Optimal Pricing for a Short Life-Cycle Product When Customer Price-Sensitivity Varies Over Time

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Abstract

Technology products often experience a life-cycle demand pattern that resembles a diffusion process, with weak demand in the beginning and the end of the life cycle and high demand intensity in between. The customer price-sensitivity also changes over the life cycle of the product. We study the pre-specified pricing decision for a product that exhibits such demand characteristics. In particular, we determine the optimal set of discrete prices and the times to switch from one price to another, when a limited number of price changes are allowed. Our study shows that the optimal prices and switching times show interesting patterns that depend on the product's demand pattern and the change in the customers' price sensitivity over the life cycle of the product.

Keywords: *Discrete Pricing, Pre-specified Pricing Strategy, Product Life Cycle, Time-Varying Price Sensitivity*

1 Introduction

This paper presents and analyzes a model for optimal inter-temporal pricing of a short life-cycle product when customer sensitivity to price varies over time. A well-known example of this type of product is the microprocessor, which is a key component in personal computers. Every two years, Intel introduces a new silicon process technology onto an existing product architecture (called a "tick"), and every other year it introduces a new architecture onto an existing silicon process technology (a "tock"). This synchronized tick-tock product development strategy drives a new generation of processors to the market every year [37]. Such frequent product introduction leads to short product life cycles (often less than two years) and is accompanied by systematic price reductions throughout each product's life cycle [10]. Figure 1 illustrates the price path of several Intel processors over a period of eighteen months.

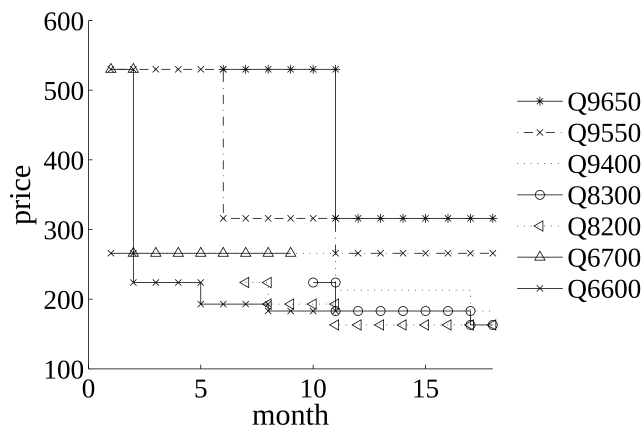

Figure 1: Sample Price Paths for Intel Processors (between March 2008 and August 2009). Data from Intel Corporation Website [19].

Price reduction over a product's life cycle has been attributed to factors such as declining production cost [41], increasing competition [12], or concerns of excess inventory [16]. As production technology matures, the unit cost of a product decreases, and therefore the product price often decreases in a cost-based pricing model. As competitive products enter into the market, companies often lower prices in order to keep their customers. With short life-cycle products, the obsolescence cost is high; when companies need to push sales within a limited time window, they often reduce price as the selling window becomes smaller. Yet another important factor that leads to price reduction over time is price discrimination based on the time of purchase. It is commonly believed that the early adopters of a product are much less price-sensitive than the population who buys later. For example, when iPhone 3G was first introduced, the bidding price for a 16G-byte iPhone 3G on e-Bay topped $1000 despite Apple's list price of $299 [36]. Apple dropped the price for the same phone to $199 only a year later in order to attract new buyers. Because of the time-varying price-sensitivity, it is optimal to charge different prices at different stages of a product's life cycle to maximize revenue.

In this paper, we consider the problem of revenue maximization for a short life-cycle product given such inter-temporal price-sensitivity variations. We investigate the optimal inter-temporal pricing policy and the impact of a product's demand pattern on the optimal pricing policy. The demand of technology products such as microprocessors often follows a bell-shaped curve. For example, the sales ramp up as the product awareness and customer confidence grow, and later decline as the product ages and a newer generation product becomes available. The exact shape of the curve depends upon many product-specific characteristics. For example, a processor with high speed and low power consumption may reach a higher demand peak, stay at the peak longer, and decline at a slower rate, compared to a product with weaker performance on these metrics. If a newly-released processor is "pin-compatible" with the existing platforms (i.e., a customer can simply plug the new processor in his computer without purchasing additional hardware), the adoption will be faster and the demand will peak sooner. Managers at Intel know their market and product well and understand the underlying demand pattern and its evolution. They are interested in learning how one product's inter-temporal pricing pattern should differ from another given the differences in their demand patterns. For example, Table 1 shows the actual discrete prices and their percentage reductions for several processor product categories. We observe that some products (the Q series, the T series, and the Celeron processors) show decelerating percentage reduction in price over time. Others exhibit more complex discounting patterns (the E and Z series). Managers question whether a particular pattern of price discount makes sense for each specific product – when should they be more aggressive or less aggressive on pricing given what they know about the product. In this paper, we present a model that addresses this question and provides qualitative predictions of how changes in the underlying demand pattern would affect the optimal pricing decision of a product.

Our research originates from a collaborative project with Intel. In contrast to companies who sell directly to consumers and are able to continuously change the price of their products, Intel's customers are Original Equipment Manufacturers (OEMs) who use Intel's product as a component in a product that they in turn sell to end users. Both Intel and its OEM customers desire a certain degree of short-term price stability to facilitate business planning. With stable prices, OEMs are able to better estimate market demand and can pass on the demand estimate to Intel, which is particularly helpful as the production lead time for processors is three months. As a result, Intel only makes a limited number of price changes during a product's life cycle, as exemplified in Figure 1, and such price changes

Table 1: Microprocessor Prices Used by Intel, and Corresponding Percentage Reductions (Data from Intel Corporation Website [19])

Product	Discrete Prices (In Sequence) Percent Price Reductions					
Core 2 Quad Processor	530	316	266	224	193	183
Q series, Desktop		40.4%	15.8%	15.8%	13.8%	5.2%
Core 2 Duo Processor	266	183	163	133	113	
E series, Desktop		31.2%	10.9%	18.4%	15.0%	
Core 2 Duo Processor	530	316	241	209		
T series, Mobile		40.4%	23.7%	13.3%		
Atom Processor	135	70	40	20		
Z series, Mobile		48.1%	42.9%	50.0%		
Celeron Processor	134	107	86			
Mobile		20.1%	19.6%			

are pre-specified at the beginning of the product life cycle. In this paper, we determine a price schedule that consists of a sequence of prices and a set of switching times at which the price changes from one to another. As discussed earlier, the problem is complicated by time-varying price-sensitivity of the customers and product-specific demand patterns. To our knowledge, no prior research has addressed pricing problems with these features.

In this paper, we adopt a simplified view of customers. Specifically, “customers” in this paper refer to the OEM customers, not end customers. It is true that end customers ultimately drive the demand and the price-sensitivity change. However, very few end customers directly buy processors from the manufacturers and most purchase products containing processors as components. Therefore, we treat the OEMs as the “delegate” customers whose demand reflect the end market demand and we neglect the detailed dynamics between the OEMs and end customers. That is, we assume that the OEM customers simply pass along the end-market demand and possess the same price-sensitivity characteristics as end customers.

We also assume that there is no strategic behavior from the customers. Given the complexity of information needed to predict future prices, it is reasonable to expect that individual OEMS may not act strategically. The assumption of completely non-strategic customers is somewhat restrictive; in practice certain OEM customers might be tempted to act strategically, for example, by delaying an order when anticipating a price reduction. However, the magnitude of such strategic behavior is small because most OEM customers operate with very low inventory and delaying orders exposes them to high stockout risk. Therefore, we do not consider strategic customer behaviors in this paper.

Many considerations may go into the pricing decisions including production cost, competition, substitute products, inventory and capacity. In this paper, we focus on one particular aspect of this complicated situation – how the evolution of optimal prices over time depends on the characteristics of a life cycle. We consider a case of a single monopoly firm, facing a deterministic, but time-varying price-sensitive demand for a single product. The optimal price path is determined at the beginning of the horizon, prior to the release of the product, and is based on the best demand estimate at the time of decision-making. While we recognize that this model is a simplification of the reality, it captures the first-order effects of the demand pattern and price sensitivity, and it represents the first stage of a hierarchical decision-making process popular in practice.

One of the models often used for life-cycle demand is the diffusion model, first proposed by Bass [4]. The original Bass diffusion model assumes that new product adoption starts with some innovators who adopt the product, and then those customers who have purchased the product can influence other potential buyers to adopt (also known as the *word-of-mouth* effect). Many researchers extend the Bass model to include the impact of price. Bass et al. [6] categorize these extensions into *current-effect* models [5, 20, 39] and models with *carry-through* property [22, 23, 29, 33, 24]. The *carry-through* models extend the impact of price into future demands whereas the *current-effect* models allow the effect of price to apply only to instantaneous adoption rate at a given time. Although models with the *carry-through* property are sometimes desirable as they capture the dependency of demand on historic prices, they are intractable and difficult to analyze.

In the majority of this paper we adopt a *current-effect* model. We assume that the diffusion of new product awareness follows a given pattern and is independent of price. Each individual who becomes newly aware determines whether or not to purchase the product and this purchasing decision is strongly influenced by the price of the product. This is similar to Speece and MacLachlan [39], who incorporated price by multiplying the cumulative adoption by a price function, Li and Shen [25], who consider a variation of the Bass model by assuming that the word-of-mouth effect is driven by customers who already own the product as well as those who have considered purchasing, and He et al. [18], who model the instantaneous demand rate as a product of the current product awareness (which follows a diffusion-like differential equation) and the current price. Our discussion with Intel indicates that the diffusion of new product awareness is driven by many more prevailing factors than price (such as product performance, features, compatibility, advertisement, expert reviews, etc). The customer's final purchasing decision will be greatly affected by price, which forms the basis of our demand model.

Using this model, we develop procedures to determine the timing and magnitude of price changes that will maximize revenue over a fixed life cycle when a given number of price changes are allowed. Based on analytical results and numerical examples, we develop qualitative guidelines for inter-temporal price-setting in this environment. Later in Section 6.2, we relax the assumption of separable price effect and explore a demand model with *carry-through* price effect. We find that in addition to mark-down strategies, the optimal pricing solution reflects an "early promotion" strategy which is used to speed up the initial adoption and amplify product diffusion. The insights from this analysis can help managers determine how to price a product differently from others based on the projected demand pattern and price sensitivity path for that product.

1.1 Previous Related Work

While the literature of pricing and revenue management has been vast and growing, we highlight papers related to the following key features used in our model.

(1) *Time-Varying Price Sensitivity.* Earlier work on inter-temporal pricing problems focuses on price discrimination [41]. When a monopolist faces a market of consumers with heterogenous valuations for a product, it first sets a high price in order to reap a high premium from those who are more willing to pay, and later reduces the price to attract those who are less willing to pay (but more willing to wait) to improve the total revenue. Charging different prices based on the time of purchase thus provides a means of mitigating the information asymmetry between the seller and the buyers. Harris and Raviv [17] introduce a capacity limit to the above problem and derive the optimal inter-temporal price-discrimination schemes. Besanko and Winston [7] incorporate consumer rationality into the pricing model, and derive the equilibrium pricing strategy for the monopolist given that consumers optimally respond to its pricing policy by weighing the benefit of immediate purchase against that of waiting. In these papers, customers buy at different times because they have different valuations for the same product. This is similar to what we assume in this paper that customers who purchase a product at different stages of a product's life cycle have different price sensitivities; however we examine a discrete pricing problem with a restriction on the number of price changes.

More recently, Zhao and Zheng [47] show a price monotonicity property for fashion goods with limited inventory when customers' reservation price shifts monotonically over time, and they characterize the optimal discrete price policies. Xu and Hopp [46] study the dynamic pricing policies with time-varying price sensitivity and strategic customer behavior, and they show that the optimal prices form a supermartingale if the price sensitivity increases over time and a submartingale otherwise. Elmaghraby et al. [13], Su [43], and Aviv and Pazgal [2] study strategic customer behaviors when a customer's valuation of a product changes over time. We complement this literature on pricing with time-varying customer valuation by incorporating price-sensitivity changes into a *product life cycle* and studying its impact on a product's optimal life-cycle pricing pattern. A paper worth noting is Li and Shen [25], which studies the timing of a product line extension. They consider two variants of a product, a high-end version and a low-end version, which are introduced at different times to target customer segments with different valuations for the product.

(2) *Pre-Announced Pricing Strategy.* Many papers in the dynamic pricing literature consider adaptive pricing actions due to factors that change stochastically over time such as inventory [16]. In this paper, for the reasons mentioned previously, we consider pricing decisions that are made at the beginning of the planning horizon and we do not consider any recourse action. There are a handful of papers which study a pre-announced pricing strategy under which the price path for the entire horizon is fixed at the beginning. Bitran and Mondschein [8] study a pricing strategy with pre-announced fixed percentage discount per time period in retail stores. Elmaghraby et al. [13] investigate the optimal pre-announced markdowns when a company has limited supply and faces strategic customers with multi-unit demand. They consider the optimal number of markdowns and the level of each markdown. Aviv and Pazgal [2] study both inventory-contingent and pre-announced discount strategies with strategic customers and compare their performances. We also point to the literature on price commitment in the presence of strategic customer behaviors, for example, by Su and Zhang [44] whose results suggest that price and quantity commitment to consumers can improve retail profits (price commitment in this case implies that the retailer commits to a high price and does not discount), and by Cachon and Swinney [9] who show that price commitment is generally not as good as dynamic discounting. This literature focuses on contexts with limited supply and often price commitment is compared with dynamic pricing adjustment. In comparison to our paper, these papers do not consider the timing of discounts. We also note that even though most of the papers in this stream of literature incorporate the finiteness of the selling horizon, the diffusion demand pattern – that is characteristic of many products with a short life cycle – has not been explicitly modeled in any of the existing models, to our knowledge.

(3) *Limited Price Change.* In this paper, we consider a pricing policy in which the total number of price changes is fixed. Feng and Gallego [14] study a similar problem with an added restriction of at most two discrete prices over the planning horizon; in this case, the optimal decision consists of the pair of prices and the switching time. Feng and Gallego [15] consider the optimal switching times from one price path to another price path when the menu of price paths are given. Furthermore, Netessine [31] explores various complementarity conditions between prices and switching times to identify conditions for a unique timing solution for a given set of prices. Our paper extends these works by characterizing both the optimal switching times *and* the optimal sequence of prices for any given total number of price changes.

(4) *Multiplicative, Separable Demand.* In this paper, we adopt a *current-effect* demand

model which has a multiplicative separable function form. Examples of this demand form are seen in both normative and empirical work. These models are able to empirically fit the sales and price data quite well [6] and are more tractable than models that allow price to effect future demand. Smith and Achabal [38] use a demand rate function that is multiplicative separable in seasonal effect, inventory effect and price sensitivity to obtain a closed-form pricing solution. Bass [5] extends the original Bass diffusion model to the case where pricing is an endogenous control. He models the demand rate as a multiplicative separable form of the adoption rate and price sensitivity, which leads to tractable analysis and useful insights. Jain and Rao [20] propose a model in which price affects the effective market potential. This model is similar to Bass [5] as the price affects only the sales function but not the basic diffusion process. Speece and MacLachlan [39] incorporate the effect of price by multiplying a price factor with the cumulative adoption. Recently, He et al. [18] study cooperative advertising and pricing decisions between a retailer and a wholesaler, using a demand model that is multiplicatively separable in awareness and price.

In this paper, we consider the inter-temporal pricing problem with a limited number of price changes. We investigate the effect of time-varying demand pattern on pricing decisions. We observe that the optimal prices and price-switching times are closely related to the shape of the demand pattern. Furthermore, we capture that the customers' *price sensitivity* increases during the life cycle, i.e., the composition of the customer base changes from a pro-technology population to a more value-conscious one over a product's life cycle. To our knowledge, neither the operations management nor the marketing researchers have studied the interaction of life-cycle demand characteristics with inter-temporal price-sensitivities as a driver for pricing strategies. Lastly, we note that the model does not limit its application to diffusion patterns, but applies to any general demand pattern caused by non-price-related characteristics such as seasonality and a macroeconomic cycle.

1.2 Summary of Results and Organization

We describe in Section 2 a general model and solution approach for solving the optimal prices and switching times for any fixed number of price changes. We then specialize this model to a case where the price-sensitivity parameter increases linearly in time in Section 3. We obtain a closed-form solution when the nominal demand pattern is stationary throughout a product's life cycle as a benchmark. We show in this benchmark case that the optimal prices follow a constant percentage reduction over time and that the time duration of each price is the same on a *logarithmic* scale. When the nominal demand pattern is of a general unimodal form, but still with linear price sensitivity change, the results show a stark comparison with the benchmark case of stationary demand: In particular, the percentage price discount between two successive prices decreases (i.e., the relative price change becomes smaller) before the peak of the demand pattern and increases after this peak. This explains some companies' practice of heavy discounting toward the latter half of a product's life cycle. In addition, on the logarithmic time scale, price changes are shown to concentrate more in the region where the demand pattern peaks (instead of being evenly distributed on the same logarithmic time scale as in the benchmark case), given that the total number of price changes is fixed.

In Section 4, we relax the linearity assumption on price sensitivity and study the optimal pricing policy when the customers' price sensitivity may change over time in a nonlinear fashion. We show properties of the optimal prices and switching times when price sensitivity increases either concavely or convexly in time and contrast that with the linear case.

In Section 5, we examine cases that are not fully characterized analytically using numerical methods to further explore the impact of life-cycle demand pattern and time-varying price sensitivity on the optimal pricing decision. In addition, we consider the performance improvement of the optimal discrete-price policy as the number of allowed price changes increases, as well as the impact of demand pattern and price-sensitivity on the performance improvement. This provides useful insights on the decision of how many times a firm should adjust the price of a product.

