Pricing Decisions during Inter-generational Product Transition

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Abstract

How should companies price products during an inter-generational transition? High uncertainty in a new product introduction often leads to extreme cases of demand and supply mismatches. Pricing is an effective tool to either prevent or alleviate these problems. We study the optimal pricing decisions in the context of a product transition in which a new generation product replaces an old one. We formulate the dynamic pricing problem and derive the optimal prices for both the old and new products. Our analysis sheds light on the pattern of the optimal prices for the two products during the transition and on how product replacement, along with several other dynamics including substitution, external competition, scarcity and inventory, affect the optimal prices. We also determine the optimal initial inventory for each product and discuss a heuristic method.

Keywords: dynamic pricing; product transition; new product introduction; Multinomial Logit model

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1. Introduction

In high-tech industries, a company periodically replaces the current product with a newer generation product. In many cases, this transition does not occur instantaneously but rather involves a transition period during which the company sells both products. The introduction of a new product creates high uncertainty in both demand and the supply. If many new features are added to the new generation product, it is difficult to predict its acceptance by the customers. Furthermore, there is uncertainty in how smoothly the suppliers handle the technological or production changes for the new product, which often results in delays in the new product release.

This paper is motivated by collaborative work with a telecommunications equipment manufacturer. In this industry, the replenishment lead time is about 18 weeks: 13 weeks for procuring components plus 5 weeks for production and testing at the contract manufacturer. A product transition starts with the release of the new product and ends with the old product demand dropping to a negligible level, and usually lasts from a few weeks to a few months. With an 18-week lead time, any replenishment order placed during the transition will not arrive before the transition ends. Therefore, once the transition starts, there is little chance to correct the initial inventory decisions, even if a demand-supply mismatch becomes evident. Consequently, the company often runs out of the product that customers want while having excess of the other.

For example, a chip supplier issued an end-of-life notice for the chipset used in one of the company's wireless products, Blofeld, which drove the company to introduce the next generation product Blofeld II. The company had expected the transition to be quick and did not stock up inventory for the old chipset. Unfortunately, there were unanticipated design issues around the new chipset that delayed the release of Blofeld II. Consequently, the company kept selling Blofeld long after the scheduled release date of Blofeld II, creating a shortage of Blofeld. To counteract such supply risks, operations managers tend to add large inventories for the old product. However, a generous supply cushion can result in excess inventory of the old product at the end of the transition, which was the case for another transition at the same company. This time, they were phasing out a product with high sales volume and had purchased a large inventory for the old product to avoid any supply gap and lost sales. Ironically, when the new generation product was delivered on schedule, it left the managers in another dilemma. If they were to release the new product, Sultan II, as scheduled, they might be stuck with excess inventory of Sultan, the old product. If they delayed the release of Sultan II, they would avoid this problem; but this would be a costly option because Sultan II had a better margin than Sultan. Eventually the company decided to delay the new product introduction in selected sales regions and forego some of the margin benefits to alleviate the excess problem.

The countermeasures for addressing a demand-supply mismatch during a product transition are very limited due to the long lead time. We study in Li et al. (2010) the option of product substitution: When one product is depleted, a company may offer the other one as a substitute. Pricing is another option: The managers can manipulate the prices of the two products to mitigate the risk of demand

and supply mismatch. If sales of the old product are sluggish during the transition, they could discount it. If the new product does not sell well, managers may increase the price of the old product to make the new appear more attractive.

In this paper, we study the single-firm pricing problem for a product transition in which the firm introduces a new product to replace an existing, old product. We term this an inter-generational product transition, and contrast this with the case of a "completely new" product that has no direct predecessor and is designed to meet a new set of customer needs. For example, the replacement of Canon's camera model PowerShot SD700 by SD800 is an inter-generational transition, whereas the first introduction of the "iPhone" is not. While both cases are important for business success, the former is a day-to-day problem facing decision makers in technology companies and is the focus of this paper.

We consider a finite time horizon that starts when the new product is introduced and that ends when the transition finishes, namely when the demand of the old product has dropped to a negligible level. We are given or determine the initial inventories of the old and new products, and there is no option for replenishment during the transition. In the transition, the demand of the old product gradually phases out while the new phases in and will continue to be sold beyond the transition. Due to the similarities of the two products, any pricing decision for one product affects the demand of both products. Therefore, we determine the optimal prices for the two products simultaneously, as a function of time and inventory.

This pricing problem differs from previously studied dynamic pricing problems in that the two products not only compete with each other as two substitute products, but one is on a path to replace the other. As such we need to adapt existing demand models to capture this phenomenon of product replacement. The existing literature on multi-product pricing problems, as we review in the next section, treats product value as constant over time and allows the products' demands to vary with prices. In contrast, during a product transition, the old product becomes less attractive over time whereas the newer generation becomes more attractive; as customers learn about and gain confidence in the new product, they increasingly view the old product as obsolete. In contrast to the existing literature, we must determine the dynamic prices, accounting for the substitution between the old and the new products as well as this dynamic change in tastes.

In addition to solving for the optimal dynamic prices, we study the effect on the optimal price trajectories of the two products due to product replacement, due to product substitution, due to external competition, due to scarcity, and due to inventory.

We find that the transition from the old product to the new one forces a company to price both products lower during the transition than outside of the transition window due to the replacement effect. We also find that the customers' preference for the no-purchase option (buying neither the old nor the new product) can affect the optimal prices significantly and may dominate the price trend. The larger a company's market share, the more it is affected by changes in the no-purchase option.

In addition, we demonstrate how certain product or market characteristics, such as the speed of the transition, customers' price sensitivity, and the speed of product obsolescence, affect the pricing decisions. For instance, contrary to our intuition, high price sensitivity leads to less price swings during the transition due to the effect of product replacement. Previous dynamic-pricing literature has not considered product replacement, and hence does not find this dynamic.

Proper inventory planning for product transition is critical as shown in the motivating examples. In this paper, we also consider the initial inventory decision along with the dynamic pricing problem in a product transition.

The remainder of this paper is organized as follows: In Section 2, we review the relevant literature. In Section 3, we specify the problem and the demand model. We solve for the optimal prices in Section 4 and identify the key dynamics influencing the optimal prices in Section 5. In Section 6, we determine the optimal initial inventory and present a heuristic method. We conclude with a discussion on the limitations of the model and future research possibilities. The proofs are in the Appendix, available as Online Supplement.

2. Literature Review

Gallego and van Ryzin (1997) is one of the first to study dynamic pricing problems for multiple products. They do not explicitly model the demand relationships among the products but instead assume a generic set of demand functions. Bitran and Caldentey (2003) also give a generic formulation of the multiple-product pricing problem and provide an optimality condition. In general, these generic formulations can say very little about the optimal policy. Maglaras and Meissner (2006) extend the Gallego and van Ryzin (1997) model to consider a joint revenue management and capacity allocation problem for multiple products that share capacity. Zhang and Cooper (2009) consider the pricing problem for substitutable flights for an airline, using a similar demand model. Our paper differs from this literature in that we need to model both the substitution between the two generations of products, and the product replacement as the demand transitions from the old product to the new product. Thus, we need to formulate and consider demand models that allow the integration of both effects.

Existing literature considers several ways to model demands for substitutable products. The Multinomial Logit (MNL) model is first proposed by Luce (1959). Under the MNL model, the customer's purchase probability ρ_i depends on his/her utility from each product u_i through $\rho_i = e^{u_i} / \sum_k e^{u_k}, k \in K$ where K is the set of possible customer options. The MNL model has been used to predict individual choices (McFadden 1986), as well as aggregate market share for new products (Berry 1994).

Many pricing models for multiple products use the MNL choice model. However, we are not aware of any papers that consider product replacement as occurs during an inter-generational product transition.[1] This product replacement typically induces two contrasting S-shaped demand functions, as the new product replaces the old. This phenomenon creates interesting pricing dynamics that have not been previously studied by researchers.

Hanson and Martin (1996) study the product line pricing decisions using the MNL model and show that the profit function is not quasi-concave in prices. They use a "path-following" computation approach to obtain the optimal prices. It is a single-period model and they do not consider inventory in the pricing decision. Aydin and Porteus (2008) study a single-period multi-product newsvendor problem with both inventory and pricing decisions. They show that the first-order condition leads to a unique optimal solution if the objective function is separately quasi-concave in each product price and the second order cross partials are zero at the points where the first order condition is met. We expand their method to address a multi-period dynamic pricing problem.

Song and Xue (2007) study a multi-period pricing problem with a replenishment decision in each time period, using several alternative demand models including the MNL model. They show that the value function is concave in the market shares and that the optimal decision in each period consists of a not-to-order list, base stock levels, and target market shares. In our model, there is a single replenishment opportunity at the beginning of the planning horizon and there are no opportunities for subsequent replenishments within the transition period, due to the long lead time.

Dong et al. (2009) use the MNL model to study dynamic pricing decisions for substitutable products with a single selling season. As in Song and Xue (2007), they consider market share as decision variables. They identify inventory scarcity and product quality difference as the two key driving forces for the optimal prices. They assume product quality to be constant and show that in the absence of inventory scarcity, dynamic pricing is unnecessary. Suh and Aydin (2009) consider a similar problem, focusing on the analytical insights on the effect of the remaining time and a product's

[1] There has been previous research on revenue management with diminishing product values, but not with MNL demand (for example, Zhao and Zheng 2000, Su 2007, Aviv and Pazgal 2008).

own stock level, as well as the substitute product's stock level on pricing. They too assume that the inherent product quality is constant over time and thus the changing price of a product can be attributed entirely to the scarcity of this product, or the substitute product, or both. In contrast, we show that the optimal prices of the two successive product generations exhibit time-dependent behavior even in the absence of the inventory scarcity effect. We derive a recursive formula of the optimal prices for any given time and inventory and examine how the replacement of one generation of product by the other affects the optimal pricing decisions. In particular, we find that when we exclude other factors such as inventory scarcity and competition, the inter-generational replacement causes the prices of both products to decrease initially, followed by a gradual price recovery; this behavior is driven by the time-varying valuation of the two products in the transition. Furthermore, we examine how certain product or market characteristics such as the speed of the product transition, the speed of product obsolescence, and the customers' price sensitivity affect the optimal pricing decision.

Other papers related to our work include Kornish (2001), who studies the pricing problem for a monopolist with frequent product upgrades but sells only the latest generation of product in any period, and Ferguson and Koenigsberg (2007), who use a linear aggregate demand model to derive the optimal pricing and stocking decisions when a company sells the newly-replenished units in the presence of left-over units from last period. Goettler and Gordon (2009) study the dynamic pricing and investment decisions for a product that competes in a duopoly with a MNL demand model. Xu and Hopp (2004) consider a pricing problem for a single product with one or multiple retailers and derive the equilibrium pricing policies for the retailers. In this paper, we study the *centralized* pricing decision, i.e., a company that sells two generations of products and thus has to maximize the total expected profit from the two products.

