf{q})}{\partial q_{jk}} = p_{jk} + \sum_{\ell=1}^{M_k} q_{\ell k} \frac{\partial p_{\ell k}(\mathbf{q})}{\partial q_{jk}}$. Recalling the expression of $\partial p_{\ell k}(\mathbf{q})/\partial q_{jk}$ given in (14), we obtain

$$\begin{aligned} q_{\ell k}\frac{\partial p_{\ell k}(\mathbf{q})}{\partial q_{jk}} &= -\frac{1}{b_k}\left[\frac{q_{\ell k}}{1-\sum_i Q_i}\right] - \frac{1}{b_k}\frac{1-\tau_k}{\tau_k}\left[\frac{q_{\ell k}}{1-\sum_i Q_i}\right] - \frac{1}{b_k}\mathbf{1}_{\{\ell=j\}} - \frac{1}{b_k}\frac{1-\tau_k}{\tau_k}\frac{q_{\ell k}}{Q_k} \\ &= -\frac{1}{b_k\tau_k}\left[\frac{q_{\ell k}}{1-\sum_i Q_i}\right] - \frac{1}{b_k}\mathbf{1}_{\{\ell=j\}} - \frac{1}{b_k}\frac{1-\tau_k}{\tau_k}\frac{q_{\ell k}}{Q_k} \ . \end{aligned}$$

Since $\sum_{\ell=1}^{M_k} q_{\ell k} = Q_k$, it follows

$$\begin{aligned} \frac{\partial R_k(\mathbf{q})}{\partial q_{jk}} &= p_{jk} - \frac{1}{b_k\tau_k}\left[\frac{Q_k}{1-\sum_i Q_i}\right] - \frac{1}{b_k} - \frac{1}{b_k}\left(\frac{1-\tau_k}{\tau_k}\right) \\ &= p_{jk} - \frac{1}{b_k\tau_k}\left\{\left[\frac{Q_k}{1-\sum_i Q_i}\right] + 1\right\} \ . \end{aligned}$$

By setting the partial derivative of $\Gamma_k(\mathbf{q}) = R_k(\mathbf{q}) - C_k(\mathbf{q})$ to zero at $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$, we obtain

$$\hat{p}_{jk} = \frac{1}{b_k\tau_k}\left(1 + \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\right) + c_{jk} \ . \tag{19}$$

The above identity (19) is expressed in terms of both the equilibrium price and the equilibrium quantity vector. We use the relationship between the price vector and the market share vector given in (4) to express it in terms of the equilibrium quantity vector $\hat{\mathbf{q}}$ only. By equating two expressions of $\hat{p}_{jk} = p_{jk}(\hat{\mathbf{q}})$ given in (4) and (19),

$$\frac{1}{b_k\tau_k}\left(1 + \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\right) + c_{jk} = \frac{a_{jk}}{b_k} + \frac{1}{b_k}\left[\log(1-\sum_\ell \hat{Q}_\ell) - \log \hat{q}_{jk}\right] + \frac{1}{b_k}\frac{1-\tau_k}{\tau_k}\left[\log(1-\sum_\ell \hat{Q}_\ell) - \log \hat{Q}_k\right] \ .$$

Multiplying both sides by b_k and rearranging terms,

$$\begin{aligned} a_{jk} - \frac{1}{\tau_k} - b_k c_{jk} &= \frac{1}{\tau_k}\cdot\frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i} - \left[\log(1-\sum_\ell \hat{Q}_\ell) - \log \hat{q}_{jk}\right] - \left(\frac{1}{\tau_k} - 1\right)\left[\log(1-\sum_\ell \hat{Q}_\ell) - \log \hat{Q}_k\right] \\ &= \frac{1}{\tau_k}\left[\log\frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i} + \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\right] + \log\frac{\hat{q}_{jk}}{\hat{Q}_k} \ . \end{aligned}$$

Exponentiate the above equation and sum over j to obtain

$$\sum_{j=1}^{M_k} e^{a_{jk}-1/\tau_k - b_k c_{jk}} = \sum_{j=1}^{M_k}\left(\exp\left[\log\frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i} + \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\right]\right)^{\frac{1}{\tau_k}}\frac{\hat{q}_{jk}}{\hat{Q}_k} = \left(\exp\left[\log\frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i} + \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\right]\right)^{\frac{1}{\tau_k}} ,$$

where the last equality follows from $\sum_{j=1}^{M_k} \hat{q}_{jk} = \hat{Q}_k$. Thus,

$$\left(\sum_{j=1}^{M_k} e^{a_{jk}-1/\tau_k - b_k c_{jk}}\right)^{\tau_k} = \exp\left[\log\frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i} + \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\right] = \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\cdot\exp\left[\frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i}\right] \ .$$