We discuss further generalizations including nonlinear price-demand relationships and the "carry-through" effect of price in Section 6 and we conclude in Section 7 by summarizing our findings from the model.

2 The Model and the Solution Approach

2.1 The Model

We consider the pricing problem of a product during its life cycle. Following the release of the product, the demand is often initially weak due to the lack of awareness or the perceived uncertainty of quality, but then it increases and reaches its peak before it fades away due to saturation. During the life cycle, not only does the magnitude of demand change, but the consumer sensitivity to price also changes, typically increasing over time. Customers chasing the latest cutting-edge technology are the least price-sensitive and adopt the product early in a product's life cycle; those with a preference for "value" are more price-sensitive and purchase it later. Thus, the revenue depends not only on the size of the underlying demand, but also on the consumer sensitivity to price. We incorporate into our model the effect of these two features and determine the path of the optimal price during the planning horizon that maximizes the total revenue.

Let $[0, T]$ denote the planning horizon, where the release of the product occurs at time 0. We use t to represent the time instance corresponding to the product's age (i.e., time since release). Let $r(p, t)$ denote the revenue rate function, which depends on both the current price p and the product's age t. Our objective is to determine the price path $p(\cdot)$ that maximizes the total revenue during the planning horizon:

$$\int_{t=0}^{T} r(p(t), t)dt \ . \tag{2.1}$$

As discussed earlier in the introduction, this is a *current-effect* model, similar to Bass [5], Speece and MacLachlan [39], and He et al. [18]. In maximizing the total revenue function given in (2.1), we first consider two extreme scenarios with respect to the constraints on the price path. In the first scenario, a constant price should be maintained through the entire planning horizon. Then, the total revenue is a single-dimensional function, and thus can be optimized quite easily. In the second scenario, the price may change continuously over time without any inter-temporal restriction, and we refer to this model as the *continuous pricing* problem. In this case, it is optimal to simply maximize $r(p, t)$ over p for each $t \in [0, T]$ separately, which is again a single-dimensional problem. While continuous pricing is possible in certain settings, it is inappropriate in business-to-business settings in which the direct customers are OEMs who prefer price stability in their planning, as discussed earlier. For the remainder of the paper, we focus on the *discrete pricing* problem, where the number of possible price changes is limited and fixed.

The discrete pricing problem is in general analytically intractable, and poses computational difficulties. As a result, we introduce specific functional forms for the demand and revenue functions that our industry collaborators considered to be a reasonable simplification and that yield a tractable analysis. In particular, we build upon the literature of *current-effect* diffusion models to allow separability of product diffusion and price, and we extend it to incorporate both the underlying diffusion pattern and evolving customer price sensitivity into a single framework.

Assumption 1. There exist a positive function $h(t)$ and a positive continuous increasing function $b(t)$ such that $r(p, t) = h(t) \cdot g(p, t)$ where $g(p, t) = d(p, t) \cdot p$ and $d(p, t) = a - b(t)p$.

Here, the instantaneous revenue function consists of two factors. The factor $h(t)$ represents the underlying *demand pattern*, which indicates, for example, the progression of product awareness, the growth in customer confidence, and the aging of the product over time (losing appeal to newer products). There are a number of models developed for this purpose, notable examples of which are diffusion-based models (see the introduction). The other factor $g(p,t)$ represents the normalized revenue rate. To model the price-demand relationship, we use a linear form of price-demand relationship $d(p,t) = a - b(t)p$, which is one of the most widely-used in the literature [32, 27]. Here, we refer to $b(t)$ as the *price sensitivity parameter*, which varies with time to reflect the evolving price sensitivity over the planning horizon. Note that one may interpret $h(t) \cdot a$ as the maximum possible demand at time t (i.e., the demand when price approaches zero).

2.2 Discrete Pricing Problem: Solution Approach

We now consider the discrete pricing problem in which there can be n distinct prices over the planning horizon, where $n \geq 2$ is finite. We need to decide not only what values should be chosen for the set of these n prices, but also how to partition the planning horizon into n intervals, one for each price. The objective is to maximize the total revenue over the horizon. In this section, we formulate this problem as an optimization problem. This problem is a multi-dimensional problem without an easily identifiable structure. However, we show that it can be interestingly reduced to an optimization problem over only a single variable such that it can be easily solved using a simple search procedure.

We introduce some notation and basic properties. For fixed t, the choice of p that maximizes $g(p,t)$ also maximizes the revenue rate $r(p,t) = g(p,t) \cdot h(t)$, and is given by $p^*(t) = \arg\max_p r(p,t) = a/[2b(t)]$. As the price sensitivity parameter $b(t)$ is increasing in t, the instantaneous optimal price $p^*(t)$ is unique, positive and decreasing in t. The monotonicity of $p^*(t)$ in t, that the price declines over time, is commonly observed in the high-tech industry. Now, for any interval $[t_1, t_2]$, where $0 \leq t_1 \leq t_2 \leq T$, define the optimal single price in the interval by $p^*(t_1, t_2) = \arg\max_p R(p, t_1, t_2)$, where

$$R(p, t_1, t_2) \quad = \quad \int_{t_1}^{t_2} r(p,t)dt \ .$$

Then, it can be shown that $p^*(t_1, t_2)$ is decreasing in both t_1 and t_2. The proof of this result is based on the concavity of $r(p,t)$ in p and the above-mentioned monotonicity property of $p^*(t)$, and appears in Appendix A.1. Later in this paper, we will further investigate additional properties of how the optimal price evolves over time.

Multi-Dimensional Optimization for Maximizing Revenue

We denote by $\{p_i \mid i = 1, \ldots, n\}$ the sequence of n prices, and by $\{\tau_i \mid i = 1, \ldots, n-1\}$ the sequence of switching times, where price switches from p_i to p_{i+1} at time τ_i. We require $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n-1} \leq T$. For the n-price discrete pricing problem, the total revenue during the planning horizon $[0, T]$ given in (2.1) can be written using the following notation:

$$V(\tau_1, \ldots, \tau_{n-1}, p_1, \ldots, p_n) \;=\; \sum_{i=1}^{n} R(p_i, \tau_{i-1}, \tau_i) \;,$$

where we define $\tau_0 = 0$ and $\tau_n = T$ for notational convenience. Note that the total revenue V is a function of both $(\tau_1, \ldots, \tau_{n-1})$ and $(p_1, \ldots, p_n)$, a set of $(2n-1)$ decision variables. Yet, it is possible to write it as a function of $(\tau_1, \ldots, \tau_{n-1})$ only – by substituting the optimal choice of $(p_1, \ldots, p_n)$ for given $(\tau_1, \ldots, \tau_{n-1})$. In fact, as we shall see below, the optimal p_i depends only on τ_{i-1} and τ_i, the switching times that define the interval in which p_i is in effect.

Proposition 2.1. *Fix $(\tau_1, \ldots, \tau_{n-1})$. Under Assumption 1, the value of $(p_1, \ldots, p_n)$ maximizing $V(\tau_1, \ldots, \tau_{n-1}, p_1, \ldots, p_n)$ is given by $p_i = p^*(\tau_{i-1}, \tau_i)$ for each $i \in \{1, \ldots, n\}$, where*

$$p^*(\tau', \tau'') \;=\; \frac{a}{2 \cdot \overline{b}(\tau', \tau'')} \quad \text{and} \quad \overline{b}(\tau', \tau'') = \frac{\int_{\tau'}^{\tau''} b(t)h(t)dt}{\int_{\tau'}^{\tau''} h(t)dt} \;. \tag{2.2}$$

The proof of this proposition is in Appendix A.2. This proposition can be interpreted as follows. Note that $\overline{b}(\tau', \tau'')$ is a weighted average of $b(t)$ over the interval $[\tau', \tau'']$ where the weight is given by the life-cycle effect $h(t)$. Thus, by the Intermediate Value Theorem, there exists $\overline{t}(\tau', \tau'')$ in the interval $[\tau', \tau'']$ such that

$$\overline{b}(\tau', \tau'') \;=\; b(\overline{t}(\tau', \tau'')) \;, \tag{2.3}$$

and we refer to $\overline{t}(\tau', \tau'')$ as the *analytic center* of the interval $[\tau', \tau'']$. Thus, the optimal price depends on the interval only through the analytic center of the interval $[\tau', \tau'']$, and the optimal price of the interval, $p^*(\tau', \tau'')$, indeed maximizes the revenue rate $r(p, \overline{t}(\tau_{i-1}, \tau_i))$ at the analytic center. Furthermore, it can be shown that the analytical center $\overline{t}(\tau', \tau'')$ is increasing in both of its arguments (i.e. the boundaries of the interval).

Now, by substituting optimal values of prices in each interval based on Proposition 2.1, we can write the objective function in terms of $(\tau_1, \ldots, \tau_{n-1})$ only, a set of $(n-1)$ decision variables. Define

$$\tilde{V}(\tau_1, \ldots, \tau_{n-1}) \;=\; \sum_{i=1}^{n} R(p^*(\tau_{i-1}, \tau_i), \tau_{i-1}, \tau_i) \;. \tag{2.4}$$

While this function is not a separable function, it exhibits a property that perturbing τ_i does not affect the revenue outside the interval $[\tau_{i-1}, \tau_{i+1}]$. We can exploit this property to develop an iterative method in which each τ_i is repeatedly optimized.

From the first-order and second-order necessary conditions for optimality, we obtain a property that, at the price switching time τ_i, the instantaneous revenue rates are equal under the two prices p_i and p_{i+1}. That is, the revenue curves $r(p^*(\tau_{i-1}, \tau_i), t)$ and $r(p^*(\tau_i, \tau_{i+1}), t)$, as functions of t, should intersect at $t = \tau_i$. (Otherwise, it would increase the total revenue by expediting or delaying the switching time.) Furthermore, we can show that, at this point, $r(p^*(\tau_i, \tau_{i+1}), t)$ must cross $r(p^*(\tau_{i-1}, \tau_i), t)$ from below, i.e., the next price $p^*(\tau_i, \tau_{i+1})$ starts becoming better than the previous price $p^*(\tau_{i-1}, \tau_i)$. These necessary conditions are unfortunately not sufficient for optimality, and in fact $\tilde{V}$ may even not be quasi-concave with respect to some τ_i. It can be shown however that the quasi-concavity property is guaranteed by a technical condition, which for example is satisfied by constant $h(t)$ and linear $b(t)$ functions. See Appendix A.3 for details.

Single-Dimensional Search Approach for Maximizing Revenue

We have formulated the revenue objective as a function of $(n-1)$ decision variables (instead of $(2n-1)$ variables), which is still not straightforward to solve particularly when n is large. However, it is possible to formulate the revenue objective as a function of a single variable only, using the special structure that we have identified in Proposition 2.1. We now demonstrate this reformulation such that the objective function can be maximized using a *single-dimensional* algorithmic approach.

We first show the following proposition, which provides necessary optimality conditions in terms of the switching times only. It shows how the optimal switching time τ_i (which is for the price transition from p_i to p_{i+1}) is related to the analytic centers of the interval with price p_i and the interval with price p_{i+1}. Recall the definition of $\overline{b}(\tau', \tau'')$ from (2.2). The proof of this proposition is in Appendix A.4.

Proposition 2.2. *Suppose Assumption 1 holds. Then, the optimal switching times in an n-price model satisfy: for $i = 1, \ldots, n-1$,*

$$\frac{b(\tau_i)}{\overline{b}(\tau_{i-1}, \tau_i)} + \frac{b(\tau_i)}{\overline{b}(\tau_i, \tau_{i+1})} \quad = \quad 2 \ . \tag{2.5}$$

Observe that $\overline{b}(\tau_{i-1}, \tau_i)$ is a function of τ_{i-1} and τ_i, and that $\overline{b}(\tau_i, \tau_{i+1})$ is similarly a function of τ_i and τ_{i+1}. Therefore, an important consequence of Proposition 2.2 is the relationship among the three consecutive switching times $\{\tau_{i-1}, \tau_i, \tau_{i+1}\}$. If both τ_{i-1} and τ_i are fixed, we can use this relationship to determine the value of τ_{i+1}. Based on this idea, if the first switching time is given, then the entire sequence of switching times can be defined. More specifically, for a fixed positive value $\theta \in (0, T)$, we construct a sequence $(\hat{\tau}_1(\theta), \ldots, \hat{\tau}_{n-1}(\theta))$ as follows. Let $\hat{\tau}_0(\theta) = 0$ and $\hat{\tau}_1(\theta) = \theta$. For any given pair of $\hat{\tau}_{i-1}(\theta)$ and $\hat{\tau}_i(\theta)$, we recursively define $\hat{\tau}_{i+1}(\theta)$ via (2.5). If such $\hat{\tau}_{i+1}(\theta)$ does not exist, then we set $\hat{\tau}_{i+1}(\theta) = \cdots = \hat{\tau}_n(\theta) = \infty$. We are particularly interested in the values of θ satisfying $\hat{\tau}_n(\theta) = T$ since otherwise any solution with $\tau_1 = \hat{\tau}_1(\theta) = \theta$ does not satisfy the optimality condition in Proposition 2.2. (For a graphical illustration of constructing $(\hat{\tau}_1(\theta), \ldots, \hat{\tau}_{n-1}(\theta))$, please see Appendix A.5.)

Therefore, we propose the following method to identify the candidates for optimal solutions. First, identify all possible values of θ such that $\hat{\tau}_n(\theta) = T$. In our computation, there is only one value of θ satisfying this condition; however, in general, $\hat{\tau}_n(\theta)$ is not monotonic in θ, and thus it is possible that there may be multiple values of θ with this property. This step amounts to finding all zeros of a single-dimensional function, which is not difficult computationally. Next, for each identified θ, we find $\hat{p}_i$ values based on (2.2), and we evaluate the performance of each candidate to select the best solution.

Summarizing this section, we have formulated the objective function for the discrete pricing problem. While this function depends on multiple variables, we have shown that the optimal solution can be obtained by performing a *single* dimensional search, regardless of n, the number of discrete prices.

3 Analysis of Models with Linear Price Sensitivity $b(t)$

In Section 2, we have considered general approaches for numerically finding the optimal solution. In this section, for analytical tractability and ease of demonstration, we restrict our attention to the case where the sensitivity parameter $b(t)$ is a linear function of time t, i.e., $b(t) = \beta_0 + \beta_1 t$, where β_0 and β_1 are nonnegative constants. This allows us to understand the impact of the demand pattern more clearly. Later, in Section 4, we extend the analysis to a general nonlinear $b(t)$. Under linear $b(t)$, customers' sensitivity towards price increases proportionally with the product's age. The older the product becomes, the less willing a customer is to pay. In addition, we define $m = \beta_1/\beta_0$. Thus, a higher value of m indicates a greater change of the price sensitivity as a function of time t.

Given that $b(t)$ is linear, we first examine, in Section 3.1, the case where the life-cycle effect is absent, and use results and insights obtained in this case as a benchmark. Then, in Section 3.2, we extend to the general case where the life-cycle effect is of a unimodal pattern, and we illustrate the impact of the demand pattern on optimal pricing decisions by comparing the results in the general model against the benchmark case of constant $h(t)$.

3.1 A Benchmark Case of Constant $h(t)$

One of our objectives in Section 3 is to understand the impact of the demand pattern on the optimal price path (both in terms of the set of prices and timing). For the benchmark case where the life cycle effect is absent, i.e., $h(t)$ is constant throughout the planning horizon, we obtain a closed-form solution for the optimal switching times and prices. (Note that we let $h(t) = 1$ without loss of generality.)

Proposition 3.1 (Linear $b(t)$ and Constant $h(t)$)**.** *Suppose Assumption 1 holds, and assume $h(t) = 1$ and $b(t) = \beta_0 + \beta_1 t = \beta_0(1 + mt)$. For the n-price model, the optimal switching times $(\tau_1, \ldots, \tau_{n-1})$ and the optimal prices $(p_1, \ldots, p_n)$ are given by the following:*

$$\tau_i \;=\; \frac{(1+mT)^{i/n} - 1}{m} \qquad \textit{and} \qquad p_i \;=\; \frac{a/\beta_0}{(1+mT)^{i/n} + (1+mT)^{(i-1)/n}} \;.$$

We comment on some interesting properties of the optimal solution for the n-price problem in this benchmark case. The optimal switching time τ_i is an exponential function of the switching index i. It implies that price switching is more frequent at the early part of the product life cycle and becomes more infrequent, i.e., $\tau_i - \tau_{i-1} < \tau_{i+1} - \tau_i$. This phenomenon can be explained by the fact that the customer population shifts over time to become less willing to pay; thus, for a given number of allowed price changes, we change prices more frequently in the early part of the life cycle in order to capture as much of the surplus as possible.

In the two-price case (i.e., $n = 2$), the above argument implies that the only switching time τ satisfies $\tau \leq 0.5T$, indicating that price changes are more important at the earlier stage of the life cycle. Furthermore, we can examine how the optimal switching time depends on a measure of how fast the price sensitivity change, namely $[b(T) - b(0)]/b(0)$ (which simplifies to $\beta_1 T/\beta_0$):

$$\tau \;\approx\; \begin{cases} 0.477T & \text{if } \beta_1 T/\beta_0 = 0.2 \\ 0.414T & \text{if } \beta_1 T/\beta_0 = 1.0 \\ 0.290T & \text{if } \beta_1 T/\beta_0 = 5.0 \;. \end{cases} \tag{3.1}$$

This shows that the optimal switching time τ occurs earlier in the life cycle if the demand sensitivity changes more rapidly.