3. Problem Description and Demand Model

We present a dynamic programming model that addresses the pricing decisions for a product upgrade during the transition period of length T. We make the following assumptions:

(i) The transition from the old product to the new one starts at time 0 and is completed within time T; thus by time T the demand of the old product has become negligible.

(ii) There is no option for inventory replenishment during the period $[0, T]$.

Given the above, we solve for the optimal prices for the old and new products during the transition period $[0, T]$ as a function of both time and inventory. The justification for (ii) is that the replenishment time is long relative to the transition period.

We adopt the Multinomial Logit (MNL) consumer choice model, whereby a customer chooses one option among a set of alternatives. In this case, the options are the old product, the new product, and no purchase. Assume that a customer's utility of purchasing product i (i=1 refers to the old product and i=2 refers to the new product) at time t is $u_i(r_i,t) = a_i(t) - g(r_i) + \varepsilon_i$, where r_i is the selling price of product i, $g(r_i)$ is the disutility of paying r_i and $a_i(t)$ is the time-varying attribute(s) that affects the customer's utility. We also assume that a customer's utility of not purchasing any product at time t is $u_0(t) + \varepsilon_0$. If the disturbances $\varepsilon_i, i = 0,1,2$ are independent and identically distributed Gumbel random variables with distribution function $F(x) = e^{-e^{-x}}$, the MNL model gives the probability that a customer purchases product i at time t: $\rho_i(\mathbf{r},t) = \dfrac{e^{a_i(t)-g(r_i)}}{e^{a_1(t)-g(r_1)} + e^{a_2(t)-g(r_2)} + e^{u_0(t)}} \quad i = 1,2,$ where $\mathbf{r} = (r_1, r_2)$ is the price vector; then the probability of no-purchase at time t is $\rho_0(\mathbf{r},t) = 1 - \rho_1(\mathbf{r},t) - \rho_2(\mathbf{r},t)$ (McFadden 1973). By using the same $g(\cdot)$ function for both products, we are assuming that a customer's disutility toward price, or equivalently the utility toward money, is the same for both products. The no-purchase option allows us to explore the pricing problem in a monopolistic situation, as well as in a competitive market. In the monopoly case, we interpret the no-purchase utility as the customer's utility of not obtaining any product; in the case with competition, the no-purchase utility equates to the customer's reservation utility for other market options (buying a competitor's product) or not buying any product, whichever is higher. We consider both time-invariant and time-increasing $u_0(t)$.

Economists developed the MNL model to describe an individual's choice behavior when facing a set of alternatives. Melnikov (2001) introduces time-related attributes to the choice model to model the inter-temporal demand substitution in computer and printer products. In this paper, we introduce an attribute term $a_i(t)$ to characterize the time-varying customer preference for the old and new products. As customers shift their preference from the old product to the new one, the impact on demand is reflected through an increase in $a_2(t)$ and a decrease in $a_1(t)$.

Existing OM papers that use the MNL model often assume constant a_i and u_0, and $g(r_i) = r_i$ throughout the planning horizon. In order to capture the unique dynamics in product transition, we allow for time-dependent $a_i(t)$ and $u_0(t)$, and use a more general $g(r_i)$. Realistically, the utility function may depend strongly on factors other than time and price such as the advertising effort or complementary product offerings. We do not differentiate these factors and treat them as either a constant term (if they do not vary with time) in $a_i(t)$, or simply another generic force that contributes to the time trend in $a_i(t)$.

Assumption 1. *We assume that customer arrival is a homogeneous Poisson process with rate* λ.

Hence the old and new products have Poisson demand with time-varying rates

$$\lambda_i(\mathbf{r},t) = \lambda\rho_i(\mathbf{r},t) = \lambda \frac{e^{a_i(t)-g(r_i)}}{e^{a_i(t)-g(r_i)} + e^{a_j(t)-g(r_j)} + e^{u_0(t)}} \quad i = 1,2 \tag{1}$$

and the no-purchase rate is

$$\lambda\rho_0(\mathbf{r},t) = \lambda(1-\rho_1(\mathbf{r},t)-\rho_2(\mathbf{r},t)). \tag{2}$$

This demand model is relatively simple and intuitive. The fact that the time factor and the price factor are separable within each exponent term leads to significant analytical tractability. In addition, it generates a logistic demand pattern that is often observed in practice. For example, when

$a_1(t) = 5 - 0.0125t$, $a_2(t) = 0.0125t$, $u_0(t) = 0$, $g(r) = r$, $r_1 = r_2 = 3$ and $\lambda = 0.1$, the above demand model generates a demand pattern shown in Figure 1.

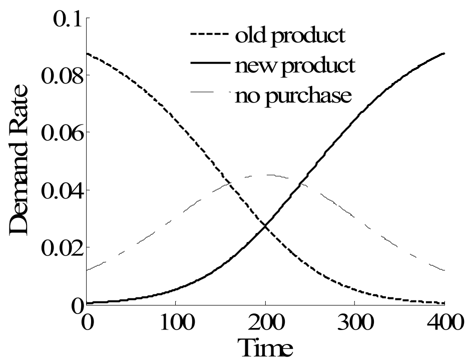


Figure 1: Demand Pattern under Equation (1)

We observe from Figure 1 that, for a time-invariant reservation utility $u_0(t)$, the total demand rate of the two products drops to the lowest level when the two products have equal market shares. This represents a period of time when the customers' preference for the old product is significantly reduced by the introduction of the new product, while the new product itself has not yet gained full acceptance from the customers. As a result, customers cannot decide which product to buy and thus are more likely to resort to the no-purchase option.

When a product stocks out, a customer then chooses between buying the other product and the no-purchase option. Thus, the purchase probabilities when a particular product runs out are:

$$\bar{\rho}_i(r_i,t) = \frac{e^{a_i(t)-g(r_i)}}{e^{a_i(t)-g(r_i)} + e^{u_0(t)}} \text{ and } \bar{\rho}_0(r_i,t) = \frac{e^{u_0(t)}}{e^{a_i(t)-g(r_i)} + e^{u_0(t)}}, \ i = 1,2. \quad t \in [0,T] \tag{3}$$

Implicitly, we assume that running out of a product is equivalent to setting an infinitely high price for that product (i.e., $\lim_{r\to\infty} g(r) = \infty$). Consequently, its demand is proportionally split between the other product and the no-purchase option. This also follows directly from the Independence from Irrelevant Alternatives (IIA) property of the MNL model, which states that the ratio of any two choices within a choice set is not affected by the presence of other choices. In this case, IIA implies that $\bar{\rho}_i(r_i,t)/\bar{\rho}_0(r_i,t) = \rho_i(\mathbf{r},t)/\rho_0(\mathbf{r},t)$.

4. Optimal Dynamic Prices

In the analysis that follows, we use the "baby Bernoulli process" approximation of the Poisson process (Gallager 1999) to discretize the finite planning horizon, similar to the approach taken in Bitran and Mondschein (1997). We choose the length of each discrete time period such that the probability of more than one demand arrival in each time period is nearly zero. We then assume that there can be at most one demand in each time period. Given this time period, we let T denote the number of time periods in the planning horizon. Then we rescale the parameter λ so that within each time period t, the probability of no customer arrival is 1-λ, and the probability of exactly one customer arrival is λ. Therefore, for each time period t, $\lambda\rho_i(\mathbf{r},t) = \lambda_i(\mathbf{r},t)$ is the probability that a demand occurs for the old product (i=1) or for the new product (i=2); the probability of a customer arrival and no purchase is $\lambda\rho_0(\mathbf{r},t)$.

Let $V_t(x_1,x_2)$ be the value-to-go at the beginning of period t if the company has x_i units of old product (i=1) and new product (i=2), and makes optimal price decisions at t and thereafter.

We define $V_t(x_1,x_2)$ recursively:

$$V_t(x_1,x_2) = \max_{\mathbf{r}} h_t(\mathbf{r},x_1,x_2)$$

$$\begin{aligned} \text{where} \quad h_t(\mathbf{r},x_1,x_2) \equiv\ & \lambda_1(\mathbf{r},t)(r_1 + V_{t+1}(x_1-1,x_2)) \\ & + \lambda_2(\mathbf{r},t)(r_2 + V_{t+1}(x_1,x_2-1)) \\ & + (\lambda\rho_0(\mathbf{r},t) + 1-\lambda)V_{t+1}(x_1,x_2) \qquad \forall x_1,x_2 > 0 \end{aligned} \tag{4}$$

$$\begin{aligned} h_t(\mathbf{r},0,x_2) \equiv\ & \lambda\bar{\rho}_2(r_2,t)(r_2 + V_{t+1}(0,x_2-1)) \\ & + (\lambda\bar{\rho}_0(r_2,t) + 1-\lambda)V_{t+1}(0,x_2) \qquad \forall x_2 > 0 \end{aligned} \tag{5}$$

$$\begin{aligned} h_t(\mathbf{r},x_1,0) \equiv\ & \lambda\bar{\rho}_1(r_1,t)(r_1 + V_{t+1}(x_1-1,0)) \\ & + (\lambda\bar{\rho}_0(r_1,t) + 1-\lambda)V_{t+1}(x_1,0) \qquad \forall x_1 > 0 \end{aligned} \tag{6}$$

The terminal value is the salvage value of products left over after T: $V_{T+1}(x_1,x_2) = s_1x_1 + s_2x_2$ where s_i is the unit salvage value of a product at the end of the transition. The salvage value for the new product reflects the value depreciation of the new product. It does not necessarily imply that the company will salvage any left over units of the new product. Presumably any leftover new product is just kept in inventory and sold later. Hence the value of the inventory at *T* is the replacement value for the new product, i.e., the manufacturing cost for the new product at time *T*. Let the production cost for product *i* at time *0* be c_i. Then the salvage value s_2 could be c_2, reflecting no cost reduction over *T*, or a fraction of c_2, reflecting cost reduction from learning.

The problem is to find the optimal prices r_1 and r_2 for each (t, x_1, x_2) combination. The value function $h_t(\mathbf{r}, x_1, x_2)$ as defined in equations (4)-(6) is not jointly concave or quasi-concave in prices. Hanson and Martin (1996) give a counter example to the joint quasi-concavity. To circumvent this problem, Song and Xue (2007), as well as, Dong et al. (2009) use an inverse demand function of the MNL model and show that the value function becomes jointly concave in the market shares. In this paper, we follow the Aydin and Porteus (2008) approach to show that, even though joint quasi-concavity does not hold, the first-order condition yields a unique price vector, and that it is optimal for the problem we consider. In the following, we first solve the first-order necessary condition, and then show that it is also sufficient.