Since the leftmost side of the above equality is $\overline{A}_k$ (see equation (7)), it follows from the definition of the Lambert W function that

$$W\left(\overline{A}_k\right) = \frac{\hat{Q}_k}{1-\sum_i \hat{Q}_i} \ . \tag{20}$$

By substituting this expression into (19), we obtain the required expression for $\hat{p}_{jk}$.

Now, from (20),

$$1+\sum_{i=1}^{K} W(\overline{A}_i) \;=\; 1+\frac{\sum_{i=1}^{K}\hat{Q}_i}{1-\sum_{i=1}^{K}\hat{Q}_i} \;=\; \frac{1}{1-\sum_{i=1}^{K}\hat{Q}_i} \,.$$

Applying this to (20), we obtain

$$\hat{Q}_k \;=\; W(\overline{A}_k)\cdot\left(1-\sum_{i}\hat{Q}_{i=1}^{K}\right) \;=\; \frac{W(\overline{A}_k)}{1+\sum_{i=1}^{K}W(\overline{A}_i)} \,,$$

which is the required expression for $\hat{Q}_k$. To get the expression for $\hat{q}_{jk}$, (2) and (3) imply

$$\hat{q}_{jk} \;=\; \hat{Q}_k\cdot\hat{q}_{j|k} \;=\; \hat{Q}_k\cdot\frac{e^{a_{jk}-b_k\hat{p}_{jk}}}{\sum_{\ell=1}^{M_\ell}e^{a_{\ell k}-b_k\hat{p}_{\ell k}}} \;=\; \hat{Q}_k\cdot\frac{e^{a_{jk}-b_k c_{jk}}}{\sum_{\ell=1}^{M_\ell}e^{a_{\ell k}-b_k c_{\ell k}}} \,,$$

where the last equality follows from the fact that $\hat{p}_{1k}=\cdots=\hat{p}_{M_k k}$ (which follows from the first part of this theorem). □

A.3 Proof of Lemma 1

Proof. We first show that $V(x)<W(x)$. For any $x\in(0,\infty)$, let $w=W(x)$ and $v=V(x)$. Then, $w\exp(w)=x=v\exp(v/(1-v))$. Since $v\in(0,1)$, we have $v/(1-v)>v$. Thus, $w\exp(w)=v\exp(v/(1-v))>v\exp(v)$. Since ye^y is strictly increasing in y, we conclude that $w>v$.

Next we show that $W(x)\le(1+\lambda)V(x/(1+\lambda))$ for $\lambda\ge W(x)$. For any $x\in(0,\infty)$, now let $w=W(x)$ and $v=V(x/(1+\lambda))$. Then, $w\exp(w)=x=(1+\lambda)\cdot v\exp(v/(1-v))$. We wish to show that $w\le(1+\lambda)v$. Let $\lambda\ge w$. Then, by rearranging terms, we obtain $w/(1+\lambda-w)\le w$, which implies

$$w\exp(w/(1+\lambda-w)) \;\le\; w\exp(w) \;=\; (1+\lambda)\cdot v\exp(v/(1-v)) \,.$$

Assume, by way of contradiction, that $w>(1+\lambda)v$. Then, the above inequality implies $\exp(w/(1+\lambda-w))\le\exp(v/(1-v))$, i.e., $w/(1+\lambda-w)\le v/(1-v)$. However, the assumption $w>(1+\lambda)v$ implies both $v<w/(1+\lambda)$ and $(1-v)>1-w/(1+\lambda)=(1+\lambda-w)/(1+\lambda)$. By dividing the first inequality with the second inequality, we obtain $v/(1-v)<w/(1+\lambda-w)$, which is a contradiction. Thus, we conclude $w\le(1+\lambda)v$, as required.