Not only are the price changes more frequent at the beginning of the life cycle, but also the drop in prices is greater at that time; mathematically, $p_i - p_{i-1} > p_{i+1} - p_i > 0$. These characteristics are further demonstrated in the following corollary of Proposition 3.1: Under the assumptions of Proposition 3.1, the optimal price points for the n-price model satisfy, for each $i \in \{1, \ldots, n-1\}$,

$$p_{i+1}/p_i = (1+mT)^{-1/n} \;, \tag{3.2}$$

and, furthermore, the optimal switching times satisfy, for $i \in \{0, \ldots, n-1\}$,

$$\log(1 + m\tau_{i+1}) - \log(1 + m\tau_i) \;=\; [\log(1+mT)]/n \;. \tag{3.3}$$

From this result, the ratio between two adjacent prices under the optimal policy turns out to be a constant. This suggests that, in the absence of the life-cycle effect ($h(t) = 1$), the price points should be chosen such that the *percentage* reduction between two adjacent prices stays constant. (See Appendix A.7 for the proof.)

In the pricing problem, the total number of price changes is fixed for the entire planning horizon. Thus, the question we address is how to distribute the fixed number of price changes along the time horizon. To characterize the optimal switching times, we define a *logarithmic time scale* $\log(b(t))$. On the logarithmic time scale, equation (3.3) shows that, under the constant $h(t)$ benchmark case, the duration for each price point remains the same in the *logarithmic* scale.[1] Furthermore, it implies

$$\frac{b(\tau_{i+1}) - b(\tau_i)}{b(\tau_i)} \quad = \quad [b(T)/\beta_0]^{1/n} - 1 \; . \tag{3.4}$$

(See Appendix A.8 for the proof.) Equation (3.4) indicates that the optimal strategy to segment the customers based on price sensitivity is to partition the customers such that the proportional increase in price sensitivity at subsequent switching points is a constant, which is the right-side expression of (3.4).

The observations identified here will serve as a benchmark basis in the remainder of the paper.

3.2 General Unimodal $h(t)$

Products exhibiting life-cycle demand characteristics typically go through a demand ramp up, followed by a ramp down. Therefore, a unimodal $h(t)$ is general enough to include most demand patterns with diffusion characteristics. In this section, we characterize the optimal solution for the discrete pricing problem with n prices and show how the demand pattern affects the ratio of subsequent prices and the time duration of each price, when compared to the benchmark case of constant $h(t)$ – which we have discussed in Section 3.1.

The following theorem is the main analytical result of Section 3.

Theorem 3.2 (Linear $b(t)$ and Unimodal $h(t)$)**.** *Suppose Assumption 1 holds, and assume* $b(t) = \beta_0 + \beta_1 t = \beta_0(1 + mt)$. *Suppose that* $h(t)$ *is unimodal, i.e., there exists* $\hat{t}$ *such that* $h(t)$ *increases in* t *in the interval* $[0, \hat{t}]$ *and decreases in* t *in the interval* $[\hat{t}, T]$. *The* n*-price model satisfies the following properties.*

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[1] On the original time scale, however, equal length of $\log(b(t(i))) - \log(b(t(i-1)))$ always implies increasing length of $\tau_i - \tau_{i-1}$ by the concavity and monotonicity of the logarithmic function and linearity of $b()$.

(a) *The optimal prices* $(p_1, \ldots, p_n)$ *satisfy:*

$$\frac{p_i}{p_{i-1}} \leq \frac{p_{i+1}}{p_i} \quad \textit{if } \tau_i \leq \hat{t}, \quad \textit{and} \tag{3.5}$$
$$\frac{p_i}{p_{i-1}} \geq \frac{p_{i+1}}{p_i} \quad \textit{if } \tau_{i-1} \geq \hat{t}\ . \tag{3.6}$$

(b) *Furthermore, the optimal switching times* $(\tau_1, \ldots, \tau_{n-1})$ *satisfy:*

$$\log(1+m\tau_i) - \log(1+m\tau_{i-1}) \geq \log(1+m\tau_{i+1}) - \log(1+m\tau_i) \quad \textit{if } \tau_i \leq \hat{t}, \quad \textit{and} \tag{3.7}$$
$$\log(1+m\tau_i) - \log(1+m\tau_{i-1}) \leq \log(1+m\tau_{i+1}) - \log(1+m\tau_i) \quad \textit{if } \tau_{i-1} \geq \hat{t}. \tag{3.8}$$

The results in equations (3.7) and (3.8) provide a stark comparison with the benchmark case when $h(t)$ is constant. Recall that under the assumption of constant $h(t)$, the optimal price points should follow a constant *percentage* discount pattern: each time we make a price change, we apply a certain fixed percentage discount (equation (3.2)). Also, the duration for each price point is of "equal" length on a logarithm scale of time (Proposition 3.1 and equation (3.3)). When the demand pattern $h(t)$ is *not* a constant, these results no longer hold, which we elaborate below.

First, equations (3.5) and (3.6) show that the price discount between two successive prices depends on the demand pattern $h(t)$, in a way such that if demand is increasing, then the percentage price discount decreases (i.e., the relative price change is smaller); similarly, if demand is decreasing, then the percentage price discount increases.

Second, equations (3.7) and (3.8) show that, on the logarithmic time scale, price changes should be more concentrated in the region where the demand pattern peaks, given that the total number of price changes is fixed. More specifically, before the peak, the duration for each price point (equivalently, the switching time interval) becomes smaller in time, and after the peak, the switching interval becomes larger in time – both in the log scale.[2]

Third, equation (3.7) is equivalent to

$$\frac{b(\tau_i) - b(\tau_{i-1})}{b(\tau_{i-1})} \geq \frac{b(\tau_{i+1}) - b(\tau_i)}{b(\tau_i)}\ ,$$

which implies that the percentage increase in *price sensitivity* between adjacent segments becomes smaller before the peak; we can similarly show that this percentage becomes bigger after the peak. This result contrasts with equation (3.4) and indicates that the optimal

[2]On the original time scale, however, this relationship does not necessarily hold. In particular, increasing length of $\log(b(t(i)) - \log(b(t(i-1)))$ always implies increasing length of $\tau_i - \tau_{i-1}$ by the concavity and monotonicity of the logarithmic function and linearity of $b(\cdot)$. However, decreasing length of $\log(b(t(i)) - \log(b(t(i-1)))$ does not necessarily correspond to decreasing length of $\tau_i - \tau_{i-1}$. Therefore, on the original time scale, the switching intervals become shorter before the peak, but the trend is undecided after the peak.

strategy to segment the customers is to divide the customers such that the percentage increase in price sensitivity at subsequent switching points is decreasing before the peak and increasing after the peak.

While the above-mentioned differences from the benchmark case arise from the demand pattern $h(t)$, our numerical experiments indicate that one of the insights identified previously for the constant demand pattern – that the price switching times tend to concentrate in the earlier stages of a product life cycle – tends to hold here as well. While it is not easy to formalize this because of the compounding effect of the demand pattern, we can show that the price change occurs before the middle of the horizon if there is only one price change allowed: Under the conditions of Theorem 3.2, if $h(t)$ is symmetric around $t = T/2$ and it is log-concave, then the optimal switching time τ of the two-price model satisfies

$$\tau \;\leq\; T/2\,. \tag{3.9}$$

takes advantage of the monotonicity property of *mean-advantage-over-inferior* functions under log-concavity [3], and it appears in Appendix A.9. Many commonly known diffusion models such as the Bass model are log-concave. (See, for example, Sengupta and Nanda [35] for discussion on the sufficient conditions for log-concavity.)

Now, for an illustrative purpose, suppose that $h(t)$ is given by the probability density function $N(\mu, \sigma^2)$, a normal distribution with mean μ and standard deviation σ. Let $\mu = T/2$ and $\sigma = T/6$ such that the demand pattern $h(t)$ peaks at the center of the life cycle (when $t = T/2$). Then, as we vary $[b(T) - b(0)]/b(0) = \beta_1 T/\beta_0$ as before, it can be shown that the optimal switching times are as follows:

$$\tau \;\approx\; \begin{cases} 0.491T & \text{if } \beta_1 T/\beta_0 = 0.2 \\ 0.468T & \text{if } \beta_1 T/\beta_0 = 1.0 \\ 0.425T & \text{if } \beta_1 T/\beta_0 = 5.0\,. \end{cases} \tag{3.10}$$

See Appendix A.10 for the proof of (3.10). This result again demonstrates a pattern that the switching times should occur earlier in the life cycle as the change in price sensitivity (i.e., $\beta_1 T/\beta_0$) increases. This is consistent with the earlier result where $h(t)$ remains constant over the life cycle. Yet, compared to the constant $h(t)$ case shown in (3.1), the above result shows that the switching time is closer to the center of the life cycle where the demand pattern peaks.

We end this section with a remark that the proof of Theorem 3.2 can be easily extended to the case where $h(t)$ may not be unimodal. For *any* general continuous $h(t)$, we can divide the planning horizon into an alternating sequence of time intervals during which $h(t)$ is increasing or decreasing, and show that for the time intervals during which $h(t)$ is increasing, equations (3.5) and (3.7) hold, and for time intervals during which $h(t)$ is decreasing, equations (3.6) and (3.8) hold.

4 Analysis with Nonlinear Price Sensitivity $b(t)$

In Section 3, we derived several properties regarding the optimal prices and their timing (Proposition 3.1 and Theorem 3.2) under the assumption that the customers' price sensitivity $b(t)$ changes *linearly* in time. In this section, we consider the impact of price sensitivity by considering the case where it increases in a more general manner.

As before (the benchmark case in Section 3.1), we first consider the impact of nonlinear price sensitivity $b(t)$ by fixing the demand pattern $h(t)$ at a constant. In particular, if the price sensitivity increases in a convex or concave manner, we examine what would happen to the price changes and their timing, compared to the benchmark case with linear price sensitivity. Intuitively, convex price-sensitivity increase implies that the rate of price-sensitivity increase accelerates with the age of the product, or equivalently, the decline in customers' willingness to pay becomes steeper as the product grows older; the opposite holds for concave price-sensitivity increase. We characterize these two cases because any nonlinear increasing $b(t)$ can be treated as intervals of concave or convex increasing functions.

Proposition 4.1 (Nonlinear $b(t)$ and Constant $h(t)$)**.** *Suppose Assumption 1 holds, and assume $h(t) = 1$ in the n-price model. Then, the optimal prices $(p_1, \ldots, p_n)$ and switching times $(\tau_1, \ldots, \tau_{n-1})$ satisfy the following.*

(a) If $b(t)$ is concave in t, then $(p_{i-1} - p_i)/p_{i-1}$ and $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ decrease in i.

(b) If $b(t)$ is convex in t, then $(p_{i-1} - p_i)/p_{i-1}$ and $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ increase in i.

Recall from Section 3.1 that both $(p_{i-1} - p_i)/p_{i-1}$ and $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ remain constant independent of i in the linear $b(t)$ case, a result that can also be implied by the above theorem. Proposition 4.1 provides insights that are intuitive in hindsight but not necessarily straightforward initially. As the customers' price sensitivity increases through time in a concave manner, there is less need to provide a steep price discount to cater to the increasingly slowing changing customer sensitivity (i.e., the percentage price reduction $(p_{i-1} - p_i)/p_{i-1}$ becomes smaller), and furthermore, the percentage change in price sensitivity at switching times becomes smaller each time (i.e., $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ becomes smaller). A similar argument can be made for the case of a convex price sensitivity. We remark that the results of Proposition 4.1 can be extended such that if $b(t)$ follows an S-shaped curve or a rotated S-shaped curve in which the concavity and convexity of $b(t)$ can switch in time, or if $b(t)$ is of a general nonlinear form which comprises of intervals of concave and/or convex functions.

Now, we consider a more general case where $b(t)$ may be non-linear and $h(t)$ may not be constant. By combining Theorem 3.2 and Proposition 4.1, we can obtain the following result. (The proofs of Proposition 4.1 and Corollary 4.2 are in Appendix A.12.)

Corollary 4.2 (Nonlinear $b(t)$ and Non-constant $h(t)$). *Under Assumption 1, the optimal prices $(p_1, \ldots, p_n)$ and switching times $(\tau_1, \ldots, \tau_{n-1})$ satisfy the following.*

(a) If $h(t)$ is increasing and $b(t)$ is concave in t, then $(p_{i-1} - p_i)/p_{i-1}$ and $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ decrease in i.

(b) If $h(t)$ is decreasing and $b(t)$ is convex in t, then $(p_{i-1} - p_i)/p_{i-1}$ and $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ increase in i.

Summarizing the analytical results in Sections 3 and 4, we see that increasing underlying demand pattern $h(t)$ and less and less rapidly changing price sensitivity $b(t)$ have a similar impact on the relative amount of price change and the frequency of price changes in a log-price-senstivity scale. Corollary 4.2 is still an incomplete characterization of the optimal price and switch times since it does not discuss the case of increasing $h(t)$ and convex $b(t)$, or the case of decreasing $h(t)$ and concave $b(t)$. In these cases, a simple clean-cut result cannot be obtained and we demonstrate this with more details in Section 5.2 with numerical studies.

5 Sensitivity Analysis: Numerical Results

Since the primary modeling features of this paper are the demand pattern during a product's life cycle and the customer price sensitivity that evolves in time, we continue in this section to explore their impact on the optimal decisions using numerical examples. In particular, we seek to complement the analytical study in Sections 3 and 4 by examining cases that are not fully characterized analytically.

- We have established in Section 3 that price switching times follows two trends: (i) they concentrate more in the peak region of the demand pattern $h(t)$; (ii) the optimal pricing switching tends to be earlier in the life cycle than later, for symmetric $h(t)$. In Section 5.1, we study the impact of a general unimodal $h(t)$ by varying the levels of asymmetry and width of the peak while keeping the price sensitivity $b(t)$ at a linear form.

- We have allowed both increasing and decreasing demand pattern $h(t)$, and both convex and concave price sensitivity $b(t)$. In Section 4, we have obtained analytical results for the combinations of (i) increasing $h(t)$ and concave $b(t)$ and (ii) decreasing $h(t)$ and convex $b(t)$. In Section 5.2, we numerically examine the other combinations of $h(t)$ and $b(t)$: (iii) increasing $h(t)$ and convex $b(t)$, and (iv) decreasing $h(t)$ and concave $b(t)$.

- Finally, in Section 5.3, we address the restriction of discrete pricing (limiting the total number of price changes) by comparing the performance of discrete pricing to continuous pricing.

Experiment Set-Up. While there are a large number of life cycle demand patterns available in the literature (for example, Mead and Islam [28] summarize 29 models), we choose three demand patterns that are commonly used. Each of these demand patterns, listed in Table 2, is a unimodal bell-shaped curve, in which we can vary parameters to control both the location and "width" of the peak, the mode of the demand pattern $h(t)$.

Table 2: Three Types of Demand Patterns $h(t)$

Type	Parameters	$h(t)$	Literature
Normal	$\mu, \sigma > 0$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$	Rogers [34], Stapleton [40]
Simple Logistic	$\gamma > 0, k > 0$	$\frac{k\gamma e^{-kt}}{(1+\gamma e^{-kt})^2}$	Stone [42], Tanner [45]
Bass	$\gamma > 0, k > 0$	$\frac{ke^{-kt}}{1+\gamma e^{-kt}}[1 + \frac{\gamma(1-e^{-kt})}{1+\gamma e^{-kt}}]$	Bass [4], Mahajan and Peterson [26]

For the Normal demand pattern, we use the following parameter values:

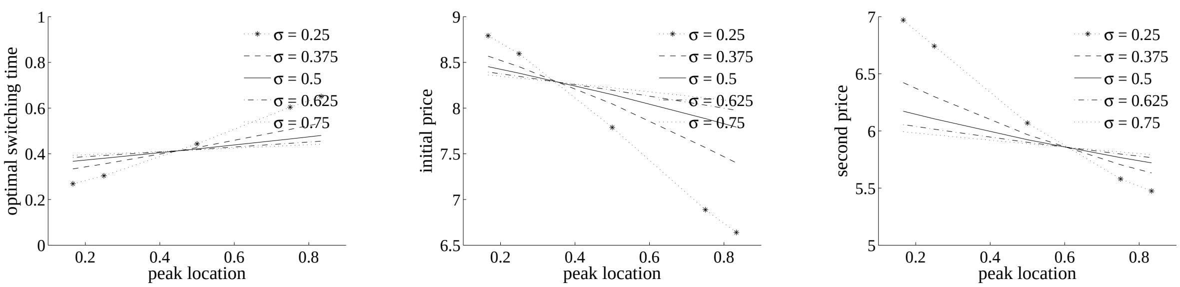
- Location of the peak: $\mu \in \{T/6,\ T/4,\ T/2,\ 3T/4,\ 5T/6\}$.
- Width of the peak: $\sigma \in \{T/4,\ 3T/8,\ T/2,\ 5T/8,\ 3T/4\}$.
- Inter-temporal price sensitivity: $m = \beta_1/\beta_0 \in \{1, 5, 20, 35, 50\}$. Here, we vary the value of β_1 while keeping the value of β_0 constant at 10 and the parameter a in the linear demand equation (see Assumption 1) at 200.

Without loss of generality, we use $T = 1.0$. We have computed the optimal solution for each of $5 \times 5 \times 5 = 125$ parameter combinations using the Normal demand pattern, and we report our findings and insight in this section. We also tested the logistic and Bass demand patterns with similar parameter values (by choosing γ and k such that the location and width of the peak match the Normal demand patterns listed above) – while these results are not reported here for the simplicity of presentation, we have obtained similar results as the Normal demand pattern, providing the robustness of our findings.

5.1 Impact of Demand Pattern (Peak Location and Width)

We first consider the impact of the peak location and peak width of the demand pattern, denoted in our model by μ and σ respectively, on the optimal decision for the discrete pricing model. In this section, we restrict our attention to the case of $n = 2$ discrete prices for the ease of illustration.