Solving the first-order condition for equation (4), we obtain:

$$\rho_i(\mathbf{r},t) + \frac{\partial \rho_j(\mathbf{r},t)}{\partial r_i}(r_j - \Delta_j V_{t+1}(x_1,x_2)) + \frac{\partial \rho_i(\mathbf{r},t)}{\partial r_i}(r_i - \Delta_i V_{t+1}(x_1,x_2)) = 0 \quad i = 1,2,\ j \neq i \quad (7)$$

where $\Delta_1 V_{t+1}(x_1,x_2) \equiv V_{t+1}(x_1,x_2) - V_{t+1}(x_1 - 1, x_2)$ and $\Delta_2 V_{t+1}(x_1,x_2) \equiv V_{t+1}(x_1,x_2) - V_{t+1}(x_1, x_2 - 1)$ are the marginal value of inventory for the old and new product respectively.

From equation (1), we obtain

$$\frac{\partial \rho_0(\mathbf{r},t)}{\partial r_i} = \rho_0 \rho_i g'(r_i), \frac{\partial \rho_i(\mathbf{r},t)}{\partial r_i} = \rho_i(\rho_i - 1) g'(r_i) \text{ and } \frac{\partial \rho_i(\mathbf{r},t)}{\partial r_j} = \rho_i \rho_j g'(r_j) \quad (8)$$

Substituting (8) into (7) and treating equation (7) as two linear equations with two unknowns $r_i - \Delta_i V_{t+1}(x_1,x_2), i = 1,2$, we can solve equation (7) and rewrite the first order condition as:

$$r_i - \Delta_i V_{t+1}(x_1, x_2) = \frac{1}{\rho_0(\mathbf{r}, t)} [g'(r_i)^{-1} - (g'(r_i)^{-1} - g'(r_j)^{-1}) \rho_j(\mathbf{r}, t)] \qquad (9)$$

Assumption 2. *The disutility function* $g(\cdot)$ *is continuous and twice differentiable and satisfies* $g'(r_i) + g''(r_i) / g'(r_i) > 0$.

Assumption 2 is a technical assumption that is satisfied by many increasing utility functions.

Lemma 1. *Under Assumption 2, we have*

i) *For any given* r_i, $h_t(\mathbf{r}, x_1, x_2)$ *is strictly quasi-concave in* r_j $\forall x_1 \geq 0, x_2 \geq 0, t \in [0, T]$ *and* $i, j = 1, 2,\ \ i \neq j$.

ii) *The cross price effect* $\frac{\partial^2 h_t(\mathbf{r}, x_1, x_2)}{\partial r_i \partial r_j} = 0$ *whenever* $\frac{\partial h_t(\mathbf{r}, x_1, x_2)}{\partial r_1} = 0$ *and* $\frac{\partial h_t(\mathbf{r}, x_1, x_2)}{\partial r_2} = 0$.

Proposition 1. *Let* $\hat{\mathbf{r}} = (\hat{r}_1, \hat{r}_2)$ *solve the first order condition of equations (4)-(6) where* $\hat{r}_1, \hat{r}_2 \in [0, \infty)$ *. Then it is the unique maximizer of* $h_t(\mathbf{r}, x_1, x_2)$ $\forall x_1 \geq 0, x_2 \geq 0, t \in [0, T]$.

Therefore, we can obtain the optimal solution by solving the first-order condition in equation (9)

Economists often assume a quasi-linear utility function to simplify problems and obtain tractable solutions by eliminating the effect of initial wealth (Mas-Colell et al. 1995). In the analysis that follows, we assume that the customers' utilities are linear with respect to money, i.e., the disutility function $g(r_i)$ is a linear function.

Assumption 3. $g(r_i) = \beta r_i$ *where* $\beta > 0$.

With Assumption 3 we can reduce condition (9) to

$$r_i - \Delta_i V_{t+1}(x_1, x_2) = \frac{1}{\beta \rho_0(\mathbf{r}, t)} \qquad (10)$$

The term $r_i - \Delta_i V_{t+1}(x_1, x_2)$ is the marginal gain from selling a unit of product *i* at time *t*. Intuitively, the optimal price is set such that the company is indifferent between selling an old and selling a new. Solving the above equation for r_i (see proof of Proposition 2), we obtain:

$$r_i^*(t,x_1,x_2)=\Delta_i V_{t+1}(x_1,x_2)+\frac{1}{\beta}\{1+W(e^{a_i(t)-u_0(t)-1-\beta\Delta_i V_{t+1}(x_1,x_2)}+e^{a_j(t)-u_0(t)-1-\beta\Delta_j V_{t+1}(x_1,x_2)})\}$$

where W is the Lambert's W function, i.e., $W(x)$ solves the equation $we^w = x$ for w as a function of x. We can do a similar analysis for the cases when one of the products runs out. Substituting the optimal prices into equations (4)-(6) yields a recursive formula for computing the value function $V_t(x_1,x_2)$. We summarize these results in Proposition 2.

Proposition 2. *Under Assumption 3, the optimal price of the old and new product at time t for a given inventory level* (x_1,x_2) is

$$r_i^*(t,x_1,x_2)=\Delta_i V_{t+1}(x_1,x_2)+\frac{1}{\beta}\{1+W(Z)\} \tag{11}$$

where $Z=\begin{cases} e^{a_i(t)-u_0(t)-1-\beta\Delta_i V_{t+1}(x_1,x_2)}+e^{a_j(t)-u_0(t)-1-\beta\Delta_j V_{t+1}(x_1,x_2)} & \forall x_1,x_2>0 \\ e^{a_2(t)-u_0(t)-1-\beta\Delta_2 V_{t+1}(0,x_2)} & \forall x_1=0,x_2>0 \\ e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)} & \forall x_1>0,x_2=0 \end{cases}$

and we obtain $V_t(x_1,x_2)$ *using the following recursive equations:*

$$V_{T+1}(x_1,x_2)=s_1x_1+s_2x_2 \tag{12}$$

$$V_t(x_1,x_2)=V_{t+1}(x_1,x_2)+\frac{\lambda}{\beta}W(Z) \tag{13}$$

We note from (13) that the marginal value of time is $\Delta_t V(x_1,x_2)\equiv V_t(x_1,x_2)-V_{t+1}(x_1,x_2)=\frac{\lambda}{\beta}W(Z)$. We can now express the optimal prices as:

$$r_i(t,x_1,x_2)=\Delta_i V_{t+1}(x_1,x_2)+\frac{1}{\lambda}\Delta_t V(x_1,x_2)+\frac{1}{\beta} \tag{14}$$

Thus the optimal price is determined by the marginal value of inventory and the marginal value of time. Note that the factor $1/\lambda$ in the second term represents a conversion to per unit price.

In addition, comparing equations (10) and (11), we obtain

$$W(Z)=\frac{\rho_1(\mathbf{r}^*,t)+\rho_2(\mathbf{r}^*,t)}{\rho_0(\mathbf{r}^*,t)} \tag{15}$$

Therefore, we can interpret the term $W(Z)$ as the ratio of the company's market share against the competition at time *t*, assuming we follow the optimal pricing policy.

Heretofore we have not made any specific assumptions on $a_i(t)$, the time-varying attribute. In fact, the solution given in Proposition 2 applies to any two substitutable products with time-varying attributes and sold by the same company. To derive structural properties of the optimal dynamic prices in the context of a product transition, we will make some additional assumptions regarding $a_i(t)$ and $u_0(t)$. In what follows, we examine in a progressive manner the various factors and dynamics that affect the optimal prices of the two products during the transition. Throughout this paper, we use both analytical and numerical examples to develop the insights on how certain dynamics and factors effect price.

5. Factors and Dynamics Affecting the Optimal Price

We start with a simple base case. We let both $a_i(t)$ and $u_0(t)$ be constant, and consider infinite supply of inventory. Corollary 1 follows directly from equation (11).

Corollary 1. *If* $a_i(t) = a_i$, $u_0(t) = u_0$, $x_1, x_2 \to \infty$, *the optimal prices are constants throughout the planning horizon:* $r_i^* = s_i + \frac{1}{\beta}(1 + W(e^{a_1 - u_0 - 1 - \beta s_1} + e^{a_2 - u_0 - 1 - \beta s_2}))$.

This is consistent with findings from existing dynamic pricing literature with unconstrained supply and time-invariant attributes. The base case does not necessarily correspond to any realistic situation. However, we can infer the various factors and dynamics affecting the optimal price by comparing the optimal solution under more complex cases with the base case in Corollary 1.

In Sections 5.1-5.3, we assume ample inventory and thus the effect of inventory scarcity is absent. In Sections 5.4 and 5.5, we discuss the cases of limited inventory.

5.1 Effect of Product Replacement

The function $a_i(t)$ represents the change of customers' attitude toward a product after the new product introduction, independent of the products' prices. We expect $a_i(t)$ to vary over the transition period, and it could take on various functional forms.

Assumption 4. *We assume that* $a_i(t)$ *is given by* $a_1(t) = a_0 - kt$ *and* $a_2(t) = kt$ *where* $a_0, k > 0$ *are known constants.*

As justification for this assumption, we consider the behavior of the market shares for the two products. Under the MNL model, this ratio at time *t* is $\rho_2 / \rho_1 = G(\mathbf{r})e^{a_2(t)-a_1(t)}$ where $G(r) = e^{-(g(r_2)-g(r_1))}$. Given that $a_i(t)$ is linear, the market share ratio in the MNL model has the same form as for the Fisher and Pry (1971) model, for which the market share of the old and new technologies (products) is $e^{2\alpha(t-t_0)}$ where t_0 is the time at which the new and the old have equal market shares, and α is a constant that signifies the rate of substitution. We can also show that Assumption 4 is consistent with the Norton and Bass (1987) model: If we assume that the old and new products have the same customer population, then the market share ratio in the Norton-Bass model differs from the MNL model by a constant. Both the Fisher-Fry model and the Norton-Bass model perform well on empirical data. Thus, we contend that the linear assumption of $a_i(t)$ is reasonable.

As shown in Figure 1, Assumption 4 generates a logistic demand pattern commonly observed during a product transition. The new product will, over time, replace the old product. The magnitude of *k* represents the rate of the transition, i.e., how quickly the new product replaces the old; it may depend on multiple factors including, but not limited to, product capability, timing of the new product introduction, marketing effort and macroeconomic environment. For example, Intel generates a PTI score for each transition based on the assessment of these constituent factors. The resulting PTI score is a direct indicator of the transition rate. According to Jay Hopman (Hopman 2005),

> *"If all vectors are scored down the middle, the product transition should be expected to unfold at a rate on par with the average of past transitions. Hotter scores predict a faster transition, colder scores a slower transition."*

The Intel approach includes price and external competition, along with many other factors in the PTI scores. In our model, prices are decision variables and we model external competition implicitly through $u_0(t)$. We consolidate all other factors into *k*.

The following proposition highlights the effect of $a_i(t)$ on the optimal prices.