Finally, we show that $(1+\lambda)V(x/(1+\lambda))\le x$. Again let $v=V(x/(1+\lambda))$. Since $\exp(v/(1-v))\ge 1$, it follows that $x=(1+\lambda)\cdot v\exp(v/(1-v))\ge(1+\lambda)\cdot v$. □

A.4 Proof of Theorem 4

Proof. Since $q_{jk}=q_{j|k}Q_k$, it follows from (2) and the definition of $I_k=\log\sum_{j=1}^{M_k}e^{a_{jk}-b_kp_{jk}}$,

$$\begin{aligned}
\frac{\partial q_{jk}}{\partial p_{jk}} &= \frac{\partial q_{j|k}}{\partial p_{jk}}\cdot Q_k \;+\; q_{j|k}\cdot\frac{\partial Q_k}{\partial p_{jk}} \;=\; \frac{\partial q_{j|k}}{\partial p_{jk}}\cdot Q_k \;+\; q_{j|k}\cdot\frac{\partial Q_k}{\partial I_k}\cdot\frac{\partial I_k}{\partial p_{jk}}\\
&= \left\{b_kq_{jk}(q_{j|k}-1)\right\} \;+\; q_{j|k}\cdot[\tau_kQ_k(1-Q_k)]\cdot\left[-b_kq_{j|k}\right]\\
&= b_kq_{jk}(q_{j|k}-1)-b_k\tau_kq_{jk}q_{j|k}(1-Q_k) \,.
\end{aligned}$$

Similarly, for $i \neq j$, obtain $\partial q_{ik}/\partial p_{jk} = b_k q_{i|k} q_{jk} - b_k \tau_k q_{i|k} q_{jk}(1-Q_k)$. Since $\Gamma_k = \sum_{j=1}^{M_k} (p_{jk} - c_{jk}) q_{jk}$,

$$\frac{\partial \Gamma_k}{\partial p_{jk}} = q_{jk} + \sum_{i=1}^{M_k} (p_{ik} - c_{jk}) \frac{\partial q_{ik}}{\partial p_{jk}} = q_{jk} - b_k(p_{ik} - c_{jk}) q_{jk} + \sum_{i=1}^{M_k} (p_{ik} - c_{jk})[b_k q_{i|k} q_{jk} - b_k \tau_k q_{i|k} q_{jk}(1 - Q_k)] \ .$$

Setting this expression to 0 and simplifying it, it follows $b_k(\tilde{p}_{jk} - c_{jk}) = 1 + b_k[1 - \tau_k(1 - \tilde{Q}_k)] \sum_{i=1}^{M_k} (\tilde{p}_{ik} - c_{jk}) \tilde{q}_{i|k}$. (We use tilde to indicate that it is the price-competition equilibrium solution.) Since the right side of this equation is independent of j, it follows that $\tilde{p}_{1k} = \cdots = \tilde{p}_{M_k k}$, which we denote by $\tilde{p}_k$. Since $\tilde{q}_{1|k} + \cdots + \tilde{q}_{M_k|k} = 1$, we obtain $b_k(\tilde{p}_k - c_{jk}) = 1 + b_k[1 - \tau_k(1 - \tilde{Q}_k)] \cdot (\tilde{p}_{jk} - c_{jk})$. Thus, $(\tilde{p}_k - c_{jk}) = 1/\left[b_k \tau_k (1 - \tilde{Q}_k)\right]$.

We claim

$$\tilde{p}_k = \frac{1}{b_k} \log \sum_{i=1}^{M_k} \exp(a_{ik}) + \frac{1}{b_k \tau_k} \left[\log(1 - \sum_{\ell} \tilde{Q}_\ell) - \log \tilde{Q}_k \right] \ .$$

To prove this claim, substitute (2) and (3) into (4) to obtain

$$\tilde{p}_{jk} = \frac{a_{jk}}{b_k} + \frac{1}{b_k} \left[\log(1 - \sum_{\ell=1}^{K} \tilde{Q}_\ell) - \log \left(\frac{\tilde{Q}_k \cdot e^{a_{jk}}}{\sum_{i=1}^{M_k} e^{a_{ik}}} \right) \right] + \frac{1}{b_k} \frac{1 - \tau_k}{\tau_k} \left[\log(1 - \sum_{\ell=1}^{K} \tilde{Q}_\ell) - \log \tilde{Q}_k \right] .$$

Then, since $a_{jk} = \log(\exp(a_{jk}))$, the above equality implies the claim.