The location of the peak determines where a large portion of demand can be expected, and thus has a direct positive effect on the optimal switching time. One implication of Theorem 3.2 is that price switching should concentrate more near the peak region. Therefore, in a two-price policy, if the peak occurs later in the life cycle, then the optimal switching time is delayed, and the optimal price in each segment decreases as it caters to more price-sensitive customers (consistent with the monotonicity result of $p^*(t_1, t_2)$ in Section 2.2). See Figure 2 for an illustration.



(a) Switching Time (b) First Price (c) Second Price

Figure 2: Impact of Demand Pattern on the Optimal Decision in a 2-price Case: $m = 1.0$

Our finding leads to a better understanding of the interaction between the location of the demand peak and the pricing strategy. If demand peaks early in a product's life cycle, it strengthens the need to set an initial high price and then discount early (known as the *price skimming* strategy); in comparison, if demand peaks later in a product's life cycle, it is optimal to start with a lower price and delay price reduction (*price penetrating* strategy). Thus, in an industry where demand peaks in the early part of the life cycle (for example, in the film industry where pre-release promotion stimulates early demand), the firm sets a high initial price for the product (compare movie tickets to DVD rentals); by the same logic, for many household items whose release rarely grabs customer attention, the firm maintains a steady price throughout the life cycle.

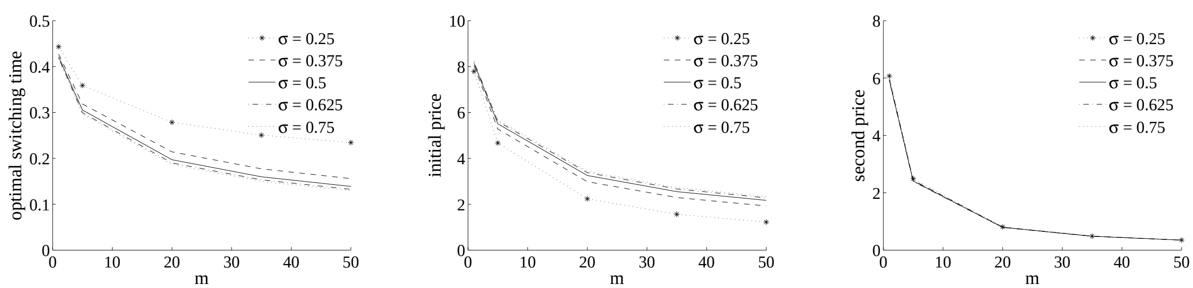
In addition, Figure 2 shows that the strength of our earlier observation depends on the width of the peak, i.e., the less spread the demand pattern is (smaller σ), the more closely the switching time follows the peak location.

5.2 Impact of Inter-Temporal Price Sensitivity

While we have examined the impact of demand pattern $h(t)$ in Section 5.1, we now turn our attention to the impact of the inter-temporal price sensitivity $b(t)$ on the optimal decision (both the switching time and prices), holding the demand pattern $h(t)$ to a fixed curve. We investigate both linearly-changing price sensitivity and nonlinear price sensitivity. With linear price sensitivity, we demonstrate how the speed of the price-sensitivity change affects the switching time using a two-price example. Then, we study the case in which the customers' price sensitivity evolves over time in various nonlinear ways and show how the optimal price reductions and the optimal switching times are affected using a 10-price example.

Linear Price Sensitivity

Since the price sensitivity parameter is given by $b(t) = \beta_0 + \beta_1 t$, which starts at β_0 and increases by $\beta_1 T$ during the life cycle, $m = \beta_1/\beta_0$ represents the proportional increase in price sensitivity. The value of m often depends on the market positioning of each product; for example, Intel may brand a certain processor for a small group of pro-performance segment in which case m is small while a product may be sold across the entire spectrum of the customer base in which case the value of m may be large.



(a) Switching Time (b) First Price (c) Second Price

Figure 3: Impact of m on the Optimal Decision in a 2-price Case: $\mu = 0.5$

In Figure 3, we show the optimal decisions as a function of m. We fix the location of the peak in the center of the life cycle (at $\mu = 0.5$), and each line corresponds to a "width" σ of the peak. In Figure 3(a), we observe that when price sensitivity increases rapidly (high value of m), the optimal switching point occurs early in the life cycle. Such a phenomenon can be explained by the need to generate as much revenue as possible from the price-insensitive customers in the earliest part of a life cycle. We also observe that when demand is concentrated in the middle of the life cycle (small value of σ), the switching time is later because there is less need to focus on the earliest part of the life cycle that has fewer customers. These observations are consistent with equation (3.9) and the numerical examples in Section 3.

Nonlinear Price Sensitivity

Without the linear assumption on $b(t)$, the analytical result regarding the optimal pricing policy given in Section 4 is less conclusive except for the case of constant $h(t)$ (Proposition 4.1) and the two cases characterized in Corollary 4.2. In this section, we investigate the cases where the customers' price sensitivity may take on various curvatures, given that the life cycle demand pattern is a unimodal curve.

We use the following exponential family of price sensitivity functions $b(t)$:

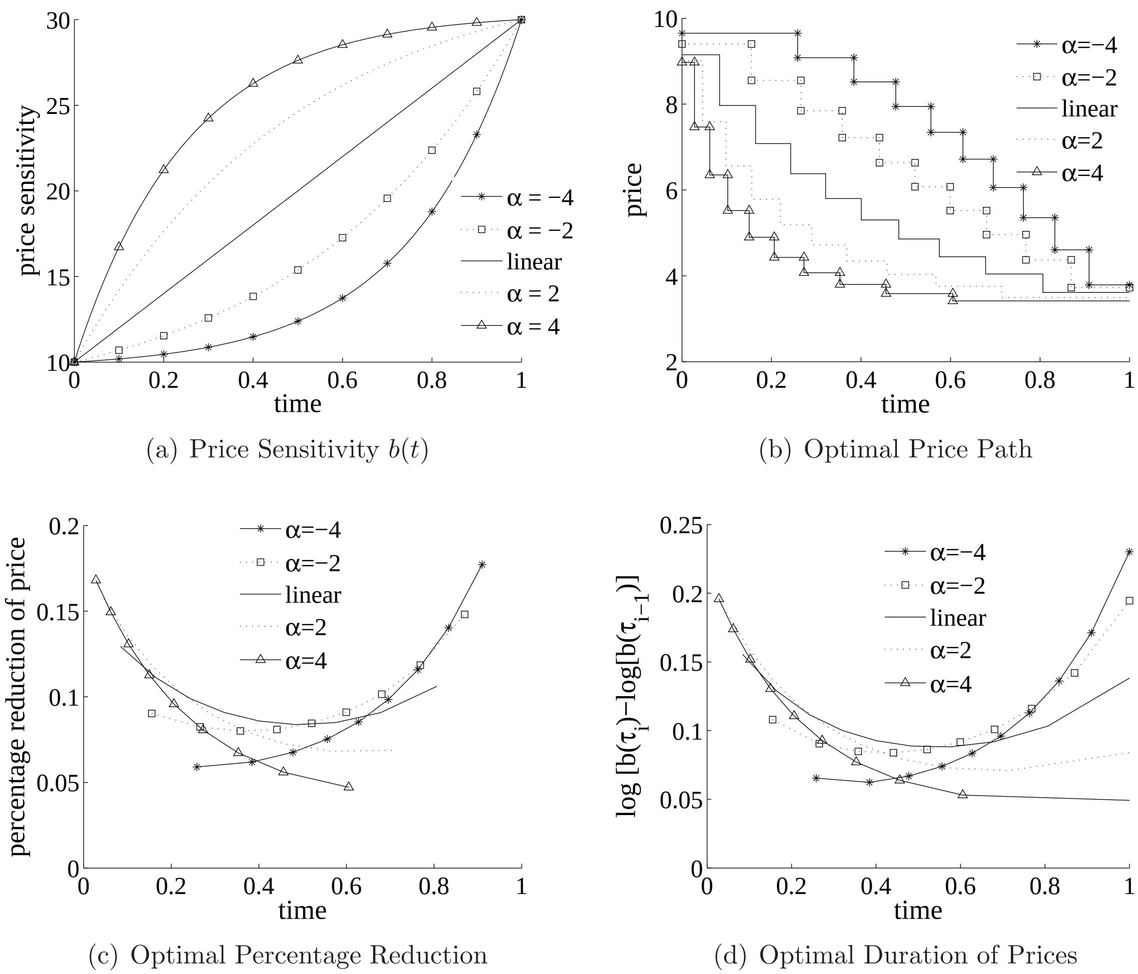
$$b(t) \quad = \quad b_0 + (b_T - b_0) \cdot \frac{1 - e^{-\alpha t}}{1 - e^{-\alpha}}$$

where b_0 and b_T are constants, and α is a curvature parameter given by $-b''(t)/b'(t)$. By varying the values of α, we obtain curves with different curvatures but fixed values at time 0 and time T (which are b_0 and b_T respectively) – see Figure 4(a) for a set of price sensitivity curves with $b_0 = 10$ and $b_T = 30$. The curve in the middle is the case of linear $b(t)$ ($\alpha = 0$), and the others are either concave ($\alpha > 0$) or convex ($\alpha < 0$).

We use the general $b(t)$ function described above to solve the optimal 10-price problem while we fix the demand pattern $h(t)$ at a normal shaped curve with mean $\mu = 0.5$ and standard deviation $\sigma = 0.25$. Figure 4(b) shows the optimal pricing policy. Note that concave (convex) $b(t)$ means that the price sensitivity is less and less (more and more) rapidly changing. One clear observation is that for concave (convex) $b(t)$, price switchings occur earlier (later) in the product life cycle; this can be explained by the fact that concave (convex) $b(t)$ means that there are rapid changes in price sensitivity in the early (later) part of the cycle.

Figure 4(c) demonstrates a property on the optimal sequence of prices. With linear $b(t)$, percentage reduction between adjacent prices decreases before the peak of $h(t)$ (which is 0.5 in this case) and increases after the peak. When $b(t)$ is convex (i.e., $\alpha < 0$), the increasing trend starts sooner, much earlier than the peak; with concave $b(t)$, the opposite is true. Therefore, the concavity of $b(t)$ elongates the period of decreasing percentage reduction and shortens the period of increasing percentage reduction relative to the case of linear $b(t)$.

Figure 4(d) shows a property of the optimal sequence of switching times. Recall from equation (3.2) and Theorem 3.2 that with linear $b(t)$, the duration of each price point measured in the logarithmic scale of $\log[b(t)]$ is a constant if $h(t)$ is constant; the duration is increasing in the price index if $h(t)$ is increasing, and decreasing if $h(t)$ is decreasing. This is clearly verified with the linear case in Figure 4(d). When $b(t)$ is concave or convex, the



(a) Price Sensitivity $b(t)$

(b) Optimal Price Path

(c) Optimal Percentage Reduction

(d) Optimal Duration of Prices

Figure 4: Impact of Curvatures of $b(t)$: Optimal Pricing Policy in a 10-Price Case

above is not necessarily true. Given what we know of the impact of $b(t)$ on the optimal price percentage reduction, and that the percentage reduction and the switching time interval (measured by $\log[b(\tau_i)] - \log[b(\tau_{i-1})]$) ought to move in the same direction (see Proposition 4.1 and its proof, particularly Lemma A.9 in Appendix A.12), it is no surprise that we observe very similar behavior for the switching time interval (Figure 4(d)) and the percentage reduction (Figure 4(c)). In summary, concavity leads to either steeper decreases or less steep increases for the switching intervals $\log[b(\tau_i)] - \log[b(\tau_{i-1})]$. On the *original* time scale, we observe that as the degree of concavity increases, more price adjustments occur early in the life cycle (see Figure 4(b)).

The managerial implications of this analysis reinforces a simple pricing policy guideline. If the customer price sensitivity changes more rapidly in the early part of the product life cycle (i.e., in the case of concave $b(t)$), it makes sense to adjust price more frequently and to use more aggressive percentage price discounts in the early part. With a price sensitivity function that changes more rapidly in the latter half (i.e., convex $b(t)$), these frequent and aggressive changes should be reserved for later in the life cycle. Therefore, depending on the industry and product, which may have a specific price sensitivity evolution, the pricing strategy should differ accordingly.

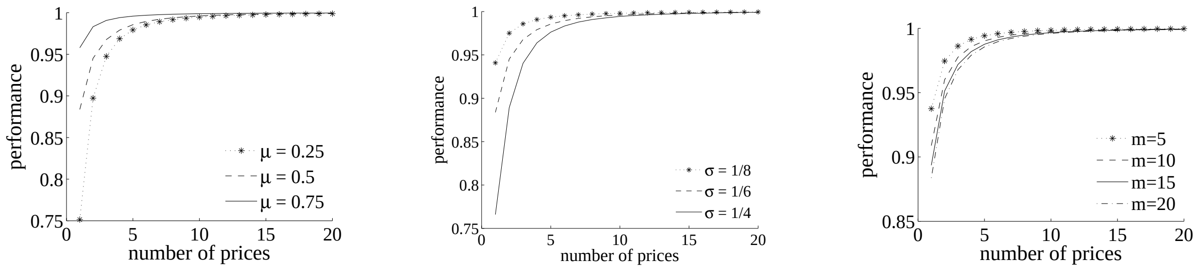
We have shown in Table 1 the paths of price discounting for several Intel products. The percentage discount is initially aggressive, then slows down over time, and in some cases rises again. Although it is tempting to draw an analogy to the observation from Figure 4(c) where the percentage discount could take either a monotonic decreasing path, or a decreasing-and-then-increasing pattern, we recognize that Intel's current price points are influenced by many other considerations in addition to what we focus on in this paper. However, being able to exemplify how important demand and market characteristics such as $h(t)$ and $b(t)$ affect the optimal price path through an analytical model and to compare this with current practice may prove useful to decision makers.

5.3 The Performance of Discrete Pricing as n Changes

In this section, we study how the restriction of limiting the number of the price changes during the planning horizon (discrete pricing) impacts the performance. We measure this by comparing it to the case without this restriction where the price can change continuously (continuous pricing). This can provide a guideline for a firm faced with the decision of how many times it should adjust the price of a product. We also investigate how this *relative performance* is affected by the shape of demand and by the price sensitivity change of the customers.

In Figure 5, we report the *relative performance* of discrete pricing as a function of the number of discrete prices. While it is easy to see that the overall life-cycle revenue is positively related to the flexibility associated with price changes (i.e., the number of allowed price changes), we emphasize how the location and width of the peak in the demand pattern affect the performance of discrete pricing. We find that the relative performance is high when the demand pattern peaks late in the life cycle, as illustrated by Figure 5(a), where each curve represents a different peak location in the life cycle. This can be explained that if demand pattern peaks early when customers are more price sensitive, there is a greater need or value of differentiated pricing tailored for changing customer price sensitivity.

Figure 5(b) suggests that the performance is high when the demand peak is narrow (small width) since demand is concentrated in a small time window during which price sensitivity does not change significantly. We also find that the performance is high when the inter-temporal price sensitivity m is small (see Figure 5(c)). This is intuitive since for a more stable price sensitivity there is less pressure for the company to adjust price over time.



(a) $m = 20$ and $\sigma = 1/8$. μ refers to the location of the peak in the demand pattern.

(b) $m = 20$ and $\mu = 1/2$. Here, σ refers to the width of the peak in the demand pattern.

(c) $\mu = 1/2$ and $\sigma = 1/6$. Here, m refers to inter-temporal price sensitivity.

Figure 5: Relative Performance of Discrete Pricing Compared to Continuous Pricing

Results in Figure 5 are useful for determining the optimal number of price changes to implement. A company may evaluate the administrative cost associated with additional price changes against the performance improvement and make the decision. We note that the performance improvement, i.e., the magnitude of benefit from an additional price change, depends on how fast price sensitivity changes with time, as well as the shape of the demand pattern.

6 Discussions

6.1 Nonlinear Demand

Thus far in this paper, we model the price-demand relationship using a linear function $d(p,t) = a - b(t)p$, which is commonly adopted in economics and management literature. For a nonlinear relationship, the analysis becomes intractable. However, we show with a numerical example that the main insights do not deviate.

In particular, we consider another commonly-adopted demand relationship: $d(p,t) = ae^{-b(t)p}$ where $b(t)$ is defined as in Proposition 3.1 with $b(t) = \beta_0 + \beta_1 t = \beta_0(1 + mt)$. The exponential demand form is widely used in the literature on pricing and revenue management (see, for example, Smith and Achabal [38], Gallego and van Ryzin [16], and Araman and Caldentey [1]). Since the parameter a does not affect the pricing and timing decision, we let $a = 1$. In our analysis in Section 2.2, we have provided an illustration using a two-price case and have applied the analytical results to obtain the optimal switching time when the demand pattern $h(t)$ is constant (Equation (3.1)) and when $h(t)$ is given by the normal density function $N(\mu, \sigma^2)$ (Equation (3.10)). We now provide a similar illustration for a nonlinear price-demand relationship using computation results.

If $h(t)$ is a constant, i.e., $h(t) = 1$, we obtain under the exponential demand that

$$\tau \approx \begin{cases} 0.477T & \text{if } \beta_1 T/\beta_0 = 0.2 \\ 0.414T & \text{if } \beta_1 T/\beta_0 = 1.0 \\ 0.290T & \text{if } \beta_1 T/\beta_0 = 5.0 \,. \end{cases} \tag{6.1}$$

Note the same switching time is obtained for the linear demand in the case that $h(t) = 1$ (equation (3.1)).[3] If $h(t)$ is given by the normal density with $\mu = T/2$ and $\sigma = T/6$, then

$$\tau \approx \begin{cases} 0.475T & \text{if } \beta_1 T/\beta_0 = 0.2 \\ 0.466T & \text{if } \beta_1 T/\beta_0 = 1.0 \\ 0.453T & \text{if } \beta_1 T/\beta_0 = 5.0 \,. \end{cases} \tag{6.2}$$

When $h(t)$ is of a normal-density shape, the specific switching time may shift, but the insights obtained from the linear demand stay true for the exponential demand. Specifically, the switching time occurs earlier in the life cycle ($\tau < 0.5T$) and, with a bell-shaped demand pattern, the switching time is closer to the demand peak than in the case that $h(t)$ is constant.