Proposition 3. *Under Assumptions 3 and 4, if the no-purchase utility does not vary with time, i.e.,* $u_0(t) = u_0$, *then*

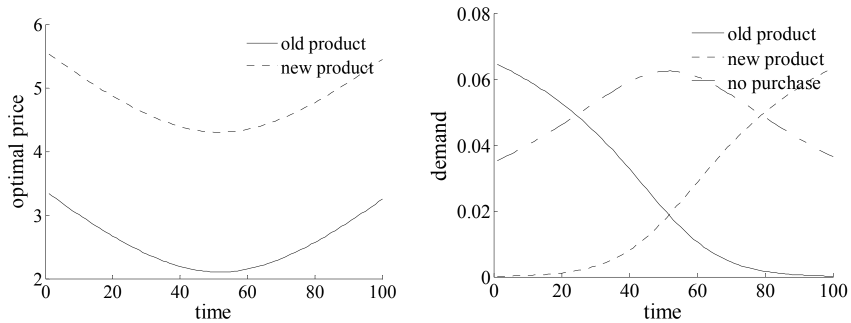
i) If $x_1 \to \infty, x_2 = 0$, $r_1^*(t, x_1, x_2)$ *decreases in t* $\forall t \in [0, T]$

ii) If $x_1 = 0, x_2 \to \infty$, $r_2^*(t, x_1, x_2)$ *increases in t* $\forall t \in [0, T]$

iii) If $x_1, x_2 \to \infty$ *,* $r_i^*(t, x_1, x_2)$ *decreases in t for* $t \in [0, \bar{t}]$ *and increases in t for* $t \in [\bar{t}, T]$*, where* $\bar{t} = \dfrac{a_0 + \beta(s_2 - s_1)}{2k}$.

As inventory increases, the marginal value of inventory $\Delta_i V_{t+1}(x_1, x_2)$ approaches the salvage value s_i. Thus, from equation (14), the optimal price is solely determined by the marginal value of time $\Delta_t V(x_1, x_2)$. For case i), $\Delta_t V(x_1, x_2)$ is an increasing function of $a_1(t)$, which decreases linearly in time by assumption. Therefore, the optimal price of the old product decreases over time. Similarly, in case ii), $\Delta_t V(x_1, x_2)$ is an increasing function of $a_2(t)$, which increases in *t*. Therefore the optimal price of the new product increases over time. When both products are available, the optimal price behavior is more complex. In case iii), $\Delta_t V(x_1, x_2)$ is an increasing function of $Ae^{a_1(t) - u_0} + Be^{a_2(t) - u_0}$ where $A = e^{-1-\beta s_1}$ and $B = e^{-1-\beta s_2}$ are constants. Thus the price trend depends on which term dominates.

We illustrate the behavior of case iii) graphically in Figure 2. In this example, we assume the inventory level to be (10,10) throughout the transition. This inventory level represents, for all practical purposes, an infinite amount of inventory since the maximum total expected demand for both products for the planning horizon is $\lambda T = 10$. We observe from the optimal demand curve that the new product gradually replaces the old as the dominating product. The no-purchase option reaches its highest level midway through the transition when neither product dominates.



Parameter Values	
T	100
a_0	4
k	0.06
β	1
λ	0.1
u_0	0
s_1	0.5
s_2	2.7

Figure 2: Optimal Prices and Demand for Given Inventory Level (10,10)

Comparing this with the base case, we see that the price trend is due to changes in $a_i(t)$. Initially, the new product is at its infancy and the old product offers the highest utility to customers (i.e., the term $Ae^{a_1(t)-u_0}$ dominates over $Be^{a_2(t)-u_0}$); the competition is essentially between the old product and the non-purchase option. As t increases, customer preference for the old product decreases as it approaches obsolescence. The overall impact is that the no-purchase option becomes relatively more attractive over time before the new product gains strong hold. Therefore, the optimal pricing strategy during the first half of the transition is to decrease price in order to compete with the no-purchase option. Later, the new product replaces the old product to become the main product that competes with the no-purchase option, i.e., the term $Be^{a_2(t)-u_0}$ dominates over $Ae^{a_1(t)-u_0}$. As the new product's attractiveness increases with time, the company ought to increase price to maximize revenue. Therefore, during the transition, the company faces higher risk of losing to customers' other options. The price dip during the transition (Figure 2) is hence a strategy to counteract that risk.

In practice, we rarely observe price increases for technology products. Therefore, the price trend in Proposition 3(iii) and Figure 2 is an isolated effect of the replacement between two generations of products. Later we show that external competition and/or the effect of inventory scarcity is likely to create a downward pricing trend.

5.2 Effect of External Competition

In the demand model from Section 3, the no-purchase option may represent different customer behaviors. In the case of a monopoly, a customer's no-purchase utility, u_0, represents his/her preference for not getting any product, old or new. Such preference may change over time. For example, technology development may make it easier (or harder) over time for a customer to get by without buying either product. In a competitive market, a customer's no-purchase utility represents the customer's preference for the best outside option (competitive product), or not buying any product (e.g., using its legacy system). If the competitor continuously improves its product attractiveness, whether through enhanced features or lower prices, the value of u_0 may increase over time. Therefore, a time-varying u_0 allows us to model the impact of these various factors on the optimal pricing strategy during a product transition.

The discrete choice model employed in this paper depends critically on the ability to estimate $u_0(t)$. Companies often study their customer base through focus groups or surveys, as well as competitors' product offerings to help them better understand customers' preference for other options (including buying from competition and not buying any). As with $a_i(t)$, the no-purchase option may depend on factors other than time and we treat these factors as either a constant term or a generic force that contributes to the time trend in $u_0(t)$.

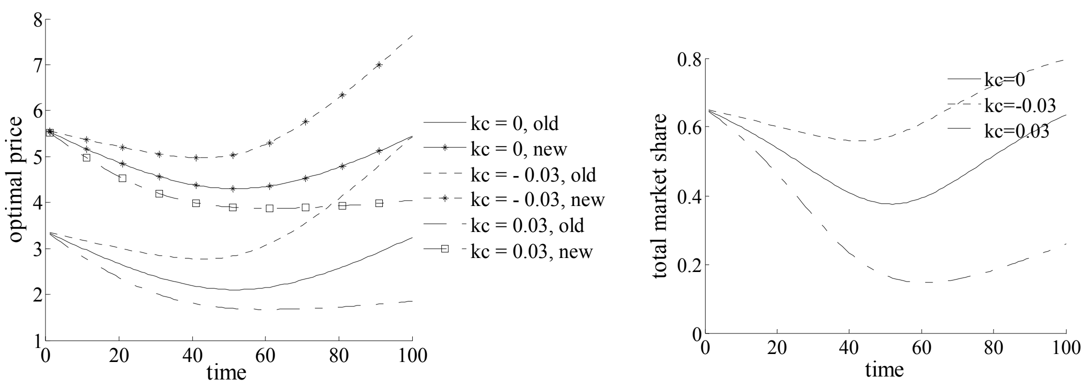
A major difficulty with statistically estimating the parameters in the MNL demand model including λ and u_0 is that, given the sales data, it is impossible to tell if a period with no sale is due to no customer arrival or due to no purchase upon arrival. A recent paper by Vulcano et al (2009) addresses this problem using an expectation-maximization (EM) method which maximizes the *expected* log-likelihood function conditioned on the current parameter estimates. Specifically, one can use the *observed* sales in a fixed time horizon, plus the *estimated* number of *arrivals* from the periods with no sales observation (conditioned on current parameter estimates), divided by the total number of time periods during that time horizon to form the next estimate of λ. In our model, both a_i and u_0 may vary with time, which requires us to estimate additional parameters in the EM method. For example, suppose Assumptions 3 and 4 hold, and $u_0(t) = u_c + k_c t$. Then the log-likelihood function will have time as an additional independent variable, and parameter terms including $\log\lambda$, β, k, k_c, u_c and a_0. Nonetheless, the same technique in Vulcano et al (2009) should extend to our problem.

From Proposition 2, it is easy to see that all else equal, the optimal price at time t for each product decreases in $u_0(t)$.

Corollary 2. *Under Assumption 4,* $\frac{\partial(\rho_1(\mathbf{r}^*,t)+\rho_2(\mathbf{r}^*,t))}{\partial u_0(t)} = -\rho_0(\mathbf{r}^*,t)(\rho_1(\mathbf{r}^*,t)+\rho_2(\mathbf{r}^*,t))$ and $\frac{\partial r_i^*(t,x_1,x_2)}{\partial u_0(t)} = -\frac{\rho_1(\mathbf{r}^*,t)+\rho_2(\mathbf{r}^*,t)}{\beta}$. *Thus the company's total market share decreases in* $u_0(t)$ $\forall t \in [0,T], x_1, x_2 \geq 0$. *The optimal prices also decrease in* $u_0(t)$ *and the rate of decrease is higher if the company has a larger total market share at time t.*

Therefore, when the no-purchase utility $u_0(t)$ increases (decreases) over time, the optimal prices of both the old and new products experience downward (upward) pressure, relative to the case of a constant $u_0(t)$. Figure 3(a) shows the optimal prices when u_0 changes linearly with time, i.e., $u_0(t) = u_c + k_c t$ with $u_c = 0$. Other parameters are the same as in Figure 2. In Figure 3(b), we show the total market share of the two products under the optimal prices. We observe that the widening of the price gaps in Figure 3(a) correlates with the gaps in market share in Figure 3(b).

The time-increasing u_0 reduces the company's market share and limits the company's ability to increase price. Indeed, a price increase is rare in practice for technology products. It is reasonable to attribute this phenomenon to a time-increasing u_0. As design and production technology advances, a customer's outside option comes in the form of a cheaper or better product, prohibiting any price increase. Although rare in practice, $u_0(t)$ may decrease in time occasionally. For instance, when the competitors are struggling with survival due to either internal or external forces, the customer's outside option might become less favorable. When this is the case, the optimal prices of the old and new products are both pushed upward due to the time-decreasing u_0.



(a) (b)

Figure 3: Optimal Prices for Linearly Changing $u_0(t)$

To summarize, the changes in the no-purchase utility over time causes the prices of the old and new products to move in the opposite direction to the changes in $u_0(t)$.

5.3 Effect of Substitution

A third factor affecting price arises from substitution between the old and new products. The optimality condition in equation (10) requires the company to be indifferent between selling the two products; thus when the price of one product decreases, the price of the other should decrease as well. Therefore, price-based substitution of the two products causes the optimal prices to move together. We also see this from equation (14): The prices of the two products are driven by $\Delta_i V_{t+1}(x_1, x_2)$, the marginal value of inventory and $\Delta_t V(x_1, x_2)$, the marginal value of time. When inventory is abundant, the former converges to s_i, thus the prices of the two products move in parallel.

Corollary 3. *If* $x_1, x_2 \to \infty$, $r_2^*(t,\infty,\infty) - r_1^*(t,\infty,\infty) = s_2 - s_1$.

We observe this result in both Figures 2 and 3(a).