Therefore,

$$\frac{1}{b_k \tau_k (1 - \tilde{Q}_k)} = \tilde{p}_k - c_{jk} = \frac{1}{b_k} \log \sum_{i=1}^{M_k} \exp(a_{ik}) + \frac{1}{b_k \tau_k} \left[\log(1 - \sum_{\ell=1}^{K} \tilde{Q}_\ell) - \log \tilde{Q}_k \right] - c_{jk} \ .$$

From algebraic transformation and the definition of $\overline{A}_k$ in (7),

$$\log \frac{\tilde{Q}_k}{1 - \sum_\ell \tilde{Q}_\ell} + \frac{\tilde{Q}_k}{1 - \tilde{Q}_k} = \tau_k \log \sum_i \exp(a_{ik} - b_k c_{jk}) - 1 = \log \overline{A}_k \ . \tag{21}$$

Exponentiating the above equation and using $\tilde{Q}_0 = 1 - \sum_\ell \tilde{Q}_\ell$, we obtain $(\tilde{Q}_k/\tilde{Q}_0) \cdot \exp\left(\frac{\tilde{Q}_k}{1-\tilde{Q}_k}\right) = \overline{A}_k$. Then, from the definition of the V function, we have $\tilde{Q}_k = V((\overline{A}_k)\tilde{Q}_0)$. From $\tilde{Q}_0 + \sum_\ell \tilde{Q}_\ell = 1$, we verify that $\tilde{Q}_0$ satisfies (10).

Finally, for the required expression for $\tilde{q}_{jk}$, use (2) and (3) along with $\tilde{Q}_k = V(\overline{A}_k \tilde{Q}_0)$ and the fact $\tilde{p}_{1k} = \cdots = \tilde{p}_{M_k k}$. For the required expression for $\tilde{p}_{jk}$, use the above expression for $\tilde{Q}_k$ as well as $(\tilde{p}_k - c_{jk}) = 1/[b_k \tau_k (1 - \tilde{Q}_k)]$. □

A.5 Proof of Corollary 2

Proof. We first compare the quantity-competition equilibrium solution to the optimal monopoly solution. It follows from the expressions of p^*_{jk} and $\hat{p}_{jk}$ given in the statements of Theorems 2 and 3 that it suffices to show $b_k \tau_k \rho^* > W(\overline{A}_k)$.From equation (8) in Theorem 2,

$$\rho^* = \sum_{\ell=1}^{K} \frac{e^{-1}}{b_\ell \tau_\ell} \left(\sum_{j=1}^{M_\ell} e^{a_{jl} - b_\ell c_{j\ell} - b_\ell \rho^*} \right)^{\tau_\ell} > \frac{e^{-1}}{b_k \tau_k} \left(\sum_{j=1}^{M_k} e^{a_{jk} - b_k c_{jk} - b_k \rho^*} \right)^{\tau_k} = \frac{e^{-1}}{b_k \tau_k} \left(\sum_{j=1}^{M_k} e^{a_{jk} - b_k c_{jk}} \right)^{\tau_k} e^{-b_k \tau_k \rho^*} \ .$$

Thus,

$$(b_k\tau_k\rho^*)\cdot\exp\left(b_k\tau_k\rho^*\right) \;>\; e^{-1}\left(\sum_{j=1}^{M_k} e^{a_{jk}-b_k c_{jk}}\right)^{\tau_k} \;=\; \overline{A}_k \;=\; W(\overline{A}_k)\cdot\exp\big(W(\overline{A}_k)\big)\;,$$

where the first equality follows from the definition of $\overline{A}_k$ given in (6), and the second equality follows from the definition of the W function. Now, since ye^y is increasing in y, we have

$$b_k\tau_k\rho^* \;>\; W(\overline{A}_k) \tag{22}$$

where the second inequality above follows since W is increasing. Thus, we conclude $p^*_{jk} > \hat{p}_{jk}$.

Since $p^*_{jk} > \hat{p}_{jk}$ holds for any j and k, it follows that $\hat{I}_k > I^*_k$ for each k, where $\hat{I}_k = \log\sum_{j=1}^{M_k} e^{a_{jk}-b_k\hat{p}_{jk}}$ and $I^*_k = \log\sum_{j=1}^{M_k} e^{a_{jk}-b_k p^*_{jk}}$. Thus, from the definition of $Q_k = e^{\tau_k I_k}/[1+\sum_{\ell=1}^K e^{\tau_\ell I_\ell}]$ in equation (2), it implies that

$$\hat{Q}_0 \;=\; 1-\sum_{k=1}^K \hat{Q}_k \;=\; \frac{1}{1+\sum_{k=1}^K e^{\tau_k\hat{I}_k}} \;<\; \frac{1}{1+\sum_{k=1}^K e^{\tau_k I^*_k}} \;=\; 1-\sum_{k=1}^K Q^*_k \;=\; Q^*_0\;.$$