6.2 "Carry-through" Effect of Price

A key assumption made in our model is that the demand pattern $h(t)$ does not depend on price and that the effect of price does not carry through to future time. This assumption is critical for tractability and provides closed-form solutions and simple approaches for obtaining the optimal solution, as well as for deriving interesting analytical results by capturing the impact of the demand pattern on pricing decisions. As discussed earlier, this assumption is consistent with industrial conditions at companies such as Intel, which motivates our research problem. However, it may not be true in other contexts in which price may have a significant impact on product visibility and thus affect the diffusion pattern $h(t)$.

To explore the impact of such "carry-through" effect on the optimal pricing and switching time decisions, we consider one of the earliest price-dependent diffusion models, which is adopted by Robinson and Lakhani [33], Dolan and Jeuland [11] and Jeuland and Dolan [21]. We use the notation α and θ to denote the innovation and imitation parameters. Let f and F be the instantaneous adoption and the cumulative adoption, respectively. The adoption rate $f(t)$ is given by

$$f(p,t) = (\alpha + \theta F(p,t))(1 - F(p,t))e^{-b(t)p}, \tag{6.3}$$

where $f(p,t) = \frac{dF(p,t)}{dt}, t \in [0,T]$. Therefore, the price at time t affects the adoption rate $f(p,t)$ at time t, and consequently the cumulative adoption $F(p,t)$, as well as future adoptions through the impact on $F(p,t)$. The revenue rate function then becomes:

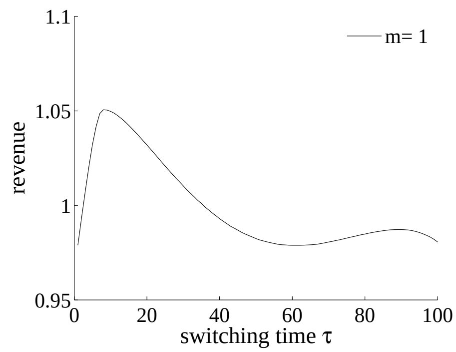
$$r(p,t) = f(p,t) \cdot p \tag{6.4}$$

[3] For the two-price problem with constant $h(t)$, the optimal switching time under a general price-demand relationship can be shown to coincide with that of the linear demand. See Appendix A.13 for detail.

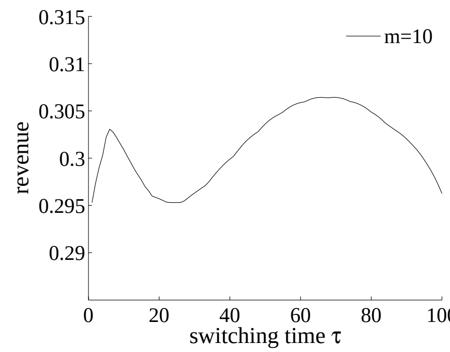
As discussed in the introduction, models with carry-through property usually do not have a closed-form adoption function. In this case, a closed-form expression of $f(p,t)$ as a function of p and t does not exist [6]. The discrete pricing problem becomes difficult to solve, and the technical properties that we have developed in this paper do not apply. However, by limiting to a simple two-price problem, we can explore the impact of carry-through on the pricing decisions through numerical means.

Assuming that price sensitivity changes in a linear fashion as given in Proposition 3.1, we compute the optimal prices (p_1, p_2), and optimal switching time τ through a two-dimensional search. We compute the adoption rate $f(p,t)$ in discrete time since it does not have a closed-form expression. Figure 6 illustrates the optimal solutions for different values of m (note that m is the rate of price sensitivity change over time). Other parameter values are $\beta_0 = 1$, $\alpha = 0.0037$ and $\theta = 0.33$; the diffusion parameters are the same as in the original Bass paper. Figures 6(a) and 6(b) show that the revenue is a bimodal function of the switching time, the first mode corresponding to dropping the price early so that the "carry-through" diffusion process can accelerate early in the life cycle, and the second mode corresponding to the case of reaping the most revenue from the price-sensitive mass market. At a lower m value, the left mode dominates and it is optimal to switch early; at a higher m value, the right mode dominates and it is optimal to switch price later. This interesting bi-modal behavior is further illustrated in Figure 6(c): For m values that are small, the optimal solution is characterized by an initial *low* price and then a switch to a higher price *early* in the planning horizon; For m values that are large, the optimal solution is characterized by a *high* initial price, followed by a price discount much *later* in the planning horizon. These can be interpreted as two commonly-observed pricing practices in the industry: For a newly-introduced product, companies sometimes use an early promotion to build up the initial adopter population and induce faster diffusion and then stop the promotion to maximize revenue. We refer to this as the "early promotion" approach, also known as the price-penetrating strategy. Alternatively, companies start with a high initial price to capture the price premium early on in the product life cycle when customers are less price sensitive, and discount the price significantly later in the life cycle to induce additional sales. We refer to this as the "late mark-down" approach, also known as the price-skimming strategy. This approach relies on inter-temporal price discrimination and works well if the market's price-sensitivity changes rapidly, i.e., m is large. Depending on which impact dominates (which appears to be determined by m only), we observe different patterns of the switching times as m changes.

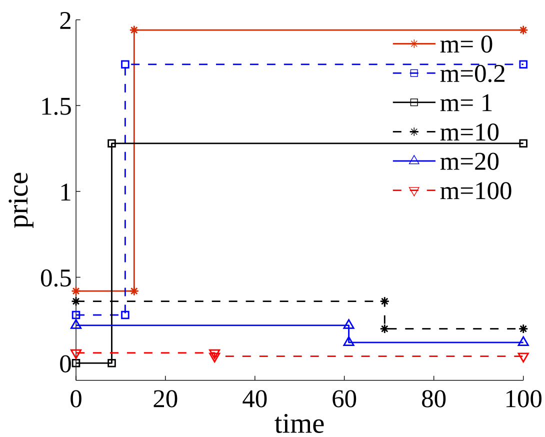
Clearly the "early promotion" approach depends critically on the assumption that the demand pattern is price dependent. The current-effect model does not capture this. However, "early promotion" is rare for technology products such as processors and we usually only observe price mark-downs for these products. Instead, strong initial adoption is often achieved through non-pricing controls, consistent with our earlier discussion. Hence the absence of "early promotion" in these markets reassures the "no carry-through" assumption.



(a) Revenue as a Function of Switching Time. Here the left mode dominates.



(b) Revenue as a Function of Switching Time. Here the right mode dominates.



(c) Optimal Prices and Switching Time. m reflects how fast price sensitivity changes with time.

Figure 6: Optimal Solutions under Price-dependent Demand Pattern

7 Conclusions

In this paper, we have addressed the dynamic pricing of a product in the high technology industry as it goes through a life cycle. In such a cycle, the underlying demand changes (usually increasing initially and decreasing eventually), and the customers' price sensitivity also evolves over time. We have considered the problem in which the price can change over time for a fixed total number of times, and both these times and the prices for each subperiod must be determined.

We have obtained a number of analytical and numerical results which can be summarized as follows. For the benchmark case of the stationary demand and linearly increasing price sensitivity, closed-form solutions for the optimal prices and switching times indicate constant percentage reduction in price over time and exponentially increasing switching times. We have studied the impact of demand pattern and price sensitivity, and have shown that, (i) the increasing trend in the demand pattern and (ii) the less and less rapidly increasing price sensitivity (i.e., concave price sensitivity) have the same qualitative impact on the optimal solution: the optimal percentage reduction in adjacent prices decreases in time. We have also obtained a similar characterization for the frequency of price changes. As a result, price switchings should concentrate more in the peak region of the demand, compared to a stationary demand case. We also have extended the analysis to the nonlinear price-demand relationship using exponential function as an example and have shown that the major insights remain true. In addition, we have considered an example of price-dependent diffusion and compute the optimal solution for the two-price problem, and have shown that this "carry-through" effect of price leads to an "early promotion" pricing strategy in which price is initially low and then increases to a higher value. Given that early price promotion is rarely observed in the focal market, we infer that the current-effect price-demand relationship is valid for our purpose.

The findings of this paper offer several "rule-of-thumb" guidelines in pricing products with distinct demand patterns: (1) Given a fixed number of price changes, reprice the product more often when demand is close to its peak than in other times; (2) Since price is more and more frequently updated as demand approaches the peak, the amount of reduction in each price update can gradually be less aggressive; (3) Adjust the decision based on the concavity of the customers' price sensitivity evolution, such that price reduction is more aggressive if price sensitivity increases in a convex manner.

To apply the results in this paper requires knowledge of the demand pattern and the price-demand relationship (more specifically, how price sensitivity changes with time), both of which are market-specific. Over time, companies develop expert knowledge on its customer population and how sensitive the customers are to price adjustments at different stages of a product life cycle. At Intel, such expert knowledge arises from market experience, from analyzing past sales and price data in the same market, as well as from focus-group studies. For example, when forecasting demand, Intel usually starts with some generic demand curve, and then modifies it based on attributes specific to that product such as price. The impact of price is derived from past sales and price data of products previously sold by Intel in this market. With sufficient data, Intel can control other product attributes to isolate the effect of price on sales and infer the price-sensitivity $b(t)$ at different stages of a product life cycle. This subsequently leads to a better and updated estimate of the price-independent demand curve $h(t)$. With the estimated demand pattern and price-sensitivity characteristics, we can apply methods in this paper and obtain reference price points and switching times, which are valuable to decision makers. Certainly Intel faces many more pricing constraints than what is captured in our model. Nevertheless, understanding the dynamics of how the life-cycle demand pattern and changing price-sensitivity affect the optimal price points and switching times enables management to make better-informed decisions in a complex business situation.

In this paper, we do not consider strategic customer behavior for the reasons discussed in the introduction. If in a different application, a large number of customers may delay their purchase anticipating a price reduction in the near future and thus strategic customer behavior cannot be neglected, the optimal pricing decisions will be affected. Since the revenue impact of such delay would be most significant during demand peak, we expect the optimal switching times to shift away from the peak time (relative to the case of no strategic behavior consideration), and also price reductions near the peak to be of a smaller magnitude.

Additionally, we have considered in this paper the pricing decisions for a product in the absence of substitutable products. For technology products, several generations of a single product family as well as many related products often coexist in the same market. An interesting question is how to address the pricing problem within this context. Efforts for these extensions are likely to include substantial development in both modeling and computation that build on the current paper, which we leave to future research.

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References

[1] V.F. Araman and R. Caldentey. Dynamic pricing for nonperishable products with demand learning. *Operations Research*, 57(5):1169–1188, 2009.

[2] Y. Aviv and A. Pazgal. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing and Service Operations Management*, 10(3):339, 2008.

[3] M. Bagnoli and T. Bergstrom. Log-concave probability and its applications. *Economic Theory*, 26:445–469, 2005.

[4] F.M. Bass. A new product growth model for consumer durables. *Management Science*, 15: 215–227, 1969.

[5] F.M. Bass. The relationship between diffusion rates, experience curves, and demand elasticities for consumer durable technological innovations. *The Journal of Business*, 53(3):S51–S67, 1980.

[6] F.M. Bass, D. Jain, and T. Krishnam. Modeling the marketing-mix influence in new-product diffusion. In V. Mahajan, E. Muller, and Y. Wind, editors, *New-product diffusion Models*. Kluwer Academic Publishers, 2000.

[7] D. Besanko and W. Winston. Optimal price skimming by a monopolist facing rational consumers. *Management Science*, 36(5):555–567, 1990.

[8] G.R. Bitran and S.V. Mondschein. Periodic pricing of seasonal products in retailing. *Management Science*, 43(1):64–79, 1997.

[9] G.P. Cachon and R. Swinney. Purchasing, pricing, and quick response in the presence of strategic consumers. *Management Science*, 55(3):497–511, 2009.

[10] R. E. Cooper. Amd v. intel: an assault on price competition. *http://support.intel.com/pressroom/legal/docs/Cooper-GCP-Mar08.pdf*, 2008.

[11] R. J. Dolan and A. P. Jeuland. Experience curves and dynamic demand models: implications for optimal pricing strategies. *Journal of Marketing*, 45(1):52–62, 1981.

[12] J. Eliashberg and A.P. Jeuland. The impact of competitive entry in a developing market upon dynamic pricing strategies. *Marketing Science*, 5(1):20–36, 1986.

[13] W. Elmaghraby, A. Gulcu, and P. Keskinocak. Designing optimal pre-announced markdowns in the presence of rational customers with multi-unit demands. *Manufacturing and Service Operations Management*, 10(1):126–148, 2008.

[14] Y. Feng and G. Gallego. Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management Science*, 41(8):1371–1391, 1995.

[15] Y. Feng and G. Gallego. Perishable asset revenue management with markovian time dependent demand intensities. *Management Science*, 46(7):941–956, 2000.

[16] G. Gallego and G. van Ryzin. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, 40(8):999–1018, 1994.

[17] M. Harris and A. Raviv. A theory of monopoly pricing schemes with demand uncertainty. *The American Economic Review*, 71(3):347–365, 1981.

[18] X. He, A. Prasad, and S. Sethi. Cooperative advertising and pricing in a dynamic stochastic supply chain: feedback stackelberg stratgegies. *Production and Operations Management*, 18 (1):78–94, 2009.

[19] Intel Corporation Website. Analyst center: Processor price list. Published Online http://www.intc.com/priceList.cfm, 2008-2009.

[20] D. C. Jain and R. C. Rao. Effect of price on the demand for durables:modeling, estimation, and findings. *Journal of Business Economics and Statistics*, 8(2):163–170, 1990.

[21] A. P. Jeuland and R. J. Dolan. An aspect of new product planning: Dynamic pricing. In A. A. Zoltners, editor, *TIMS Stutdies in the Management Sciences, Vol. 18: Special Issue on Marketing Planning Models*, pages 1–21. Elsevier Science Publishing Company, New York, 1982.

[22] S. Kalish. A new-product adoption model with price, advertising, and uncertainty. *Management Science*, 31:1569–1585, 1985.

[23] W. Kamakura and S. Balasubramanian. Long-term view of the diffusion of durables: a study of the role of price and adoption influence process via tests of nested models. *International Journal of Research in Marketing*, 5:1–13, 1988.

[24] T.V. Krishnan and D.C. Jain. Optimal dynamic advertising policy for new products. *Management Science*, 52(12):1957–1969, 2006.

[25] S. Li and Z.M. Shen. Optimal introduction timing for a product line extension with operational cost considerations. *Working Paper, http://www.ieor.berkeley.edu/ shen/webpapers/V.11.pdf*, 2008.

[26] V. R. Mahajan and R. A. Peterson. Innovation diffusion in a dynamic potential adpater population. *Management Science*, 24:1589–1597, 1978.

[27] Staelin R. McGuire, T.W. An industry equilibrium analysis of downstream vertical integration. *Marketing Science*, 2(2):161–191, 1983.

[28] N. Mead and T. Islam. Technological forecasting – model selection, model stability, and combining models. *Management Science*, 44(8):1115–1130, 1998.

[29] H.I. Mesak and W.D. Berg. Incorporating price and replacement purchases in new product diffusion models for consumer durables. *Decision Sciences*, 26(4):425–449, 1995.

[30] A. Müller and D. Stoyan. *Comparison Methods for Stochastic Models and Risks*. John Wiley & Sons, Ltd., West Sussex, England, 2002.

[31] S. Netessine. Dynamic pricing of inventory/capacity with infrequent price changes. *European Journal of Operational Research*, 174(1):553–580, 2006.

[32] D. Pekelman. Simultaneous price-production decisions. *Operations Research*, 22(4):788–794, 1974.

[33] B. Robinson and C. Lakhani. Dynamic pricing models for new product planning. *Management Science*, 21(10):1113–1122, 1975.

[34] E. M. Rogers. Diffusion of innovations. *The Free Press, New York*, 1962.

[35] D. Sengupta and A. K. Nanda. Log-concave and concave distributions in reliability. *Naval Research Logistics*, 46:419–433, 1999.

[36] A. Shah. Bidding for iphone 3g tops $1,000 on ebay. *http://www.pcworld.com/businesscenter/article/148703/bidding_for_iphone_3g_tops_1000_on_ebay.html*, 2008.

[37] S. R. Shenoy and A. Daniel. Intel architecture and silicon cadence: the catalyst for industry innovation. *ftp://download.intel.com/software/pdf/IAandSiliconCadence.pdf*, 2006.

[38] S.A. Smith and D.D. Achabal. Clearance pricing and inventory policies for retail chains.

Management Science, 44(3):285–300, 1998.

[39] M. Speece and D. MacLachlan. Forecasting fluid milk package type with a multi-generation new-product diffusion model. *IEEE Transactions on Engineering Management*, 39:169–175, 1992.

[40] E. Stapleton. The normal distribution as a model of technological substitution. *Technological Forecasting and Social Change*, 8:325–334, 1976.

[41] N. Stokey. Intertemporal price discrimination. *Quarterly Journal of Economics*, 93(3):355–371, 1979.

[42] R. Stone. Sigmoids. *Bulletin in Applied Statistics*, 7:59–119, 1980.

[43] X. Su. Intertemporal pricing with strategic customer behavior. *Management Science*, 53(5): 726–741, 2007.

[44] X. Su and F. Zhang. Strategic customer behavior, commitment, and supply chain performance. *Management Science*, 54(10):1759–1773, 2008.