5.4 Effect of Scarcity

When supplies are limited, the optimal price behavior, as well as the resulting demand, has quite different characteristics. Figure 4 demonstrates the scarcity effect on optimal prices. We hold both $a_i(t)$ and $u_0(t)$ constant to remove any time trend caused by the replacement effect or external competition. Other parameter values are as in Figure 2. The inventory level of 10 represents abundant supply and 1 scarce supply. We plot the price gaps when a product's inventory is scarce versus plenty while keeping inventory of the other product unchanged. For example, $r_2(t,10,1) - r_2(t,10,10)$ measures the impact of scarcity of the new product on its optimal price. This curve sits above zero, implying that for any given time *t*, scarcity of a product increases its own price; we term this the "within-product" impact of scarcity. The time trend of this curve shows that scarcity causes a decreasing trend in the optimal price. Similarly, $r_2(t,1,10) - r_2(t,10,10)$ measures the "cross-product" impact of scarcity.

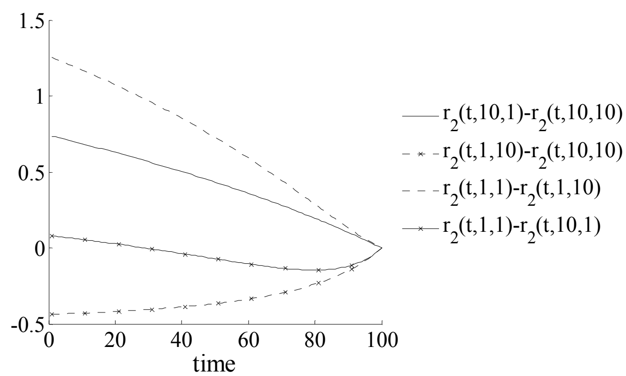


Figure 4: Impact of Scarcity on Optimal Prices

Examining the curves in Figure 4, we observe that the "within-product" scarcity effect is consistent. That is, scarcity itself increases the optimal price relative to ample inventory at any given time *t*. But for a "fixed" level of scarcity, the optimal price declines over time as the selling window becomes shorter. In the pricing literature, price decline over time due to the scarcity effect has been widely studied (e.g., Bitran and Mondschein 1997, Dong et al. 2009). The “cross-product” impact (the marked curves) indicates that scarcity of one product may increase or decrease the price of the other product. The time trend due to “cross-product” scarcity is not consistent, and in some cases, is not even monotone, as shown in the curve $r_2(t,1,1) - r_2(t,10,1)$. This non-monotonic pattern is due to a combination of the substitution between the two products and external competition, which we will explain in detail in the next section. We also observe similar behavior in the old product.

These four dynamic factors (replacement, external competition, substitution and scarcity) are critical for understanding the optimal pricing strategies *over time*. One or more of these dynamics can become the dominant force that affects the shape of the price path under specific conditions. We summarize the impact of each factor in Table 1.

Dynamic Factors	Impact on Optimal Prices
Replacement	The prices for both old and new product decrease initially, and then increase, exclusive of the impact from inventory
Scarcity	At any time, less inventory implies a higher price. But less inventory also results in a steeper price decline over time.
Substitution	The prices of the old and new product move in the same direction, all else being equal.
External Competition	Greater external competition leads to a lower price for both old and new product. As we expect external competition to increase over time, this results in increasing downward pressure on both prices.

Table 1: Impact of the Key Dynamics on Optimal Prices

5.5 Effect of Inventory

As products are sold, the inventory level for each product changes, which affects the optimal prices. When either $x_1 = 0$ or $x_2 = 0$, the relationship between r_i^* and x_i is as expected. Namely, under Assumption 4, if $x_j = 0$, $r_i^*(t, x_i)$ is non-increasing in x_i for $i, j = 1,2$, $i \neq j$. However, in the presence of the other product, it is not clear how the inventory level affects the optimal price. Numerically, we find that each product’s price still decreases with its own inventory level (Figure 5a). The impact of inventory on the optimal price of the other product is more intriguing (Figure 5b).

The vertical axis is the optimal prices of the new (Figure 5a) and old (Figure 5b) product at time *t*

= 50. Other parameters are the same as in Figure 2 except for the inventory levels. In Figure 5b, when the old product inventory is low (high), its optimal price decreases (increases) as the new product inventory increases. This pattern arises from multiple dynamics that are at play. The two products compete with the outside option; thus there is pressure to decrease price when the total inventory of the two products goes up. In the mean time, there is competition (substitution) between these two products. Increased inventory of the new product increases the risk of excess for the new product and calls for a price increase of the old to make the new appear more attractive. With more old product (e.g. $x_1 = 10$), the competitive nature of the two products are more pronounced as it becomes more likely that the company has to make a choice of which product to sell in the transition period; therefore the dominating impact is a price increase for the old product when the inventory of the new product increases.

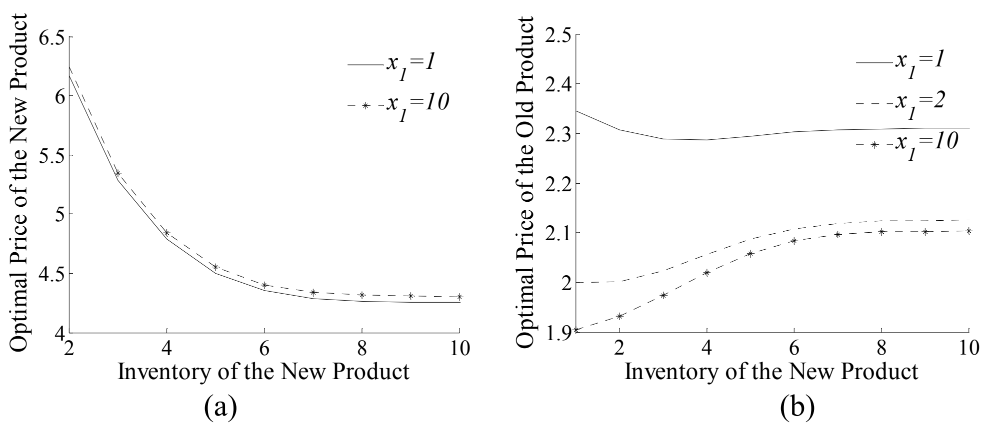


Figure 5: Optimal Price vs. Inventory

We have already seen in Figure 4 that the effect of scarcity on the *other* product's price is not monotonic. This can be explained again by the interplay of two competing forces: First, the two products together are competing against the outside option (external competition), thus the scarcity of the company's product (old or new) should cause the price to decrease over time if no sales are made. Second, the customers are making price-based substitution between the two products (substitution effect). As a result, scarcity of one product reduces the inventory pressure for the other product, causing the price of the other product to increase over time.

5.6 Impact of the Parameters on Optimal Prices

We let $T = 100$, $\lambda = 0.1$, and $a_0 = 4$. We vary the value of the price sensitivity parameter β, the transition rate k, and the salvage value s_i for each product. Specifically, we let $\beta = 0.5, 1, 2$ respectively to represent the case of low, medium, and high price sensitivity. We let $k = 0.06$, 0.12 and 0.24 to indicate slow, regular, and fast transitions. For the initial production cost at time zero, we consider two cases: i) $c_1 = 2, c_2 = 3$, and ii) $c_1 = 3, c_2 = 2$ to indicate respectively, an increased and decreased cost from the old to the new product. For the salvage value, we let the old product to be 10%, 25%, and 50% of its original production cost, the new product to be 50%, 70%, and 90% of its original cost (the new product retains a better percentage value given the earlier explanation of salvage values). The different salvage values represent fast, regular, and slow speed of product obsolescence. Table 2 summarizes the parameters used. For each set of parameters, we compute the optimal dynamic prices for any given inventory and time. In this section, we illustrate results from the numerical study for the case of abundant inventory.

β	k	(c_1, c_2)	s_1	s_2
0.5, 1, 2	0.06, 0.12, 0.24	(2,3) and (3,2)	$0.1c_1, 0.25c_1, 0.5c_1$	$0.5c_2, 0.7c_2, 0.9c_2$

Table 2: Parameters for Numerical Examples

We found that the optimal product price increases in its salvage value (equivalently, decreases with its own speed of obsolescence) due to lower overage cost, but may decrease in the other product's salvage value due to the substitution effect. In addition, we make some general observations regarding the effect of the transition rate k and the price sensitivity parameter β and illustrate them in the following examples.

In Figure 6 we show the price and demand behavior as it depends on the speed of the product transition, when $\beta = 1, c_1 = 2, c_2 = 3$, $s_1 = 0.25c_1$ and $s_2 = 0.9c_2$. We illustrate the price path for the new product only as that for the old is similar. A faster transition leads to higher prices for both products. The price increase of the new product reflects its desirability in a fast transition. Although the price of the old product follows a similar pattern as the new, its revenue impact is not significant because the demand of the old product, at the time of the price increase, was already quite low.

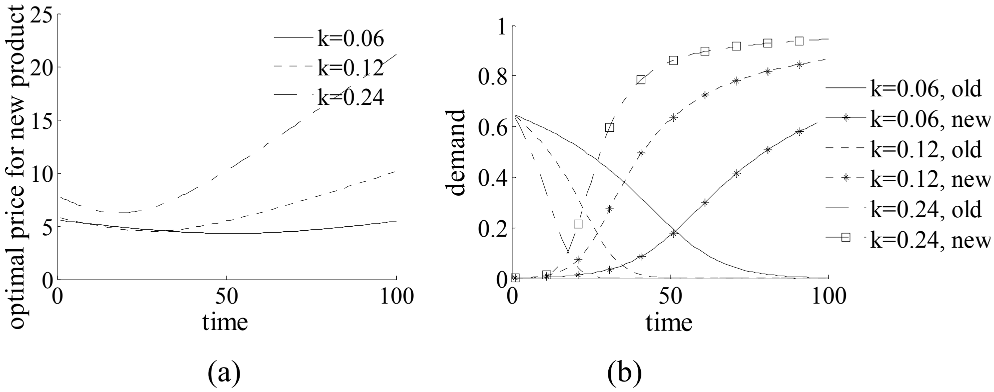


Figure 6: Optimal Price vs. Transition Rate

As the price sensitivity parameter β increases, the optimal prices exhibit much smaller changes over time, as shown in Figure 7 (with parameters $k = 0.06, c_1 = 2, c_2 = 3$, $s_1 = 0.25c_1$ and $s_2 = 0.9c_2$).