Now, we compare the price-competition equilibrium to the quantity-competition equilibrium. Let $\psi(Q_0) = Q_0 + \sum_\ell V(\overline{A}_\ell Q_0) - 1$. Since V is an increasing function, ψ is also increasing. It follows from the definition of $\tilde{Q}_0$ that $\psi(\tilde{Q}_0) = 0$. Since $\hat{Q}_k = 1/\left[1+\sum_{\ell=1}^K W(\overline{A}_\ell)\right]$ by Theorem 3, Thus

$$\begin{aligned}\psi(\hat{Q}_0)-\psi(\tilde{Q}_0) \;&=\; \frac{1}{1+\sum_\ell W(\overline{A}_\ell)} + \sum_\ell V\left(\frac{\overline{A}_\ell}{1+\sum_i W(\overline{A}_i)}\right) - 1 \\ &=\; \sum_\ell\left[V\left(\frac{\overline{A}_\ell}{1+\sum_i W(\overline{A}_i)}\right) - \frac{W(\overline{A}_\ell)}{1+\sum_i W(\overline{A}_i)}\right] \;\geq\; 0\;,\end{aligned}$$

where the inequality follows from Lemma 1 by letting $x = \overline{A}_\ell$ and $\lambda = \sum_i W(\overline{A}_i)$. Thus $\psi(\hat{Q}_0) \geq \psi(\tilde{Q}_0)$. Since ψ is increasing, we obtain $\hat{Q}_0 \geq \tilde{Q}_0$.

Finally we show $\tilde{p}_{jk} \leq \hat{p}_{jk}$ for any j and k. From the above argument,

$$W(\overline{A}_k)\cdot\hat{Q}_0 \;=\; \frac{W(\overline{A}_k)}{1+\sum_\ell W(\overline{A}_\ell)} \;\leq\; V\left(\frac{\overline{A}_k}{1+\sum_\ell W(\overline{A}_\ell)}\right) \;=\; V(\overline{A}_k\cdot\hat{Q}_0)\;.$$

It implies, along with the definitions of $V(\overline{A}_k\hat{Q}_0)$ and $W(\overline{A}_k)$,

$$\exp\left(\frac{V(\overline{A}_k\hat{Q}_0)}{1-V(\overline{A}_k\hat{Q}_0)}\right) \;=\; \frac{\overline{A}_k\cdot\hat{Q}_0}{V(\overline{A}_k\hat{Q}_0)} \;\leq\; \frac{\overline{A}_k}{W(\overline{A}_k)} \;=\; \exp(W(\overline{A}_k))\;.$$

Since $\exp(\cdot)$ is an increasing function,

$$\begin{aligned}0 \;\leq\; W(\overline{A}_k) - \frac{V(\overline{A}_k\hat{Q}_0)}{1-V(\overline{A}_k\hat{Q}_0)} \;&=\; b_k\tau_k\left\{\left[\frac{1+W(\overline{A}_k)}{b_k\tau_k}\right] - \left[\frac{1}{b_k\tau_k(1-V(\overline{A}_k\hat{Q}_0))}\right]\right\} \\ &=\; b_k\tau_k\left\{\hat{p}_{jk}-\tilde{p}_{jk}\right\}\;,\end{aligned}$$

where the last equality follows from the expressions of $\hat{p}_{jk}$ and $\tilde{p}_{jk}$ given in Theorem 3 and Theorem 4. Thus, $\hat{p}_{jk} \geq \tilde{p}_{jk}$.

□

A.6 Proof of Corollary 3

Proof. Part (a). From Theorem 2, we know that the monopoly optimal price of products within branch k are the same, i.e., $p^*_{1k} = \cdots = p^*_{M_k k}$. Similarly, from Theorem 3, we also know $\hat{p}_{1k} = \cdots = \hat{p}_{M_k k}$. Thus, the conditional probability $q_{j|k} = e^{a_{jk}} / \sum_{i=1}^{M_k} e^{a_{ik}}$ remains the same under the oligopoly and under the monopoly. Since $q_{jk} = q_{j|k} Q_k$, it is sufficient to show the existence of $\hat{k}^*_Q \in \{1, 2, \dots, K\}$ such that

$$\begin{cases} \hat{Q}_k \geq Q^*_k & \text{if } k \in \{1, \dots, \hat{k}^*_Q\} \\ \hat{Q}_k \leq Q^*_k & \text{if } k \in \{\hat{k}^*_Q + 1, \dots, K\} \ . \end{cases}$$

We will establish this by proving (i) $\hat{Q}_1 \geq Q^*_1$, and that (ii) $\hat{Q}_k \leq Q^*_k$ implies $\hat{Q}_{k+1} \leq Q^*_{k+1}$.