[45] J. C. Tanner. Long-term forecasting of vehicle ownership and road traffic. *J. Royal Statistical Society. Series A.*, 141:14–63, 1978.

[46] X. Xu and W.J. Hopp. Customer heterogeneity and strategic behavior in revenue management: a martingale approach. *Working paper*, 2004.

[47] W. Zhao and Y. Zheng. Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management Science*, 46(3):375–388, 2000.

A Appendix

A.1 Monotonicity of $p^*(t_1, t_2)$ in Section 2.2

We prove the following proposition.

Proposition A.1. *Suppose Assumption 1 holds. For any pair of* (t_1, t_2) *and* (t_1', t_2') *satisfying* $t_1 \le t_1'$ *and* $t_2 \le t_2'$*,*

$$p^*(t_1, t_2) \quad \ge \quad p^*(t_1', t_2') \ .$$

Proof. We first state the following claim: $p^*(s_1) \ge p^*(s_1, s_2) \ge p^*(s_2)$ for any s_1 and s_2 such that $s_1 \le s_2$. To prove this, recall that

$$p^*(s_1, s_2) \ = \ \arg\max_p R(p, s_1, s_2) \quad \text{where } R(p, s_1, s_2) = \int_{s_1}^{s_2} r(p, s) ds \ .$$

For any $s \in [s_1, s_2]$, the fact that $p^*(t)$ is decreasing in t implies that $p^*(s_1) \ge p^*(s) \ge p^*(s_2)$. Therefore, the concavity of $r(p, t)$ in p implies that $r(p, s)$ is increasing in p for any $p < p^*(s_2)$, and thus $p^*(s_1, s_2)$ cannot be smaller than $p^*(s_2)$. Similarly, $r(p, s)$ is decreasing in p for any $p > p^*(s_1)$, and thus $p^*(s_1, s_2)$ cannot be bigger than $p^*(s_1)$. Therefore, we complete the proof of the claim.

Now, we consider the following two cases separately depending on whether $t_2 \le t_1'$ holds or not. Suppose $t_2 \le t_1'$. Then, applying the above claim and the monotonicity of $p^*(t)$, we obtain

$$p^*(t_1, t_2) \ \ge \ p^*(t_2) \ \ge \ p^*(t_1') \ \ge \ p^*(t_1', t_2') \ ,$$

proving the required result.

Now, suppose $t_1' < t_2$. Thus, $t_1 \le t_1' < t_2 \le t_2'$. By the above claim, $p^*(t_1') \ge p^*(t_1', t_2)$. Thus, for any $t \le t_1'$, we have $p^*(t) \ge p^*(t_1', t_2)$, which implies that $r(p, t)$ is increasing in p at $p^*(t_1', t_2)$. Therefore, $R(p, t_1, t_1') = \int_{t_1}^{t_1'} r(p, t) dt$ is also increasing in p at $p^*(t_1', t_2)$. Also note that $p^*(t_1', t_2)$ is the maximizer of $R(p, t_1', t_2) = \int_{t_1'}^{t_2} r(p, t) dt$. It follows that $R(p, t_1, t_2) = R(p, t_1, t_1') + R(p, t_1', t_2)$ is increasing in p at $p^*(t_1', t_2)$, and we conclude that $p^*(t_1, t_2) \ge p^*(t_1', t_2)$. By applying a similar argument, we can prove that $p^*(t_1', t_2) \ge p^*(t_1', t_2')$. Thus, these two inequalities imply the required result. $\square$

A.2 Proof of Proposition 2.1

Proof. Since $V(\tau_1, \ldots, \tau_{n-1}, p_1, \ldots, p_n)$ is a separable function in $(p_1, \ldots, p_n)$, maximizing $V(\tau_1, \ldots, \tau_{n-1}, p_1, \ldots, p_n)$ is equivalent to maximizing $R(p_i, \tau_{i-1}, \tau_i)$ over p_i, which is a concave function maximization problem. Since

$$\int_{\tau_{i-1}}^{\tau_i} \frac{\partial}{\partial p_i} r(p_i, t) dt \ = \ \int_{\tau_{i-1}}^{\tau_i} \frac{\partial}{\partial p_i} \left[(a - b(t) p_i) \cdot p_i \right] h(t) dt \ = \ \int_{\tau_{i-1}}^{\tau_i} \left[a - 2b(t) p_i \right] h(t) dt \ ,$$

the corresponding first-order condition $\int_{\tau_{i-1}}^{\tau_i} \frac{\partial}{\partial p_i} r(p_i,t)dt = 0$ implies (2.2).

$$
\begin{aligned}
\frac{\partial}{\partial \tau_i} V(\tau_1,\ldots,\tau_{n-1},p_1,\ldots,p_n) &= \frac{\partial}{\partial \tau_i}\int_{\tau_{i-1}}^{\tau_i} r(p_1,t)dt \;+\; \frac{\partial}{\partial \tau_i}\int_{\tau_i}^{\tau_{i+1}} r(p_2,t)dt \\
&= r(p_i,\tau_i) - r(p_{i+1},\tau_i) \\
&= h(\tau_i)\left[(a - b(\tau_i)\cdot p_i)\cdot p_i\right] - h(\tau_i)\left[(a - b(\tau_i)\cdot p_{i+1})\cdot p_{i+1}\right] \\
&= h(\tau_i)\cdot(p_i^2 - p_{i+1}^2)\cdot\left[-b(\tau_i) + \frac{a\cdot(p_i - p_{i+1})}{p_i^2 - p_{i+1}^2}\right] \\
&= h(\tau_i)\cdot(p_i^2 - p_{i+1}^2)\cdot\left[-b(\tau_i) + \frac{a}{p_i + p_{i+1}}\right] . \qquad \text{(A.1)}
\end{aligned}
$$

□

A.3 Necessary Conditions for the Optimality of $\tilde{V}(\tau_1,\ldots,\tau_{n-1})$

From the first-order and second-order necessary conditions for optimality, we obtain the following proposition.

Proposition A.2. *Under Assumption 1, suppose that $(\tau_1,\ldots,\tau_{n-1})$ maximizes $\tilde{V}$, and suppose that $\tau_i \in (0,T)$ where $i \in \{1,\ldots,n-1\}$. Then, $r(p^*(\tau_{i-1},\tau_i),\tau_i) = r(p^*(\tau_i,\tau_{i+1}),\tau_i)$ and $\frac{\partial}{\partial \tau_i} r(p^*(\tau_{i-1},\tau_i),\tau_i) \leq \frac{\partial}{\partial \tau_i} r(p^*(\tau_i,\tau_{i+1}),\tau_i)$.*

Proof. For simplicity, we prove the result for $n = 2$. The case for $n > 2$ is similar. Since $n = 1$, there is only one switching point τ_1, which we denote simply by τ. For the ease of notation, we use ∇_i to denote the first order partial derivative with respect to the i_{th} argument. Suppose that τ is an optimal solution for $\tilde{V}$ such that $\tau \in (0,T)$. Then, we need to show that

$$
\begin{aligned}
r(p^*(0,\tau),\tau) &= r(p^*(\tau,T),\tau) \;, \qquad \text{and} \qquad &\text{(A.2)} \\
\nabla_2 r(p^*(0,\tau),\tau) &\leq \nabla_2 r(p^*(\tau,T),\tau) \;. &\text{(A.3)}
\end{aligned}
$$

Recall

$$
\tilde{V}(\tau) = V(\tau,p^*(0,\tau),p^*(\tau,T)) = R(p^*(0,\tau),0,\tau) + R(p^*(\tau,T),\tau,T) \;.
$$

By taking the first derivative of $\tilde{V}(\tau)$ with respect to τ, we obtain

$$
\begin{aligned}
\tilde{V}'(\tau) &= \nabla_2 p^*(0,\tau)\int_0^{\tau} \nabla_1 r(p^*(0,\tau)),t)dt \;+\; r(p^*(0,\tau),\tau) \\
&\quad +\; \nabla_1 p^*(\tau,T)\int_{\tau}^{T} \nabla_1 r(p^*(\tau,T),t)dt \;-\; r(p^*(\tau,T),\tau) \\
&= r(p^*(0,\tau),\tau) - r(p^*(\tau,T),\tau) \;, \qquad \text{(A.4)}
\end{aligned}
$$

where the last equality follows from the fact that the first and third terms are both zero by the first order condition of $p^*(t_1, t_2)$, i.e., $p = p^*(t_1, t_2)$ is a solution to

$$\frac{\partial}{\partial p} R(p, t_1, t_2) \;=\; \int_{t_1}^{t_2} \frac{\partial}{\partial p} r(p, t) dt \;=\; 0 \;.$$

Thus, the first order condition for switching time τ implies (A.2).

From (A.4),

$$\begin{aligned} \tilde{V}''(\tau) \;=\; & \nabla_2 r(p^*(0, \tau), \tau) \;+\; \nabla_1 r(p^*(0, \tau), \tau) \cdot \nabla_2 p^*(0, \tau) \\ & -\; \nabla_2 r(p^*(\tau, T), \tau) \;-\; \nabla_1 r(p^*(\tau, T), \tau) \cdot \nabla_1 p^*(\tau, T) \;. \end{aligned}$$

From Proposition A.1, we obtain $p^*(0, \tau) \geq p^*(\tau)$, and thus $\nabla_1 r(p^*(0, \tau), \tau) \leq 0$. Similarly, we obtain $p^*(\tau, T) \geq p^*(\tau)$ and thus $\nabla_1 r(p^*(\tau, T), \tau) \geq 0$. Also from Proposition A.1, $\nabla_2 p^*(0, \tau) \leq 0$ and $\nabla_1 p^*(\tau, T) \leq 0$. Therefore, the second and the fourth terms above are nonnegative. Thus, a necessary condition for $\tilde{V}''(\tau) \leq 0$ is that the first and the third terms should sum up to at most 0, a condition stated in (A.3). □

Quasi-Concavity of $\tilde{V}$ with Respect to τ_i

Under Assumption 1, the single-dimensional optimization of $\tilde{V}$ may not be quasi-concave. In this section, we identify the sufficient condition for the quasi-convexity of $\tilde{V}$ with respect to τ_i where $i \in \{1, \ldots, n-1\}$.

Suppose that we vary τ_i in $\tilde{V}(\tau_1, \ldots, \tau_{n-1})$ given in (2.4) while fixing all the other τ_j values, where $j \neq i$. For any $\tau_i \in [\tau_{i-1}, \tau_{i+1}]$, since b is an increasing function, it follows from (2.2) that $b(\tau_i)$ can be written as a convex combination of $\overline{b}(\tau_{i-1}, \tau_i)$ and $\overline{b}(\tau_i, \tau_{i+1})$. Define $\gamma(\tau_i)$ implicitly such that

$$b(\tau_i) \;=\; (1 - \gamma(\tau_i)) \cdot \overline{b}(\tau_{i-1}, \tau_i) + \gamma(\tau_i) \cdot \overline{b}(\tau_i, \tau_{i+1}) \;.$$

As τ_i increases in the interval $[\tau_{i-1}, \tau_{i+1}]$, the value of $\gamma(\tau_i)$ changes from initially 0 to eventually 1. Also, define $\rho(\tau_i)$ such that

$$\rho(\tau_i) \;=\; \frac{\overline{b}(\tau_{i-1}, \tau_i)}{\overline{b}(\tau_{i-1}, \tau_i) + \overline{b}(\tau_i, \tau_{i+1})} \;.$$

Note that $\rho(\tau_i)$ is bounded below by $b(\tau_{i-1})/[2 \cdot b(\tau_{i+1})] > 0$, and bounded above by 1/2. Therefore, $\rho(\tau_i) - \gamma(\tau_i)$ is positive when $\tau_i = \tau_{i-1}$, and it is negative when $\tau_i = \tau_{i+1}$. The necessary and sufficient condition that we identify in the following proposition is the single-crossing property of $\rho(\tau_i) - \gamma(\tau_i)$.

Proposition A.3. *Suppose Assumption 1 holds. Fix $i \in \{1, \ldots, n-1\}$, and $(\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_{n-1})$. Then, $\tilde{V}(\tau_1, \ldots, \tau_{n-1})$ is quasi-concave in $\tau_i \in [\tau_{i-1}, \tau_{i+1}]$ if and only if $\rho(\tau_i) - \gamma(\tau_i)$ crosses zero exactly once.*

Proof. We provide the proof for the two price case. The case for the n price model is similar.

Recall $r(p,t) = p \cdot (a - b(t)p) \cdot h(t)$. Recall from (2.2),

$$p^*(0,\tau) \;=\; \frac{a}{2\bar{b}(0,\tau)} \qquad \text{and} \qquad p^*(\tau,T) \;=\; \frac{a}{2\bar{b}(\tau,T)} \; .$$

From (A.4) and (2.2),

$$\begin{aligned}
\tilde{V}'(\tau) &= r(p^*(0,\tau),\tau) - r(p^*(\tau,T),\tau) \\
&= \frac{a}{2\bar{b}(0,\tau)} \cdot \left[a - b(\tau) \cdot \frac{a}{2\bar{b}(0,\tau)}\right] \cdot h(\tau) \;-\; \frac{a}{2\bar{b}(\tau,T)} \cdot \left[a - b(\tau) \cdot \frac{a}{2\bar{b}(\tau,T)}\right] \cdot h(\tau) \; .
\end{aligned}$$

Thus, the following statements are "if and only if" statements:

$$\begin{aligned}
\tilde{V}'(\tau) &\geq 0 \\
\frac{a}{2\bar{b}(0,\tau)} \cdot \left[a - b(\tau) \cdot \frac{a}{2\bar{b}(0,\tau)}\right] \cdot h(\tau) &\geq \frac{a}{2\bar{b}(\tau,T)} \cdot \left[a - b(\tau) \cdot \frac{a}{2\bar{b}(\tau,T)}\right] \cdot h(\tau) \\
\bar{b}(\tau,T) \cdot \left[1 - \frac{b(\tau)}{2\bar{b}(0,\tau)}\right] &\geq \bar{b}(0,\tau) \cdot \left[1 - \frac{b(\tau)}{2\bar{b}(\tau,T)}\right] \\
\bar{b}(\tau,T) - \bar{b}(0,\tau) &\geq \frac{1}{2} \cdot \frac{\bar{b}(\tau,T)^2 - \bar{b}(0,\tau)^2}{\bar{b}(0,\tau) \cdot \bar{b}(\tau,T)} \cdot b(\tau) \\
\bar{b}(0,\tau) \cdot \bar{b}(\tau,T) &\geq \frac{1}{2} \cdot \frac{\bar{b}(\tau,T)^2 - \bar{b}(0,\tau)^2}{\bar{b}(\tau,T) - \bar{b}(0,\tau)} \cdot b(\tau) \\
\frac{\bar{b}(0,\tau)}{\bar{b}(0,\tau) + \bar{b}(\tau,T)} \cdot \frac{\bar{b}(\tau,T)}{\bar{b}(0,\tau) + \bar{b}(\tau,T)} &\geq \frac{1}{2} \cdot \frac{b(\tau)}{\bar{b}(0,\tau) + \bar{b}(\tau,T)} \; .
\end{aligned}$$

Since $\rho(\tau) = \bar{b}(0,\tau)/[\bar{b}(0,\tau) + \bar{b}(\tau,T)]$ and $b(\tau) = (1 - \gamma(\tau)) \cdot \bar{b}(0,\tau) + \gamma(\tau) \cdot \bar{b}(\tau,T)$, the above condition is equivalent to

$$\rho(\tau) \cdot (1 - \rho(\tau)) \;\geq\; \frac{1}{2} \left[(1 - \gamma(\tau))\rho(\tau) + \gamma(\tau)(1 - \rho(\tau))\right] \; .$$

Since $\rho(\tau) \leq 1/2$ for any $\tau \in [0,T]$, the above inequality holds with equality if $\gamma(\tau) = \rho(\tau)$. In fact, we can show that the above inequality holds if and only if $\gamma(\tau) \leq \rho(\tau)$. Since $\rho(0) - \gamma(0) \geq 0$ and $\rho(T) - \gamma(T) < 0$, we can easily verify that the single-crossing property of $\rho(\theta) - \gamma(\theta)$ is sufficient and necessary to show the existence of $\hat{\tau} \in [\tau_{i-1}, \tau_{i+1}]$ such that $\tilde{V}'(\tau)$ is nonnegative in the interval $(0,\hat{\tau})$ and negative in $(\hat{\tau},T)$. □

The single-crossing property in Proposition A.3 is observed in most of the numerical examples that we have examined. If $h(t)$ is constant and $b(t)$ is a linear function of t, then the single-crossing condition in Proposition A.3 is satisfied, and we prove the following corollary.