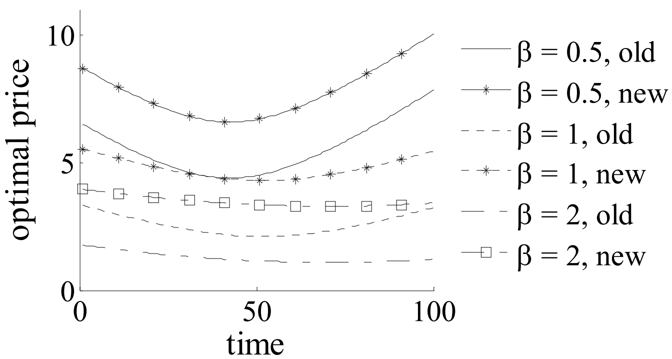


Figure 7: Optimal Price vs. Price Sensitivity

Recall from the discussion in Section 5.1 that the price dip during a transition is due to customers' preference shift from the old product to the new product, i.e., changes in $a_i(t)$. In the transition, the company faces higher risk of losing business because the customers perceive the old product as becoming obsolete and the new product has yet to prove itself being a superior option. The price dip (more specifically, a gradual decrease followed by a gradual increase) helps the company to counteract this risk. Therefore, the less price-sensitive the customers are (smaller β), the more price change is necessary to overcome that. This can also be seen mathematically from equation (11). When the supply is ample, $r_i^*(t,\infty,\infty) = s_i + \frac{1}{\beta}[1+W(Z)]$ where $Z = e^{a_1(t)-u_0(t)-1-\beta s_1} + e^{a_2(t)-u_0(t)-1-\beta s_2}$. $W(Z)$ has higher values at time 0 and time T, and lower values in between. As β increases, the price dip becomes smaller.

6. Optimal Initial Inventories

Proposition 2 gives a simple recursive algorithm for finding the value function using equations (12) and (13). We can then use the value function to easily determine the optimal initial inventories, provided we are given a procurement cost for each product. If the planning horizon is T time periods, then the largest reasonable initial inventory is T units for each product, as we can sell at most one unit per time period by assumption. Thus, we need to compute $V_t(x_1, x_2)$ for $x_1, x_2, t \in [0,T]$, which is of order $O(T^3)$. If T is large, the computation burden may become excessive.

In Table 3 we report the optimal initial inventories for various parameter combinations. The other parameter values are the same as in Section 5.6.

Case	k	s_1	s_2	c_1=2, c_2=3		c_1=3, c_2=2		Case	k	s_1	s_2	c_1=2, c_2=3		c_1=3, c_2=2	
				x_1	x_2	x_1	x_2					x_1	x_2	x_1	x_2
1	0.06	0.1c_1	0.5c_2	1	3	0	4	10	0.12	0.1c_1	0.5c_2	1	6	0	8
2	0.06	0.1c_1	0.7c_2	1	3	0	5	11	0.12	0.1c_1	0.7c_2	1	7	0	9
3	0.06	0.1c_1	0.9c_2	1	4	0	6	12	0.12	0.1c_1	0.9c_2	1	9	0	10
4	0.06	0.25 c_1	0.5c_2	1	3	0	4	13	0.12	0.25 c_1	0.5c_2	1	6	0	8
5	0.06	0.25 c_1	0.7c_2	1	3	0	5	14	0.12	0.25 c_1	0.7c_2	1	7	0	9
6	0.06	0.25 c_1	0.9c_2	1	4	0	6	15	0.12	0.25 c_1	0.9c_2	1	9	0	10
7	0.06	0.5c_1	0.5c_2	2	2	1	4	16	0.12	0.5c_1	0.5c_2	1	6	0	8
8	0.06	0.5c_1	0.7c_2	2	3	1	5	17	0.12	0.5c_1	0.7c_2	1	7	0	9
9	0.06	0.5c_1	0.9c_2	2	4	1	6	18	0.12	0.5c_1	0.9c_2	1	9	0	10

Table 3: Optimal Initial Inventories

Observations from Table 3 are consistent with our intuition: The optimal initial inventory for a product is non-decreasing in its own salvage value (for example, compare cases 1, 2 and 3). This is not surprising as the faster the speed of obsolescence, the less initial inventory the company should keep for that product. However, it may decrease in the salvage value of the other product due to the substitution effect (compare cases 1, 4 and 7 under "cost-up"). For a faster transition, we need a larger initial inventory of the new product and less initial inventory for the old product. When the inter-generational cost decreases, we stock less of the old product and more of the new product relative to the case when the inter-generational cost increases. We have run additional cases to confirm that these findings remain the same and we include them in the online Appendix.

Dong et al. (2009) propose a single-variable approximation to the problem for obtaining the initial inventory. We extend their method to incorporate the replacement effect. Specifically, we construct a pseudo product that incorporates the time-varying attributes, the costs, and the salvage values of both products: $a(t) = \ln(e^{a_1(t)} + e^{a_2(t)})$, $c = \sum_{i=1,2} c_i \theta_i$ and $s = \sum_{i=1,2} s_i \theta_i$ where $\theta_i = \frac{e^{\overline{a}_1 - \beta c_i}}{e^{\overline{a}_1 - \beta c_1} + e^{\overline{a}_2 - \beta c_2}}$ and $\overline{a}_i = (\int_0^T a_i(t)dt)/T$. We then use equations (12) and (13) for the case of a single product to find the optimal inventory $\overline{x}$ for the pseudo product and use $\theta_i \overline{x}$ as the inventory for product *i*. Table 4 shows the results and performance of this heuristic method for the "cost-up" cases shown in Table 3.

	k	s_1	s_2	*Optimal* x_1	*Optimal* x_2	*Heuristic* x_1	*Heuristic* x_2	*Heuristic Performance*	*Fixed Price old*	*Fixed Price new*	*Fixed Price Performance*
1	0.06	$0.1c_1$	$0.5c_2$	1	3	1	2	0.9961	3.60	4.70	0.8916
2	0.06	$0.1c_1$	$0.7c_2$	1	3	1	2	0.9406	3.61	4.88	0.9140
3	0.06	$0.1c_1$	$0.9c_2$	1	4	1	3	0.9696	3.61	4.92	0.9249
4	0.06	$0.25\,c_1$	$0.5c_2$	1	3	1	2	0.9953	3.68	4.70	0.9006
5	0.06	$0.25\,c_1$	$0.7c_2$	1	3	1	2	0.9407	3.69	4.88	0.9218
6	0.06	$0.25\,c_1$	$0.9c_2$	1	4	1	3	0.9696	3.69	4.92	0.9313
7	0.06	$0.5c_1$	$0.5c_2$	2	2	1	2	0.9691	3.44	5.05	0.9309
8	0.06	$0.5c_1$	$0.7c_2$	2	3	1	3	0.9810	3.44	4.89	0.9292
9	0.06	$0.5c_1$	$0.9c_2$	2	4	1	3	0.9558	3.44	4.94	0.9385
10	0.12	$0.1c_1$	$0.5c_2$	1	6	0	7	0.9942	3.12	7.29	0.7734
11	0.12	$0.1c_1$	$0.7c_2$	1	7	0	8	0.9901	3.12	7.32	0.7701
12	0.12	$0.1c_1$	$0.9c_2$	1	9	0	9	0.9970	3.12	7.44	0.7809
13	0.12	$0.25\,c_1$	$0.5c_2$	1	6	0	7	0.9888	3.23	7.28	0.7758
14	0.12	$0.25\,c_1$	$0.7c_2$	1	7	0	8	0.9850	3.23	7.32	0.7723
15	0.12	$0.25\,c_1$	$0.9c_2$	1	9	0	9	0.9922	3.23	7.44	0.7828
16	0.12	$0.5c_1$	$0.5c_2$	1	6	0	7	0.9784	3.43	7.28	0.7797
17	0.12	$0.5c_1$	$0.7c_2$	1	7	0	8	0.9753	3.43	7.32	0.7759
18	0.12	$0.5c_1$	$0.9c_2$	1	9	0	9	0.9831	3.43	7.44	0.7858

Table 4: Performance of the Heuristic Method and the Fixed-Price Policy

We define performance as the ratio of the value function evaluated at the heuristic initial inventory divided by the value function evaluated at the optimal initial inventories; in both cases, we assume we can follow the optimal pricing policy. We found that the heuristic is very effective in the ease of computation load and performance. In addition, we examine the value of dynamic pricing by evaluating the performance of a fixed-price policy where the initial inventories are set to the optimal level as in Table 3 but the prices of the two products are held constant through the transition period. We obtain the optimal fixed prices through a two-variable search. The fixed prices underperform the optimal dynamic pricing by 6% to 23%, depending on the parameters.

7. Discussions and Future Research

The main contribution of this research is to address the pricing problem in a special albeit ubiquitous industry context – inter-generational product transition. We solve for the optimal prices of the two generations of products for any given inventory level at any given time during the transition. We extend the existing literature on dynamic pricing for substitutable products to include time-varying demand patterns, namely the replacement of the new product for the old product. We characterize the behavior of the optimal prices, as it is affected by several factors, including replacement, substitution, external competition, and scarcity. Lastly, we present a heuristic method for determining the initial inventories, extending from the approximation method by Dong et al. (2009). Compared with the optimal initial inventory obtained through the enumeration method, the heuristic performs well and is very effective in reducing the computation load.

We make several simplifying assumptions so that we can develop meaningful results and insights for the complicated real problem under study. We assume a stationary customer arrival process, but our model can be adapted to permit a time-varying λ to address seasonality or other demand cycles. In fact, the solution in Proposition 2 still holds with a time-varying λ. We address a dynamic pricing problem with only two generations of products. In some instances there are multiple generations of products selling during the same time period. In that case, the MNL model allows an easy extension and we simply add more choices to the MNL model.

In the examples we supplied, the new product is considered a better product and is on a steady path to replace the old product. For the problem context we focus on, this is mostly the case as the newer product is an upgrade version and typically has better features and performances. In practice, there are cases where the new product is not necessarily better due to design flaws or other issues. Nevertheless, the model and the solution given in Proposition 2 hold true for a more general case where the products have time-varying attributes and do not require $a_i(t)$ to change in any particular direction over time. However, the results derived in Sections 5 regarding the product replacement effect may change if the $a_i(t)$ has abnormal behavior. For instance, if $a_2(t)$ drops sharply after some initial take off and never exceeds $a_1(t)$, the optimal prices will be dominated by the old product and the price trend would be similar to those we observe for a single product. In a worse case, the new product never takes off, but the mere presence of it causes the customers to believe the old product is becoming obsolete (i.e., $a_2(t)$ does not increase much but $a_1(t)$ decreases regardless), the company may be forced to monotonically price down both products over time even in the absence of a time-increasing $u_0(t)$.