From the expression of q^*_{jk} in Theorem 2 and the fact that $Q^*_k = \sum_{j=1}^{M_k} q^*_{jk}$,

$$\begin{aligned} Q^*_k &= \frac{[\sum_{i=1}^{M_k} e^{a_{ik} - b_k c_{ik} - b_k \rho^*}]^{\tau_k}}{e + \sum_{\ell=1}^{K} [\sum_{j=1}^{M_\ell} e^{a_{jl} - b_\ell c_{j\ell} - b_\ell \rho^*}]^{\tau_\ell}} \\ &= \frac{e^{-1} [\sum_{i=1}^{M_k} e^{a_{ik} - b_k c_{ik}}]^{\tau_k}}{e^{b_k \tau_k \rho^*} + \sum_{\ell=1}^{K} e^{-1} [\sum_{j=1}^{M_\ell} e^{a_{j\ell} - b_\ell c_{j\ell}}]^{\tau_\ell} e^{(b_k \tau_k - b_\ell \tau_\ell)\rho^*}} = \frac{\overline{A}_k}{e^{b_k \tau_k \rho^*} + \sum_{\ell=1}^{K} \overline{A}_\ell \cdot e^{(b_k \tau_k - b_\ell \tau_\ell)\rho^*}} \ , \end{aligned} \tag{23}$$

where the last equality follows from the definition of $\overline{A}_\ell = e^{-1} \left[\sum_{j=1}^{M_\ell} e^{a_{jl} - b_\ell c_{j\ell}}\right]^{\tau_\ell}$ given in (7). Also, from Theorem 3,

$$\begin{aligned} \hat{Q}_k &= \frac{W(\overline{A}_k)}{1 + \sum_{\ell=1}^{K} W(\overline{A}_\ell)} = \frac{\overline{A}_k \exp(-W(\overline{A}_k))}{1 + \sum_{\ell=1}^{K} A_i \exp(-W(\overline{A}_\ell))} \\ &= \frac{\overline{A}_k}{\exp(W(\overline{A}_k)) + \sum_{\ell=1}^{K} A_\ell \exp(W(\overline{A}_k) - W(\overline{A}_\ell))} \ , \end{aligned} \tag{24}$$

where the second equality holds due to the definition of the Lambert W function that $W(z) = z e^{-W(z)}$.

Suppose $k = 1$. Recall from (22) that $b_k \tau_k \rho^* > W(\overline{A}_k)$. Since both $b_1 \tau_1 \geq b_\ell \tau_\ell$ and $\overline{A}_1 \leq \overline{A}_\ell$ hold by assumption, we obtain $e^{(b_1 \tau_1 - b_\ell \tau_\ell)\rho^*} \geq e^0 \geq e^{W(\overline{A}_1) - W(\overline{A}_\ell)}$. Thus, by comparing (23) and (24), we have $\hat{Q}_1 \geq Q^*_1$, which is statement (i).

Now, for statement (ii), suppose $k \in \{1, \dots, K-1\}$. Then,

$$\begin{aligned} \overline{A}_{k+1} \cdot \left[\frac{1}{Q^*_{k+1}} - \frac{1}{\hat{Q}_{k+1}}\right] &= \left\{\exp(b_{k+1}\tau_{k+1}\rho^*) + \sum_{\ell=1}^{K} \overline{A}_\ell \exp((b_{k+1}\tau_{k+1} - b_\ell \tau_\ell)\rho^*)\right\} \\ &\quad - \left\{\exp(W(\overline{A}_{k+1})) + \sum_{\ell=1}^{K} \overline{A}_\ell \exp(W(\overline{A}_{k+1}) - W(\overline{A}_\ell))\right\} \\ &\leq \left\{\exp(b_k \tau_k \rho^*) + \sum_{\ell=1}^{K} \overline{A}_\ell \exp((b_k \tau_k - b_\ell \tau_\ell)\rho^*)\right\} \\ &\quad - \left\{\exp(W(\overline{A}_k)) + \sum_{\ell=1}^{K} \overline{A}_i \exp(W(\overline{A}_k) - W(\overline{A}_\ell))\right\} \\ &= \overline{A}_k \cdot \left[\frac{1}{Q^*_k} - \frac{1}{\hat{Q}_k}\right] \ , \end{aligned}$$

where the both equalities follow from (23) and (24), and the inequality follows from the assumptions that $\overline{A}_k \leq \overline{A}_{k+1}$ and $b_k\tau_k \geq b_{k+1}\tau_{k+1}$. Thus, if $\hat{Q}_k \leq Q^*_k$, then this inequality implies $\hat{Q}_{k+1} \leq Q^*_{k+1}$, yielding statement (ii). This completes the proof of Part (a).