Corollary A.4. *Suppose Assumption 1 holds, and that $h(t)$ is constant and $b(t)$ is linear in t. Then, for each $i \in \{1, \ldots, n-1\}$, $\tilde{V}(\tau_1, \ldots, \tau_{n-1})$ is quasi-concave in $\tau_i \in [\tau_{i-1}, \tau_{i+1}]$.*

Proof. Without loss of generality, suppose that $h(t) = 1$ for $t \in [0, T]$, and $b(t) = \beta_0 + \beta_1 t$ for nonnegative numbers β_0 and β_1. We provide the proof for the two-price case, i.e., $n = 2$; the generalization to any $n \geq 2$ follows a similar argument. Since $b(\tau) = \beta_0 + \beta_1 \tau$, it can be shown easily that

$$\begin{aligned} \overline{b}(0, \tau) &= \beta_0 + \frac{\beta_1 \tau}{2} \\ \overline{b}(\tau, T) &= \beta_0 + \frac{\beta_1 T}{2} + \frac{\beta_1 \tau}{2} \,. \end{aligned}$$

Thus, from the definition of $\gamma(\tau)$, we obtain

$$\begin{aligned} \beta_0 + \beta_1 \tau &= b(\tau) \\ &= (1 - \gamma(\tau)) \cdot \overline{b}(0, \tau) + \gamma(\tau) \cdot \overline{b}(\tau, T) \\ &= (1 - \gamma(\tau)) \cdot \left[\beta_0 + \frac{\beta_1 \tau}{2}\right] + \gamma(\tau) \cdot \left[\beta_0 + \frac{\beta_1 T}{2} + \frac{\beta_1 \tau}{2}\right] \,, \end{aligned}$$

which is simplified to

$$\gamma(\tau) = \frac{\tau}{T} \,.$$

Note that $\gamma(\tau)$ is a linear function with slope of $1/T$. Also, from the definition of $\rho(\tau)$,

$$\rho(\tau) = \frac{\overline{b}(0, \tau)}{\overline{b}(0, \tau) + \overline{b}(\tau, T)} = \frac{\beta_0 + \frac{\beta_1 \tau}{2}}{2\beta_0 + \beta_1 \tau + \frac{\beta_1 T}{2}} = \frac{1}{2} - \frac{\frac{\beta_1 T}{4}}{2\beta_0 + \beta_1 \tau + \frac{\beta_1 T}{2}} \,,$$

which is a concave increasing function of τ. Therefore, the maximum slope occurs at $\tau = 0$, at which

$$\rho'(\tau)|_{\tau=0} = \frac{\frac{\beta_1 T}{4} \cdot \beta_1}{\left[2\beta_0 + \frac{\beta_1 T}{2}\right]^2} \leq \frac{\frac{\beta_1 T}{4} \cdot \beta_1}{\left[\frac{\beta_1 T}{2}\right]^2} = \frac{1}{T} \,.$$

Therefore, $\rho(\tau) - \gamma(\tau)$ satisfies the single-crossing property. By Proposition A.3, $\tilde{V}(\tau_1, \ldots, \tau_{n-1})$ is quasi-concave in τ_i. ∎

A.4 Proof of Proposition 2.2

We first state and prove the following result.

Proposition A.5. *Fix $(p_1, \ldots, p_n)$. Under Assumption 1, the value of $(\tau_1, \ldots, \tau_{n-1})$ maximizing $V(\tau_1, \ldots, \tau_{n-1}, p_1, \ldots, p_n)$ is given by*

$$b(\tau_i) = \frac{a}{p_i + p_{i+1}} \tag{A.5}$$

for each $i \in \{1, \ldots, n-1\}$.

Proof. Consider the problem of maximizing V over τ_i while keeping all the other variables fixed. We obtain

$$\begin{aligned}
\frac{\partial}{\partial \tau_i} V(\tau_1, \ldots, \tau_{n-1}, p_1, \ldots, p_n) &= \frac{\partial}{\partial \tau_i} \int_{\tau_{i-1}}^{\tau_i} r(p_1, t) dt \; + \; \frac{\partial}{\partial \tau_i} \int_{\tau_i}^{\tau_{i+1}} r(p_2, t) dt \\
&= r(p_i, \tau_i) - r(p_{i+1}, \tau_i) \\
&= h(\tau_i) \left[(a - b(\tau_i) \cdot p_i) \cdot p_i \right] - h(\tau_i) \left[(a - b(\tau_i) \cdot p_{i+1}) \cdot p_{i+1} \right] \\
&= h(\tau_i) \cdot (p_i^2 - p_{i+1}^2) \cdot \left[-b(\tau_i) + \frac{a \cdot (p_i - p_{i+1})}{p_i^2 - p_{i+1}^2} \right] \\
&= h(\tau_i) \cdot (p_i^2 - p_{i+1}^2) \cdot \left[-b(\tau_i) + \frac{a}{p_i + p_{i+1}} \right] . \qquad \text{(A.6)}
\end{aligned}$$

Since b is an increasing function, V is quasi-concave in τ_i, and the optimal value of τ_i is given by (A.5). □

Proof of Proposition 2.2. Let p_i and p_{i+1} be as given in (2.2). Then, from (A.5) of Proposition A.5,

$$\frac{b(\tau_i)}{\overline{b}(\tau_{i-1}, \tau_i)} + \frac{b(\tau_i)}{\overline{b}(\tau_i, \tau_{i+1})} \;=\; \frac{a}{p_i + p_{i+1}} \cdot \frac{2 \cdot p_i}{a} + \frac{a}{p_i + p_{i+1}} \cdot \frac{2 \cdot p_{i+1}}{a} \;,$$

which simplifies to 2. □

A.5 Graphical Representation of Constructing $(\hat{\tau}_1(\theta), \ldots, \hat{\tau}_{n-1}(\theta))$

The method for constructing $(\hat{\tau}_1(\theta), \ldots, \hat{\tau}_{n-1}(\theta))$ can be illustrated graphically. We adopt a slightly different but still equivalent way of explaining this method. Let $\overline{t}_{i-1,i}$ be a proxy for $\overline{t}(\hat{\tau}_{i-1}(\theta), \hat{\tau}_i(\theta))$. In this method, we alternate the following two operations.

- In the first operation, we find $\hat{\tau}_i$ for given values of $\hat{\tau}_{i-1}$ and $\overline{t}_{i-1,i}$. We accomplish this by applying (2.2) such that the choice of $\hat{\tau}_i$ ensures that $\overline{t}_{i-1,i}$ is the analytic center of the interval $[\hat{\tau}_{i-1}, \hat{\tau}_i]$:

$$b(\overline{t}_{i-1,i}) \;=\; \frac{\int_{\hat{\tau}_{i-1}}^{\hat{\tau}_i} b(t) h(t) dt}{\int_{\hat{\tau}_{i-1}}^{\hat{\tau}_i} h(t) dt} \;.$$

- In the second operation, we find $\overline{t}_{i,i+1}$ for given values of $\overline{t}_{i-1,i}$ and $\hat{\tau}_i$. Recall (2.5), which is equivalent to

$$Z(\overline{t}_{i-1,i}) - Z(\hat{\tau}_i) \;=\; Z(\hat{\tau}_i) - Z(\overline{t}_{i,i+1}) \;,$$

  where we define $Z(t) = 1/b(t)$. Note that the left side of the above expression is independent of $\overline{t}_{i,i+1}$ while the right side is increasing in $\overline{t}_{i,i+1}$. This shows that the choice of $\overline{t}_{i,i+1}$ ensures that the difference in the Z function between $\overline{t}_{i-1,i}$ and $\hat{\tau}_i$ is the the same as the corresponding quantity between $\hat{\tau}_i$ and $\overline{t}_{i,i+1}$.

Figure 7 illustrates the procedure. We are interested in the values of θ such that $\hat{\tau}_n = T$ since this guarantees that the associated solution satisfies the set of first-order conditions for the optimality of V. Note that we can determine such values of θ by performing a search over a *single* dimension regardless of n.

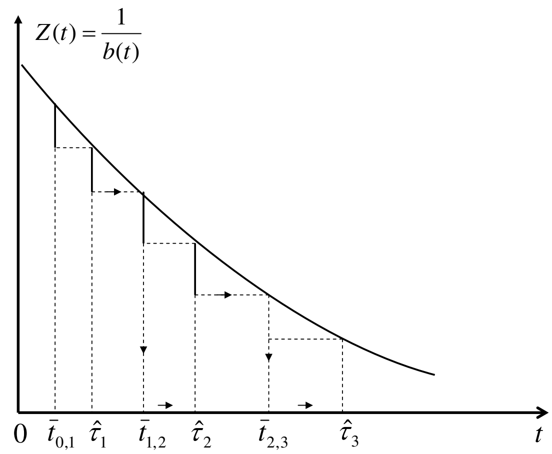

Figure 7: Iterative Algorithm for Finding $(\hat{\tau}_1(\theta), \ldots, \hat{\tau}_{n-1}(\theta))$.

A.6 Proof of Proposition 3.1

The proof of Proposition 3.1 is based on the following results.

Lemma A.6. *Under the conditions of Theorem 3.1, the optimal switching times under the n-price model, $(\tau_1, \ldots, \tau_{n-1})$, satisfy the following property: there exists c such that*

$$\frac{1 + 0.5m(\tau_{i-1} + \tau_i)}{\tau_i - \tau_{i-1}} = c, \qquad \textit{for each } i = 1, 2, \ldots, n.$$

Proof. Since $\overline{b}(\tau_{i-1}, \tau_i) = \beta_0 + \beta_1(\tau_{i-1} + \tau_i)/2$ and $\overline{b}(\tau_i, \tau_{i+1}) = \beta_0 + \beta_1(\tau_i + \tau_{i+1})/2$, Proposition 2.2 implies

$$\begin{aligned}
\frac{\beta_0 + \beta_1\tau_i}{\beta_0 + \beta_1(\tau_{i-1} + \tau_i)/2} + \frac{\beta_0 + \beta_1\tau_i}{\beta_0 + \beta_1(\tau_i + \tau_{i+1})/2} &= 2 \\
\frac{\beta_0 + \beta_1\tau_i}{\beta_0 + \beta_1(\tau_{i-1} + \tau_i)/2} - 1 &= 1 - \frac{\beta_0 + \beta_1\tau_i}{\beta_0 + \beta_1(\tau_i + \tau_{i+1})/2} \\
\frac{\beta_1(\tau_i - \tau_{i-1})/2}{\beta_0 + \beta_1(\tau_{i-1} + \tau_i)/2} &= \frac{\beta_1(\tau_{i+1} - \tau_i)/2}{\beta_0 + \beta_1(\tau_i + \tau_{i+1})/2} \\
\frac{(\tau_i - \tau_{i-1})}{1 + (\beta_1/\beta_0) \cdot (\tau_{i-1} + \tau_i)/2} &= \frac{(\tau_{i+1} - \tau_i)}{1 + (\beta_1/\beta_0) \cdot (\tau_i + \tau_{i+1})/2}.
\end{aligned}$$

Thus, we obtain the required result. □

Lemma A.7. *Under the conditions of Lemma A.6,*

$$\tau_i \;=\; \left[\left(\frac{c+0.5m}{c-0.5m}\right)^i - 1\right] / m \tag{A.7}$$

for $i = 1, \ldots, n$. Furthermore, the value of c satisfies

$$c \;=\; \frac{m}{2} \cdot \frac{\sqrt[n]{1+mT}+1}{\sqrt[n]{1+mT}-1} \;. \tag{A.8}$$

Proof. Define $z_i = m\tau_i + 1$. For the first result, we will prove the following result by induction:

$$z_i \;=\; \left(\frac{c+0.5m}{c-0.5m}\right)^i \;. \tag{A.9}$$

Clearly, this induction hypothesis implies (A.7). Since $\tau_0 = 0$ and $z_0 = 1$, the base case of (A.9) holds, and we proceed by assuming that the induction hypothesis holds for τ_{i-1}. From Lemma A.6,

$$\begin{aligned}
\frac{c}{m} &= \frac{\frac{1}{2}[(m\tau_{i-1}+1)+(m\tau_i+1)]}{m\tau_i - m\tau_{i-1}} = \frac{\frac{1}{2}[z_{i-1}+z_i]}{z_i - z_{i-1}} \\
\frac{c}{m}\cdot z_i - \frac{c}{m}\cdot z_{i-1} &= \frac{z_{i-1}}{2} + \frac{z_i}{2} \;.
\end{aligned}$$

Thus, we obtain

$$z_i \;=\; \left(\frac{c+0.5m}{c-0.5m}\right)\cdot z_{i-1} \;=\; \left(\frac{c+0.5m}{c-0.5m}\right)^i \;,$$

where the last equality follows from induction hypothesis. Thus, we complete the induction step and finish the proof of (A.7).

Now, since the the optimal value of τ_n must satisfy $\tau_n = T$, we set it to T the right side of (A.7), where $i = n$; thus,

$$1 + mT \;=\; \left(\frac{c+0.5m}{c-0.5m}\right)^n \tag{A.10}$$

for $i = 1, \ldots, n$. Then, we solve for c to obtain (A.8). □

Proof of Proposition 3.1. For $i = 1, \ldots, n$, (A.9) and (A.10) imply

$$z_i \;=\; \left(\frac{c+0.5m}{c-0.5m}\right)^i \;=\; (1+mT)^{i/n} \;,$$

where $z_i = m\tau_i + 1$. By rearranging it for τ_i, we obtain the required expression for τ_i.

From (2.2),

$$p_i \;=\; \frac{a}{2}\cdot\frac{1}{\beta_0 + \frac{\beta_1}{2}(\tau_{i-1}+\tau_i)} \;=\; \frac{a}{2}\cdot\frac{1}{\beta_0 + \frac{\beta_1}{2}(z_{i-1}+z_i-2)} \;=\; \frac{a/\beta_1}{z_{i-1}+z_i}$$

where the last equality follows from $m = \beta_1/\beta_0$. Now, substituting the above expression for z_{i-1} and z_i, we complete the proof. □

A.7 Proof of Equations (3.2) and (3.3)

Equation (3.2) follows directly from Proposition 3.1. Also, algebraic manipulation shows that the optimal switching times for the n-price model satisfy (3.3) for each $i \in \{0, \ldots, n-1\}$.

A.8 Proof of Equation (3.4)

Since $b(t) = \beta_0 + \beta_1 t$ and $m = \beta_1/\beta_0$, the left-side expression of equation (3.3) can be written as

$$\begin{aligned} \log(1 + m\tau_{i+1}) - \log(1 + m\tau_i) &= \log\left[\frac{b(\tau_{i+1})}{\beta_0}\right] - \log\left[\frac{b(\tau_i)}{\beta_0}\right] \\ &= \log\left[\frac{b(\tau_{i+1})}{b(\tau_i)}\right] = \log\left[1 + \frac{b(\tau_{i+1}) - b(\tau_i)}{b(\tau_i)}\right]. \end{aligned} \tag{A.11}$$

From (3.3) and (A.11), we obtain

$$\frac{b(\tau_{i+1}) - b(\tau_i)}{b(\tau_i)} = [1 + mT]^{1/n} - 1 = \left[\frac{b(T)}{\beta_0}\right]^{1/n} - 1 \ ,$$

which is (3.4).

A.9 Proof of Equation (3.9)

Proof. From Proposition 2.2 and equation (2.3) along with the assumption that $b(t) = \beta_0 + \beta_1 t$, the optimal switching time τ in the two-price model satisfies $\frac{1+m\tau}{1+m\bar{t}(0,\tau)} + \frac{1+m\tau}{1+m\bar{t}(\tau,T)} = 2$. Let $\Delta_L(\tau) = \tau - \bar{t}(0,\tau)$ and $\Delta_R(\tau) = \bar{t}(\tau,T) - \tau$, and rewrite the above equation as

$$\frac{1 + m\tau}{1 + m(\tau - \Delta_L(\tau))} + \frac{1 + m\tau}{1 + m(\tau + \Delta_R(\tau))} = 2 \ ,$$

which is equivalent to

$$\frac{1}{\Delta_L(\tau)} - \frac{1}{\Delta_R(\tau)} = \frac{2m}{1 + m\tau} \ .$$

Clearly, the nonnegativity of the above expression implies that $\Delta_L(\tau) \le \Delta_R(\tau)$. Furthermore, by the symmetry of $h(t)$ centered at $T/2$ and the linearity of $b(t)$, it can be verified from equations (2.2) and (2.3) that $\Delta_L(\tau) = \Delta_R(\tau)$ at $\tau = T/2$.

For $t \in [0, T]$, define $\tilde{h}(t) = h(t)/[\int_0^T h(s)ds]$. Since $h(t)$ log-concave, $\tilde{h}(t)$ is also log-concave. Define

$$\delta(\tau) = \tau - \frac{\int_0^\tau s\tilde{h}(s)ds}{\int_0^\tau \tilde{h}(s)ds} \ ,$$

which is known in the literature as the *mean-advantage-over-inferiors* function. Since $\tilde{h}(t)$ is log-concave, it follows from Theorem 5 of Bagnoli and Bergstrom (2005) that $\delta(\tau)$ is an increasing function. From

$$\begin{aligned}
\beta_0 + \beta_1 \cdot \bar{t}(\tau', \tau'') &= b(\bar{t}(\tau', \tau'')) = \bar{b}(\tau', \tau'') = \frac{\int_{\tau'}^{\tau''} b(s)h(s)dt}{\int_{\tau'}^{\tau''} h(s)ds} \\
&= \frac{\int_{\tau'}^{\tau''} [\beta_0 + \beta_1 s]h(s)ds}{\int_{\tau'}^{\tau''} h(s)ds} = \beta_0 + \beta_1 \cdot \frac{\int_{\tau'}^{\tau''} s\tilde{h}(s)ds}{\int_{\tau'}^{\tau''} \tilde{h}(s)ds} ,
\end{aligned}$$

we obtain $\bar{t}(\tau', \tau'') = \int_{\tau'}^{\tau''} s\tilde{h}(s)ds / \int_{\tau'}^{\tau''} \tilde{h}(s)ds$, and conclude that $\Delta_L(\tau) = \tau - \bar{t}(0, \tau) = \delta(\tau)$ is an increasing function. By symmetry, we can also show that $\Delta_R(\tau)$ is a decreasing function.