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Pricing Decisions during Inter-generational Product Transition

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Online Appendix

Proof of Lemma 1.

i)
From equation (1) and (8), we have

$$\frac{\partial^2 \rho_0(\mathbf{r},t)}{\partial r_i^2} = \rho_0 \rho_i [(2\rho_i - 1) g'(r_i)^2 + g''(r_i)], \tag{A-1}$$

$$\frac{\partial^2 \rho_i(\mathbf{r},t)}{\partial r_i^2} = \rho_i (\rho_i - 1)[(2\rho_i - 1) g'(r_i)^2 + g''(r_i)], \tag{A-2}$$

$$\frac{\partial^2 \rho_i(\mathbf{r},t)}{\partial r_j^2} = \rho_i \rho_j [(2\rho_j - 1) g'(r_j)^2 + g''(r_j)], \tag{A-3}$$

$$\frac{\partial^2 \rho_0(\mathbf{r},t)}{\partial r_i \partial r_j} = 2\rho_0 \rho_i \rho_j g'(r_i) g'(r_j) \tag{A-4}$$

$$\frac{\partial^2 \rho_i(\mathbf{r},t)}{\partial r_i \partial r_j} = \rho_i \rho_j g'(r_i) g'(r_j)(2\rho_i - 1) \tag{A-5}$$

Therefore,

$$\frac{\partial^2 h_t(\mathbf{r},x_1,x_2)}{\partial r_1^2} = \lambda\{2g'(r_1)\rho_1(\rho_1 - 1)$$
$$+[g''(r_1) + g'(r_1)^2(2\rho_1 - 1)][\rho_1(\rho_1 - 1)(r_1 - \Delta_1 V_{t+1}(x_1,x_2)) + \rho_1\rho_2(r_2 - \Delta_2 V_{t+1}(x_1,x_2))]\}$$

Substituting equation (7) and (8) into the second order derivative yields

$$\frac{\partial^2 h_t(\mathbf{r},x_1,x_2)}{\partial r_1^2} = \lambda\{2g'(r_1)\rho_1(\rho_1 - 1) + [g''(r_1) + g'(r_1)^2(2\rho_1 - 1)][-\frac{\rho_1}{g'(r_1)}]\}$$
$$= -\lambda\rho_1[g'(r_1) + \frac{g''(r_1)}{g'(r_1)}]$$

From Assumption 3, $\frac{\partial^2 h_t(\mathbf{r},x_1,x_2)}{\partial r_1^2} < 0$ at the points with zero slope. Similarly, we can derive that

$\frac{\partial^2 h_t(\mathbf{r},x_1,0)}{\partial r_1^2} = -\lambda\overline{\rho}_1[g'(r_1) + \frac{g''(r_1)}{g'(r_1)}] < 0$ at the points with zero slope. That is, $h_t(\mathbf{r},x_1,x_2)$ is quasi-concave in r_1.

Similarly, we show that $h_t(\mathbf{r}, x_1, x_2)$ is quasi-concave in r_2.

ii)

From equation (4), we derive

$$\frac{\partial^2 h_t(\mathbf{r}, x_1, x_2)}{\partial r_1 \partial r_2} = \lambda \frac{\partial^2 \rho_1}{\partial r_1 \partial r_2}(r_1 + V_{t+1}(x_1 - 1, x_2)) + \lambda \frac{\partial \rho_1}{\partial r_2} + \lambda \frac{\partial^2 \rho_2}{\partial r_1 \partial r_2}(r_2 + V_{t+1}(x_1, x_2 - 1)) + \lambda \frac{\partial \rho_2}{\partial r_1} + \lambda \frac{\partial^2 \rho_0}{\partial r_1 \partial r_2} V_{t+1}(x_1, x_2)$$

Substituting equation (8), (A-4), and(A-5) into the above and simplify, we have

$$\frac{\partial^2 h_t(\mathbf{r}, x_1, x_2)}{\partial r_1 \partial r_2} = \lambda \rho_1 \rho_2 g'(r_1) g'(r_2)[\frac{1}{g'(r_1)} + \frac{1}{g'(r_2)} + (2\rho_1 - 1)(r_1 - \Delta_1 V_{t+1}(x_1, x_2)) + (2\rho_2 - 1)(r_2 - \Delta_2 V_{t+1}(x_1, x_2))] \qquad \text{(A-6)}$$

Substitute equation (8) into equation (7) and simplify, we obtain the first order condition

$$\frac{1}{g'(r_1)} + (\rho_1 - 1)(r_1 - \Delta_1 V_{t+1}(x_1, x_2)) + \rho_2 (r_2 - \Delta_2 V_{t+1}(x_1, x_2)) = 0$$

$$\frac{1}{g'(r_2)} + \rho_1 (r_1 - \Delta_1 V_{t+1}(x_1, x_2)) + (\rho_2 - 1)(r_2 - \Delta_2 V_{t+1}(x_1, x_2)) = 0$$

Adding these two equations together, we have

$$\frac{1}{g'(r_1)} + \frac{1}{g'(r_2)} + (2\rho_1 - 1)(r_1 - \Delta_1 V_{t+1}(x_1, x_2)) + (2\rho_2 - 1)(r_2 - \Delta_2 V_{t+1}(x_1, x_2)) = 0$$

Therefore, from equation (A-6), we have

$\frac{\partial^2 h_t(\mathbf{r}, x_1, x_2)}{\partial r_1 \partial r_2} = 0$ whenever the first-order condition $\frac{\partial h_t(\mathbf{r}, x_1, x_2)}{\partial r_i} = 0,\ i = 1,2$ is satisfied.□

Proof of Proposition 1.

From Lemma 1, for any given r_1, $h_t(\mathbf{r}, x_1, x_2)$ is strictly quasi-concave in r_2. Thus for a given r_1, we can find a unique optimal $r_2^*(r_1)$ that maximizes $h_t(\mathbf{r}, x_1, x_2)$. We define $\widetilde{h}_t(r_1, x_1, x_2) \equiv h_t((r_1, r_2^*(r_1)), x_1, x_2)$. We now show that $\widetilde{h}_t$ is strictly quasi-concave in r_1.

For notational brevity, we drop the arguments x_1 and x_2 in the rest of the proof.

Since for a given r_1, $r_2^*(r_1)$ maximizes $h_t(\mathbf{r})$, we have $\left.\frac{\partial h_t(r_1,r_2)}{\partial r_2}\right|_{r_2=r_2^*(r_1)}=0$. Thus

$\frac{\partial \widetilde{h}_t(r_1)}{\partial r_1}=\left.\frac{\partial h_t(r_1,r_2)}{\partial r_1}\right|_{r_2=r_2^*(r_1)}+\left.\frac{\partial h_t(r_1,r_2)}{\partial r_2}\right|_{r_2=r_2^*(r_1)}\cdot\frac{dr_2^*(r_1)}{dr_1}=\left.\frac{\partial h_t(r_1,r_2)}{\partial r_1}\right|_{r_2=r_2^*(r_1)}$. We then have the second order derivative

$$\frac{\partial^2 \widetilde{h}_t(r_1)}{\partial r_1^2}=\left.\frac{\partial^2 h_t(r_1,r_2)}{\partial r_1^2}\right|_{r_2=r_2^*(r_1)}+\left.\frac{\partial^2 h_t(r_1,r_2)}{\partial r_1\partial r_2}\right|_{r_2=r_2^*(r_1)}\cdot\frac{dr_2^*(r_1)}{dr_1} \qquad \text{(A-7).}$$

Now suppose that the first order condition for $\widetilde{h}_t$ is satisfied, i.e., $\frac{\partial \widetilde{h}_t}{\partial r_1}=0$. Then $\left.\frac{\partial h_t}{\partial r_1}\right|_{r_2=r_2^*(r_1)}=0$. From Lemma 1 (i), the first term in the RHS of equation (A-7) is negative; from Lemma 1 (ii), the second term in the RHS of equation (A-7) equals 0. Therefore, $\frac{\partial^2 \widetilde{h}_t}{\partial r_1^2}<0$ whenever $\frac{\partial \widetilde{h}_t}{\partial r_1}=0$, i.e., $\widetilde{h}_t$ is strictly quasi-concave in r_1. If $\hat{\mathbf{r}}=(\hat{r}_1,\hat{r}_2)$ satisfies the first-order condition for h_t, it is easy to see that $\hat{r}_2=r_2^*(\hat{r}_1)$ and that $\hat{r}_1$ satisfies the first-order condition for $\widetilde{h}_t$. Given that $\widetilde{h}_t$ is strictly quasi-concave in r_1, $\hat{r}_1$ is the unique maximizer of $\widetilde{h}_t$ and thus the price vector $\hat{\mathbf{r}}=(\hat{r}_1,\hat{r}_2)$ is the unique maximizer for h_t.□

Proof of Proposition 2.

From equation (10), we have $r_1-\Delta_1 V_{t+1}(x_1,x_2)=r_2-\Delta_2 V_{t+1}(x_1,x_2)$ and thus

$r_2=r_1-\Delta_1 V_{t+1}(x_1,x_2)+\Delta_2 V_{t+1}(x_1,x_2)$.

Rewrite equation (10) for $i=1$ as $r_1-\Delta_1 V_{t+1}(x_1,x_2)=\frac{1}{\beta}(1+e^{a_1(t)-\beta r_1-u_0(t)}+e^{a_2(t)-\beta r_2-u_0(t)})$

Substituting $r_2=r_1-\Delta_1 V_{t+1}(x_1,x_2)+\Delta_2 V_{t+1}(x_1,x_2)$ into the above equation, we obtain

$r_1-\Delta_1 V_{t+1}(x_1,x_2)=\frac{1}{\beta}(1+e^{a_1(t)-\beta r_1-u_0(t)}+e^{a_2(t)-\beta r_1+\beta\Delta_1 V_{t+1}(x_1,x_2)-\beta\Delta_2 V_{t+1}(x_1,x_2)-u_0(t)})$

Simple algebraic transformation of the above yields $Ye^{Y} = Z$

where $Y \equiv \beta r_1 - \beta\Delta_1 V_{t+1}(x_1, x_2) - 1$ and $Z \equiv e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,x_2)} + e^{a_2(t)-u_0(t)-1-\beta\Delta_2 V_{t+1}(x_1,x_2)}$

By the definition of the Lambert's *W* function, we have $Y = W(Z)$, i.e.,

$\beta r_1 - \beta\Delta_1 V_{t+1}(x_1, x_2) - 1 = W(Z)$

Hence $r_1^* = \Delta_1 V_{t+1}(x_1, x_2) + \dfrac{1}{\beta}[1 + W(Z)]$.

From equation (4), we have

$$V_t(x_1, x_2) = V_{t+1}(x_1, x_2) + \lambda\rho_1(\mathbf{r}^*, t)(r_1^* - \Delta_1 V_{t+1}(x_1, x_2)) + \lambda\rho_2(\mathbf{r}^*, t)(r_2^* - \Delta_2 V_{t+1}(x_1, x_2))$$

Substituting condition (10) into the above yields

$$V_t(x_1, x_2) = V_{t+1}(x_1, x_2) + \frac{\lambda}{\beta}\frac{\rho_1(\mathbf{r}^*, t) + \rho_2(\mathbf{r}^*, t)}{\rho_0(\mathbf{r}^*, t)})$$

From equation (11),

$$\frac{\rho_i(\mathbf{r}^*, t)}{\rho_0(\mathbf{r}^*, t)} = \exp(a_i(t) - u_0(t) - \beta r_i^*)$$
$$= \exp(a_i(t) - u_0(t) - 1 - \beta\Delta_i V_{t+1}(x_1, x_2) - W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,x_2)} + e^{a_2(t)-u_0(t)-1-\beta\Delta_2 V_{t+1}(x_1,x_2)}))$$

Therefore,

$$\frac{\rho_1(\mathbf{r}^*, t) + \rho_2(\mathbf{r}^*, t)}{\rho_0(\mathbf{r}^*, t)} = Ze^{-W(Z)} = W(Z)$$

where $Z \equiv e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,x_2)} + e^{a_2(t)-u_0(t)-1-\beta\Delta_2 V_{t+1}(x_1,x_2)}$

Hence, $V_t(x_1, x_2) = V_{t+1}(x_1, x_2) + \dfrac{\lambda}{\beta} W(Z)$.