Part (b). Following the same argument as (a), it suffices to show the existence of $\hat{\bar{k}}_Q \in \{0,1,2,\ldots,K\}$ such that

$$\begin{cases} \hat{Q}_k \geq \tilde{Q}_k & \text{if } k \in \{1,\ldots,\hat{\bar{k}}_Q\} \\ \hat{Q}_k \leq \tilde{Q}_k & \text{if } k \in \{\hat{\bar{k}}_Q+1,\ldots,K\}\ . \end{cases}$$

Since $\sum_\ell \hat{Q}_\ell \leq \sum_\ell \tilde{Q}_\ell$ (Corollary 2), there exists at least one firm j such that $\hat{Q}_j \leq \tilde{Q}_j$ Thus, it suffices to show that $\hat{Q}_k \leq \tilde{Q}_k$ implies $\hat{Q}_{k+1} \leq \tilde{Q}_{k+1}$, for any $k \in \{1,\ldots,K-1\}$.

From (21) and (20) along with the definition of the W function,

$$\log\frac{\tilde{Q}_{k+1}}{\tilde{Q}_k} + \frac{\tilde{Q}_{k+1}-\tilde{Q}_k}{(1-\tilde{Q}_{k+1})(1-\tilde{Q}_k)} \;=\; \log\overline{A}_{k+1} - \log\overline{A}_k \;=\; \log\frac{\hat{Q}_{k+1}}{\hat{Q}_k} + \frac{\hat{Q}_{k+1}-\hat{Q}_k}{1-\sum_\ell \hat{Q}_\ell}$$

Assume, by a contradiction, we have $\hat{Q}_k \leq \tilde{Q}_k$ but $\hat{Q}_{k+1} > \tilde{Q}_{k+1}$. Then, since $\tilde{Q}_{k+1}/\tilde{Q}_k < \hat{Q}_{k+1}/\hat{Q}_k$, we must have

$$\frac{(\tilde{Q}_{k+1}-\tilde{Q}_k)}{(1-\tilde{Q}_{k+1})(1-\tilde{Q}_k)} \;>\; \frac{\hat{Q}_{k+1}-\hat{Q}_k}{1-\sum_\ell \hat{Q}_\ell} \;\geq\; \frac{(\hat{Q}_{k+1}-\hat{Q}_k)}{(1-\hat{Q}_k-\hat{Q}_{k+1})}$$

Since $\hat{Q}_k \leq \tilde{Q}_k \leq \tilde{Q}_{k+1} < \hat{Q}_{k+1}$ (where the middle inequality comes from Theorem 4 and the condition $A_k \leq A_{k+1}$), each of the expressions in any bracket is nonnegative. Thus, the ζ value, defined below, should be positive:

$$\zeta \;=\; (\tilde{Q}_{k+1}-\tilde{Q}_k)\cdot(1-\hat{Q}_k-\hat{Q}_{k+1}) - (\hat{Q}_{k+1}-\hat{Q}_k)\cdot(1-\tilde{Q}_{k+1})\cdot(1-\tilde{Q}_k)\ .$$

The ζ value, as a function of $\tilde{Q}_k$ is linear and the value of $\tilde{Q}_k$ should belong to the interval $[\hat{Q}_k, \tilde{Q}_{k+1}]$. At the left endpoint of this interval,

$$\zeta\big|_{\tilde{Q}_k=\hat{Q}_k} \;=\; (\tilde{Q}_{k+1}-\hat{Q}_k)\cdot(1-\hat{Q}_k-\hat{Q}_{k+1}) - (\hat{Q}_{k+1}-\hat{Q}_k)\cdot(1-\tilde{Q}_{k+1})\cdot(1-\hat{Q}_k)\ .$$