Similarly, we can show that this holds when one product runs out. □

Proof of Proposition 3.

From equations (11),

$$r_1^*(t, \infty, 0) = s_1 + \frac{1}{\beta}[1 + W(e^{a_1(t)-u_0-1-\beta s_1})] \text{ and } r_2^*(t, 0, \infty) = s_2 + \frac{1}{\beta}[1 + W(e^{a_2(t)-u_0-1-\beta s_2})]$$

As $a_1(t)$ decreases in *t* and $a_2(t)$ increases in *t*, we have $r_1^*(t, \infty, 0)$ decreases in *t* and $r_2^*(t, 0, \infty,)$ increases in *t*.

From equation (11), $r_i^*(t, \infty, \infty) = s_i + \dfrac{1}{\beta}[1 + W(Z)]$ where $Z = e^{a_i(t)-u_0(t)-1-\beta s_i} + e^{a_j(t)-u_0(t)-1-\beta s_j}$.

From Assumption 5, $\dfrac{da_1(t)}{dt} = -k$ and $\dfrac{da_2(t)}{dt} = k$.

We then have $\dfrac{dr_i^*(t,\infty,\infty)}{dt} = \dfrac{1}{\beta_r}\dfrac{kW(Z)}{Z(1+W(Z))}(-e^{a_1(t)-u_0-1-\beta s_1} + e^{a_2(t)-u_0-1-\beta s_2})$.

Therefore, $\dfrac{dr_i^*(t,\infty,\infty)}{dt} \le 0$ iff $a_1(t) - \beta s_1 \ge a_2(t) - \beta s_2$, or equivalently $t \le \dfrac{a_0 + \beta(s_2 - s_1)}{2k}$.

That is, $r_i^*(t,x_1,x_2)$ decreases in t for $t \in [0,\bar{t}]$ and increases in t for $t \in [\bar{t},T]$, where

$\bar{t} = \dfrac{a_0 + \beta(s_2 - s_1)}{2k}$. □

Proof of Corollary 2.

From equation (11),

$$\frac{\partial r_i^*(t,x_1,x_2)}{\partial u_0(t)} = \frac{1}{\beta}\frac{\partial W(Z)}{\partial u_0(t)} = \frac{1}{\beta}\frac{W(Z)}{(1+W(Z))Z}\frac{\partial Z}{\partial u_0(t)} = \frac{1}{\beta}\frac{W(Z)}{(1+W(Z))Z}(-1)Z = -\frac{W(Z)}{\beta(1+W(Z))}$$

From equation (15), $W(Z) = \dfrac{\rho_1(\mathbf{r}^*,t) + \rho_2(\mathbf{r}^*,t)}{\rho_0(\mathbf{r}^*,t)}$, thus $\dfrac{\partial r_i^*(t,x_1,x_2)}{\partial u_0(t)} = -\dfrac{\rho_1(\mathbf{r}^*,t) + \rho_2(\mathbf{r}^*,t)}{\beta}$.□

Proof of Proposition 4.

From equation (11), $r_1^*(t,x_1,0) = \Delta_1 V_{t+1}(x_1,0) + \dfrac{1}{\beta}[1 + W(Z_1)]$ where $Z_1 \equiv e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)}$

For the Lambert's W function, $W(Z_1) = \ln Z_1 - \ln W(Z_1)$. Thus we have

$$r_1^*(t,x_1,0) = \Delta_1 V_{t+1}(x_1,0) + \frac{1}{\beta}[1 + \ln Z_1 - \ln W(Z_1)] = \frac{1}{\beta}[a_1(t) - u_0(t) - \ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)})]$$

Thus $r_1^*(t,x_1,0)$ is increasing in $\Delta_1 V_{t+1}(x_1,0)$.

Next we show by induction that $\Delta_1 V_{t+1}(x_1,0)$ is non-increasing in x_1.

When $t = T$, $\Delta_1 V_{T+1}(x_1,0) = s_1$, thus the base case is true.

Assume for induction that $\Delta_1 V_{t+1}(x_1,0)$ is non-increasing in x_1, we show that $\Delta_1 V_t(x_1,0)$ is non-increasing in x_1.

From equation (13),

$$V_t(x_1,0) = V_{t+1}(x_1,0) + \frac{\lambda}{\beta}W(Z_1) = V_{t+1}(x_1,0) + \frac{\lambda}{\beta}[\ln Z_1 - \ln W(Z_1)]$$

$$= (1-\lambda)V_{t+1}(x_1,0) + \lambda V_{t+1}(x_1-1,0) + \frac{\lambda}{\beta}[a_1(t) - u_0(t) - 1 - \ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)})]$$

We then have

$$\Delta_1 V_t(x_1+1,0) = (1-\lambda)\Delta_1 V_{t+1}(x_1+1,0) + \lambda \Delta_1 V_{t+1}(x_1,0) - \frac{\lambda}{\beta}\ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1+1,0)}) + \frac{\lambda}{\beta}\ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)})$$

Also, deriving directly from equation (13), we have:

$$\Delta_1 V_t(x_1,0) = \Delta_1 V_{t+1}(x_1,0) + \frac{\lambda}{\beta}W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)}) - \frac{\lambda}{\beta}W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1-1,0)})$$

Thus

$$\begin{aligned}\Delta_1 V_t(x_1+1,0) - \Delta_1 V_t(x_1,0) = {} & (1-\lambda)[\Delta_1 V_{t+1}(x_1+1,0) - \Delta_1 V_{t+1}(x_1,0)] \\ & - \frac{\lambda}{\beta}\ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1+1,0)}) + \frac{\lambda}{\beta}\ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)}) \\ & - \frac{\lambda}{\beta}W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)}) + \frac{\lambda}{\beta}W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1-1,0)})\end{aligned}$$

By induction assumption, $\Delta_1 V_{t+1}(x_1+1,0) < \Delta_1 V_{t+1}(x_1,0) < \Delta_1 V_{t+1}(x_1-1,0)$. As *W*() is an increasing function, $\ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1+1,0)}) > \ln W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)})$ and $W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1,0)}) > W(e^{a_1(t)-u_0(t)-1-\beta\Delta_1 V_{t+1}(x_1-1,0)})$.

Therefore, $\Delta_1 V_t(x_1+1,0) - \Delta_1 V_t(x_1,0) < 0$, i.e., $\Delta_1 V_{t+1}(x_1,0)$ is non-increasing in x_1. We already show that $r_1^*(t,x_1,0)$ is increasing in $\Delta_1 V_{t+1}(x_1,0)$; thus $r_1^*(t,x_1,0)$ is non-increasing in x_1.

Similarly, we can show that $r_2^*(t,0,x_2)$ is non-increasing in x_2 .□

Additional Numerical Cases for Table 3.

			c_1=2, c_2=3		c_1=3, c_2=2					c_1=2, c_2=3		c_1=3, c_2=2	
k	s_1	s_2	x_1	x_2	x_1	x_2	k	s_1	s_2	x_1	x_2	x_1	x_2
0.06	0.1c_1	0.5c_2	1	3	0	4	0.12	0.1c_1	0.5c_2	1	6	0	8
0.06	0.1c_1	0.6c_2	1	3	0	5	0.12	0.1c_1	0.6c_2	1	7	0	8
0.06	0.1c_1	0.7c_2	1	3	0	5	0.12	0.1c_1	0.7c_2	1	7	0	9
0.06	0.1c_1	0.8c_2	1	3	0	5	0.12	0.1c_1	0.8c_2	1	8	0	9
0.06	0.1c_1	0.9c_2	1	4	0	6	0.12	0.1c_1	0.9c_2	1	9	0	10
0.06	0.2c_1	0.5c_2	1	3	0	4	0.12	0.2c_1	0.5c_2	1	6	0	8
0.06	0.2c_1	0.6c_2	1	3	0	5	0.12	0.2c_1	0.6c_2	1	7	0	8
0.06	0.2c_1	0.7c_2	1	3	0	5	0.12	0.2c_1	0.7c_2	1	7	0	9
0.06	0.2c_1	0.8c_2	1	3	0	5	0.12	0.2c_1	0.8c_2	1	8	0	9
0.06	0.2c_1	0.9c_2	1	4	0	6	0.12	0.2c_1	0.9c_2	1	9	0	10

0.06	$0.3c_1$	$0.5c_2$	1	3	0	4	0.12	$0.3c_1$	$0.5c_2$	1	6	0	8
0.06	$0.3c_1$	$0.6c_2$	1	3	0	5	0.12	$0.3c_1$	$0.6c_2$	1	7	0	8
0.06	$0.3c_1$	$0.7c_2$	1	3	0	5	0.12	$0.3c_1$	$0.7c_2$	1	7	0	9
0.06	$0.3c_1$	$0.8c_2$	1	3	0	5	0.12	$0.3c_1$	$0.8c_2$	1	8	0	9
0.06	$0.3c_1$	$0.9c_2$	1	4	0	6	0.12	$0.3c_1$	$0.9c_2$	1	9	0	10
0.06	$0.4c_1$	$0.5c_2$	2	2	1	4	0.12	$0.4c_1$	$0.5c_2$	1	6	0	8
0.06	$0.4c_1$	$0.6c_2$	1	3	1	4	0.12	$0.4c_1$	$0.6c_2$	1	7	0	8
0.06	$0.4c_1$	$0.7c_2$	2	3	1	5	0.12	$0.4c_1$	$0.7c_2$	1	7	0	9
0.06	$0.4c_1$	$0.8c_2$	2	3	1	5	0.12	$0.4c_1$	$0.8c_2$	1	8	0	9
0.06	$0.4c_1$	$0.9c_2$	1	4	1	6	0.12	$0.4c_1$	$0.9c_2$	1	9	0	10
0.06	$0.5c_1$	$0.5c_2$	2	2	1	4	0.12	$0.5c_1$	$0.5c_2$	1	6	0	8
0.06	$0.5c_1$	$0.6c_2$	2	3	1	4	0.12	$0.5c_1$	$0.6c_2$	1	7	0	8
0.06	$0.5c_1$	$0.7c_2$	2	3	1	5	0.12	$0.5c_1$	$0.7c_2$	1	7	0	9
0.06	$0.5c_1$	$0.8c_2$	2	3	1	5	0.12	$0.5c_1$	$0.8c_2$	1	8	0	9
0.06	$0.5c_1$	$0.9c_2$	2	4	1	6	0.12	$0.5c_1$	$0.9c_2$	1	9	0	10