This expression is non-positive since $\tilde{Q}_{k+1} < \hat{Q}_{k+1}$ implies both $(\tilde{Q}_{k+1}-\hat{Q}_k) \leq (\hat{Q}_{k+1}-\hat{Q}_k)$ and $(1-\hat{Q}_k-\hat{Q}_{k+1}) \leq (1-\hat{Q}_k-\tilde{Q}_{k+1}) \leq (1-\hat{Q}_k)(1-\tilde{Q}_{k+1})$. Now, at the right endpoint of the interval,

$$\zeta\big|_{\tilde{Q}_k=\tilde{Q}_{k+1}} \;=\; (\tilde{Q}_{k+1}-\tilde{Q}_{k+1})\cdot(1-\hat{Q}_k-\hat{Q}_{k+1}) - (\hat{Q}_{k+1}-\hat{Q}_k)\cdot(1-\tilde{Q}_{k+1})\cdot(1-\tilde{Q}_{k+1})\ ,$$

which simplifies to $-(\hat{Q}_{k+1}-\hat{Q}_k)\cdot(1-\tilde{Q}_{k+1})^2$, which is a non-positive quantity. Then, by the linearity of ζ in $\tilde{Q}_k$, we conclude that the ζ value is always non-positive, and this is a desired contradiction. □

A.7 Proof of Equation (12)

Proof. Now, taking a partial derivative of (11), we obtain

$$\frac{\partial J^t(\mathbf{x},\mathbf{q})}{\partial q_k} \;=\; \lambda\cdot\left[\frac{\partial R(\mathbf{q})}{\partial q_k} \;-\; \Delta_k V^{t-1}(\mathbf{x})\right] \;=\; \lambda\cdot\left[p_k(\mathbf{q}) - \frac{1}{b_k} - \sum_i \frac{q_i}{b_i}\left(\frac{1}{1-\sum_\ell q_\ell}\right) \;-\; \Delta_k V^{t-1}(\mathbf{x})\right],$$

where the last equality follows from the proof of Theorem 2. Let $\check{\mathbf{q}}^t(\mathbf{x})$ denote the choice of $\mathbf{q}$ that maximizes $J^t(\mathbf{x}, \mathbf{q})$ for given $\mathbf{x}$. (For notational simplicity, we omit the superscript t when there is no ambiguity.) By setting the above partial derivative to zero, we obtain

$$p_k(\check{\mathbf{q}}) \;\; = \;\; \Delta_k V^{t-1}(\mathbf{x}) \; + \; \sum_i \frac{\check{q}_i}{b_i} \left(\frac{1}{1 - \sum_\ell \check{q}_\ell} \right) \; + \; \frac{1}{b_k} \; .$$

Multiplying the above identify by $\check{q}^*$ and summing over all products, we obtain

$$\begin{aligned} \sum_k p_k(\check{\mathbf{q}}) \check{q}_k \;\; &= \;\; \sum_k \frac{\check{q}_k}{b_k} \; + \; \frac{\sum_k \check{q}_k}{1 - \sum_\ell \check{q}_\ell} \sum_i \frac{\check{q}_i}{b_i} \; + \; \sum_k \check{q}_k \Delta_k V^{t-1}(\mathbf{x}) \\ &= \;\; \sum_k \frac{\check{q}_k}{b_k} \left(1 + \frac{\sum_k \check{q}_k}{1 - \sum_\ell \check{q}_\ell} \right) \; + \; \sum_k \check{q}_k \Delta_k V^{t-1}(\mathbf{x}) \\ &= \;\; \sum_k \frac{\check{q}_k}{b_k} \left(\frac{1}{1 - \sum_\ell \check{q}_\ell} \right) \; + \; \sum_k \check{q}_k \Delta_k V^{t-1}(\mathbf{x}) \; . \end{aligned}$$

Since we set $\check{r}(\mathbf{x}, \check{\mathbf{q}}) \; = \; \sum_k \left\{ p_k(\check{\mathbf{q}}) - V^{t-1}(\mathbf{x}) \right\} \cdot \check{q}_k$, it follows that $\check{r}(\mathbf{x}, \check{\mathbf{q}}) \; = \; \sum_k \frac{\check{q}_k}{b_k} \left(\frac{1}{1 - \sum_\ell \check{q}_\ell} \right)$.

Thus, we obtain $p_k(\check{\mathbf{q}}) \; = \; \Delta_k V^{t-1}(\mathbf{x}) \; + \; \check{r}(\mathbf{x}, \check{\mathbf{q}}) \; + \; \frac{1}{b_k}$, which is (12). $\square$