These monotonicity properties, along with the fact that $\Delta_L(\tau) \leq \Delta_R(\tau)$ and $\Delta_L(T/2) = \Delta_R(T/2)$, imply that the optimal choice of τ should satisfy $\tau \leq T/2$. □

A.10 Normal Demand Pattern $h(t)$: Proof of (3.10)

Suppose that $h(t)$ is given by the probability density function $N(\mu, \sigma^2)$, a normal distribution with mean μ and standard deviation σ. From (2.2) and (2.3),

$$\bar{t}(\tau', \tau'') = \mu - \sigma^2 \cdot \frac{f(\tau'') - f(\tau')}{F(\tau'') - F(\tau')} , \tag{A.12}$$

where f and F denote the probability density function and the cumulative density function of the normal distribution $N(\mu, \sigma)$. (The proof of (A.12) appears at the end of this section.) Thus, (2.5) for the two-price case would become

$$2 = \frac{b(\tau)}{\bar{b}(0, \tau)} + \frac{b(\tau)}{\bar{b}(\tau, T)} = \frac{1 + m\tau}{1 + m\left[\mu + \sigma^2 \frac{f(\tau) - f(0)}{F(\tau) - F(0)}\right]} + \frac{1 + m\tau}{1 + m\left[\mu + \sigma^2 \frac{f(T) - f(\tau)}{F(T) - F(\tau)}\right]} .$$

Recall $\mu = T/2$ and $\sigma = T/6$. Then, as we vary $[b(T) - b(0)]/b(0) = \beta_1 T/\beta_0$, we obtain the optimal switching times given in (3.10).

Proof of Equation (A.12) . For fixed μ and σ, let f and F be the probability density function and the cumulative density function of $N(\mu, \sigma^2)$, respectively. Let ϕ and Φ be the probability density function and the cumulative density function of the standard normal distribution. It has been well known that $\int_c^\infty t\phi(t)dt = \phi(c)$. Therefore,

$$\begin{aligned}
\int_s^\infty tf(t)dt &= \int_s^\infty \left[t \cdot \phi\left(\frac{t-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}\right] dt = \int_{\frac{s-\mu}{\sigma}}^\infty \left[(x\sigma + \mu) \cdot \phi(x)\right] dx \\
&= \sigma \int_{\frac{s-\mu}{\sigma}}^\infty x\phi(x)\, dx + \mu \int_{\frac{s-\mu}{\sigma}}^\infty \phi(x)\, dx = \sigma \cdot \phi\left(\frac{s-\mu}{\sigma}\right) + \mu \cdot \left[1 - \Phi\left(\frac{s-\mu}{\sigma}\right)\right] \\
&= \sigma^2 \cdot f(s) + \mu \cdot [1 - F(s)] .
\end{aligned}$$

Since h is the probability density function of $N(\mu, \sigma^2)$ and $b(t) = \beta_0 + \beta_1 t$, we obtain

$$\begin{aligned}
\int_{\tau'}^{\tau''} b(t)h(t)dt &= \beta_0 \cdot [F(\tau'') - F(\tau')] \ + \ \beta_1 \int_{\tau'}^{\tau''} tf(t)dt \\
&= \beta_0 \cdot [F(\tau'') - F(\tau')] \ + \ \beta_1 \cdot \left[\mu \cdot [F(\tau'') - F(\tau')] - \sigma^2 \cdot (f(\tau'') - f(\tau'))\right] \ .
\end{aligned}$$

Equation (2.2) and the fact $\int_{\tau'}^{\tau''} h(t)dt = F(\tau'') - F(\tau')$ together imply that

$$\bar{b}(\tau', \tau'') \ = \ \frac{\int_{\tau'}^{\tau''} b(t)h(t)dt}{\int_{\tau'}^{\tau''} h(t)dt} \ = \ \beta_0 \ + \ \beta_1 \cdot \left[\mu - \sigma^2 \cdot \frac{f(\tau'') - f(\tau')}{F(\tau'') - F(\tau')}\right] \ .$$

Since the definition of $\bar{t}$ in (2.3) is given by $\bar{b}(\tau', \tau'') = b(\bar{t}(\tau', \tau''))$, we obtain (A.12), as required. □

A.11 Proof of Theorem 3.2

Proof. (a) Note that

$$\frac{p_{i+1}}{p_i} \ = \ \frac{\bar{b}(\tau_{i-1}, \tau_i)}{\bar{b}(\tau_i, \tau_{i+1})} \ = \ 2 \cdot \frac{\bar{b}(\tau_{i-1}, \tau_i)}{b(\tau_i)} - 1 \ = \ 2 \cdot \frac{1 + m\bar{t}_{i-1,i}}{1 + m\tau_i} - 1 \ , \tag{A.13}$$

where the first equality follows from equation (2.2) of Proposition 2.1; the second equality follows from equation (2.5) of Proposition 2.2; and the third equality follows from the linearity assumption on $b(t) = \beta_0 + \beta_1 t$, the definition of $m = \beta_1/\beta_0$, and the fact that $\bar{t}_{i-1,i}$ is the analytic center of the interval $[\tau_{i-1}, \tau_i]$.

Suppose $\tau_i \le \hat{t}$. Since $h(t)$ is increasing in $t \in [0, \hat{t}]$, the average of τ_{i-1} and τ_i should be smaller than the analytic center $\bar{t}_{i-1,i}$ (see equation (2.2)), i.e., $(\tau_{i-1} + \tau_i)/2 \le \bar{t}_{i-1,i}$. This inequality is equivalent to the below inequality:

$$\frac{1 + m\tau_i}{1 + m\bar{t}_{i-1,i}} \ \le \ 2 - \frac{1 + m\tau_{i-1}}{1 + m\bar{t}_{i-1,i}} \ = \ \frac{1 + m\tau_{i-1}}{1 + m\bar{t}_{i-2,i-1}} \ ,$$

where the equality follows from equation (2.5). From (A.13), it implies $p_{i+1}/p_i \ge p_i/p_{i-1}$.

If $\tau_{i-1} \ge \hat{t}$, a similar argument shows $p_{i+1}/p_i \le p_i/p_{i-1}$, as required.

(b) From equation (A.5) in Proposition A.5, we obtain, for each $i = 1, 2, \ldots, n-2$,

$$\frac{b(\tau_{i+1})}{b(\tau_i)} \ = \ \frac{p_i + p_{i+1}}{p_{i+1} + p_{i+2}} \ .$$

Thus

$$\begin{aligned}
&\{\log[b(\tau_{i+1})] - \log[b(\tau_i)]\} - \{\log[b(\tau_i)] - \log[b(\tau_{i-1})]\} \\
&= \ \log \frac{b(\tau_{i-1})b(\tau_{i+1})}{b(\tau_i)^2} \ = \ \log \frac{(p_i + p_{i+1})(p_i + p_{i+1})}{(p_{i+1} + p_{i+2})(p_{i-1} + p_i)} \\
&= \ \log \frac{(1 + p_{i+1}/p_i)(1 + p_i/p_{i+1})}{(1 + p_{i+2}/p_{i+1})(1 + p_{i-1}/p_i)} \ = \ \log \left\{ \left[\frac{1 + p_{i+1}/p_i}{1 + p_{i+2}/p_{i+1}}\right] \left[\frac{1 + p_i/p_{i+1}}{1 + p_{i-1}/p_i}\right] \right\} \ .
\end{aligned}$$

If $\frac{p_i}{p_{i-1}} \leq \frac{p_{i+1}}{p_i} \leq \frac{p_{i+2}}{p_{i+1}}$, then we obtain $\frac{1+p_{i+1}/p_i}{1+p_{i+2}/p_{i+1}} < 1$ and $\frac{1+p_i/p_{i+1}}{1+p_{i-1}/p_i} < 1$, which imply that the above expression is non-positive, i.e., $\log[b(\tau_{i+1})] - \log[b(\tau_i)] \leq \log[b(\tau_i)] - \log[b(\tau_{i-1})]$. We obtain an analogous result if $\frac{p_i}{p_{i-1}} \geq \frac{p_{i+1}}{p_i} \geq \frac{p_{i+2}}{p_{i+1}}$ holds instead.

The proof follows from the fact that $\log(1+m\tau_i) = \log(b(\tau_i)/\beta_0) = \log(b(\tau_i)) - \log\beta_0$. $\square$

A.12 Proofs of Proposition 4.1 and Corollary 4.2

Recall, from (2.2), $\overline{b}(\tau', \tau'') = \int_{\tau'}^{\tau''} b(t)h(t)dt / \int_{\tau'}^{\tau''} h(t)dt$.

Lemma A.8. *Under Assumption 1, the optimal prices $(p_1, \ldots, p_n)$ satisfy*

$$\frac{1}{1+p_{i+1}/p_i} - \frac{1}{1+p_i/p_{i-1}} \;=\; \frac{b(\tau_i)+b(\tau_{i-1})}{2\overline{b}(\tau_{i-1},\tau_i)} - 1 \;. \tag{A.14}$$

Furthermore,

$$\begin{cases} p_i/p_{i-1} \;\leq\; p_{i+1}/p_i & \textit{if } [b(\tau_{i-1}) + b(\tau_i)]/2 \leq \overline{b}(\tau_{i-1},\tau_i) \\ p_i/p_{i-1} \;\geq\; p_{i+1}/p_i & \textit{if } [b(\tau_{i-1}) + b(\tau_i)]/2 \geq \overline{b}(\tau_{i-1},\tau_i). \end{cases}$$

Proof. Note that (A.13) is valid without the assumption on the linearity of $b(t)$. Thus,

$$1 + \frac{p_{i+1}}{p_i} \;=\; 2 \cdot \frac{\overline{b}(\tau_{i-1},\tau_i)}{b(\tau_i)} \qquad \text{and} \qquad 1 + \frac{p_i}{p_{i-1}} \;=\; 2 \cdot \frac{\overline{b}(\tau_{i-2},\tau_{i-1})}{b(\tau_{i-1})} \;.$$

Therefore,

$$\frac{1}{1+p_{i+1}/p_i} - \frac{1}{1+p_i/p_{i-1}} \;=\; \frac{1}{2} \cdot \left[\frac{b(\tau_i)}{\overline{b}(\tau_{i-1},\tau_i)} - \frac{b(\tau_{i-1})}{\overline{b}(\tau_{i-2},\tau_{i-1})}\right] \;.$$

Since Proposition (A.5) implies $b(\tau_{i-1})/\overline{b}(\tau_{i-2},\tau_{i-1}) = 2 - b(\tau_{i-1})/\overline{b}(\tau_{i-1},\tau_i)$, we get (A.14) as required.

Now, if $[b(\tau_{i-1}) + b(\tau_i)]/2 \leq \overline{b}(\tau_{i-1},\tau_i)$, then the expression in (A.14) is at most 0, and it follows that $p_i/p_{i-1} \leq p_{i+1}/p_i$. Similarly, $[b(\tau_{i-1}) + b(\tau_i)]/2 \geq \overline{b}(\tau_{i-1},\tau_i)$ implies $p_i/p_{i-1} \geq p_{i+1}/p_i$. $\square$

Lemma A.8 states that the percentage reduction in price, given by $(1-p_{i+1}/p_i)$, increases or decreases in i depending on the relationship between the arithmetic mean of the price sensitivities at the two adjacent switching times, $[b(\tau_i) + b(\tau_{i-1})]/2$, and the analytic mean $\overline{b}(\tau_{i-1},\tau_i)$. Suppose $h(t) = 1$. Then, from (2.2), $\overline{b}(\tau', \tau'') = \int_{\tau'}^{\tau''} b(t)dt/(\tau'' - \tau')$.

If $b(t)$ is concave in t, then it can be shown that, for any $t \in [\tau', \tau'']$,

$$b(t) \;\geq\; \frac{\tau''-t}{\tau''-\tau'} \cdot b(\tau') + \frac{t-\tau'}{\tau''-\tau'} \cdot b(\tau'').$$

Since $\int_{\tau'}^{\tau''}(\tau''-t)dt = [\tau''t - t^2/2]_{t=\tau'}^{\tau''} = [\tau''^2 - \tau''^2/2] - [\tau'\tau'' - \tau'/2] = (\tau''-\tau')^2/2$ and similarly $\int_{\tau'}^{\tau''}(t-\tau')dt = (\tau''-\tau')^2/2$, the above inequality implies

$$\frac{\int_{\tau'}^{\tau''} b(t)dt}{\tau''-\tau'} \geq \int_{\tau'}^{\tau''} \frac{\tau''-t}{(\tau''-\tau')^2}\cdot b(\tau')dt + \int_{\tau'}^{\tau''} \frac{t-\tau'}{(\tau''-\tau')^2}\cdot b(\tau'')dt = \frac{b(\tau')}{2} + \frac{b(\tau'')}{2}. \quad \text{(A.15)}$$

Thus, $\overline{b}(\tau_{i-1},\tau_i) \geq [b(\tau_{i-1}) + b(\tau_i)]/2$ holds. By Lemma A.8, it follows that $p_i/p_{i-1} \leq p_{i+1}/p_i$. Now, the following lemma shows that $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ is decreasing in i.

Lemma A.9. *Under Assumption 1, the optimal switching times $(\tau_1,\ldots,\tau_{n-1})$ satisfy the following properties.*
(a) If $\frac{p_i}{p_{i-1}} \leq \frac{p_{i+1}}{p_i} \leq \frac{p_{i+2}}{p_{i+1}}$, then $\log[b(\tau_i)] - \log[b(\tau_{i-1})] \geq \log[b(\tau_{i+1})] - \log[b(\tau_i)]$.
(b) If $\frac{p_i}{p_{i-1}} \geq \frac{p_{i+1}}{p_i} \geq \frac{p_{i+2}}{p_{i+1}}$, then $\log[b(\tau_i)] - \log[b(\tau_{i-1})] \leq \log[b(\tau_{i+1})] - \log[b(\tau_i)]$.

Proof. It follows the same argument as in the proof of Theorem 3.2(b). □

If $b(t)$ is convex in t, we can similarly show from Lemmas A.8 and A.9 that $p_i/p_{i-1} \geq p_{i+1}/p_i$ and $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ is increasing in i. This completes the proof of Proposition 4.1.

To prove Corollary 4.2, first suppose that $h(t)$ is increasing and $b(t)$ is concave and increasing in t. We first prove the following claim:

$$\int_{\tau_{i-1}}^{\tau_i} b(t)\cdot\frac{h(t)}{\int_{\tau_{i-1}}^{\tau_i} h(t)dt}dt \geq \int_{\tau_{i-1}}^{\tau_i} b(t)\cdot\frac{1}{\tau_i-\tau_{i-1}}dt\ .$$

Suppose that X_1 and X_2 are random variables defined on the interval $[\tau_{i-1},\tau_i]$ with densities given by $h(t)/\int_{\tau_{i-1}}^{\tau_i} h(t)dt$ and $1/(\tau_i-\tau_{i-1})$, respectively. Then, since $h(t)$ is increasing, we can show, for any t' and $s \geq 0$,

$$P[X_1 > s+t' \mid X_1 > t'] \geq P[X_2 > s+t' \mid X_2 > t']\ ,$$

i.e., X_2 is smaller than X_1 with respect to the hazard rate order. Since $b(\cdot)$ is an increasing function, by Theorem 1.3.8 and Theorem 1.2.8 of Müller and Stoyan [30], we obtain $E[b(X_1)] \geq E[b(X_2)]$, which proves the above claim.

Now, from the definition of $\overline{b}(\tau_{i-1},\tau_i)$ given in (2.2) and inequality (A.15), we can show from the above claim that $(p_i - p_{i-1})/p_{i-1}$ is decreasing in i. Furthermore, by Lemma A.9, $\{\log b(\tau_i) - \log b(\tau_{i-1})\}$ is also decreasing in i, and we establish part (a) of Corollary 4.2. By similar reasoning, we can establish part (b) also.

A.13 Proof of Footnote 3

Proposition A.10. *Assume $h(t) = 1$ and $b(t) = \beta_0 + \beta_1 t = \beta_0(1+mt)$. In addition, assume $d(p,t) = \lambda(b(t)p)$, where the function $\lambda(\cdot)$ satisfies the conditions (i) $\lambda(x)$ is decreasing in x,* *and (ii)* $\dfrac{x\lambda'(x)}{\lambda(x)}$ *is decreasing in* x*. For the two-price model, the optimal switching times* τ *is given by*

$$\tau^* = [(1 + mT)^{1/2} - 1]/m \ .$$

We note that conditions (i) and (ii) imply concavity of $x\lambda(x)$ and are satisfied by many common price-demand relationships including the linear and the exponential demand functions.

Proof. Let p_1 and p_2 be the prices in the first and second period, respectively. The revenues during $[0, \tau]$ and during $[\tau, T]$ are given by

$$\begin{aligned} R_1(p_1, \tau) &= \int_0^\tau p_1 \lambda(b(t)p_1) dt \ \text{ and} &\text{(A.16)} \\ R_2(p_2, \tau) &= \int_\tau^T p_2 \lambda(b(t)p_2) dt \ . &\text{(A.17)} \end{aligned}$$

Taking first order derivative of equation (A.16) with respect to p_1 yields:

$$\begin{aligned} \frac{\partial R_1}{\partial p_1} &= \int_0^\tau \left[\lambda(b(t)p_1) + p_1 \frac{d\lambda}{d(b(t)p_1)} b(t) \right] dt \\ &= \int_0^\tau \lambda(b(t)p_1) dt + \int_0^\tau \frac{b(t)}{\beta_0 m} \left[p_1 \frac{d\lambda}{d(b(t)p_1)} \frac{db(t)}{dt} \right] dt \\ &= \int_0^\tau \lambda(b(t)p_1) dt + \frac{1}{\beta_0 m} \int_0^\tau b(t) d\lambda(b(t)p_1) \\ &= \int_0^\tau \lambda(b(t)p_1) dt + \frac{1}{\beta_0 m} \left[b(t)\lambda(b(t)p_1)|_0^\tau - \int_0^\tau \lambda(b(t)p_1) db(t) \right] \\ &= \int_0^\tau \lambda(b(t)p_1) dt + \frac{1}{\beta_0 m} b(t)\lambda(b(t)p_1)|_0^\tau - \int_0^\tau \lambda(b(t)p_1) dt \\ &= \frac{1}{\beta_0 m} b(t)\lambda(b(t)p_1)|_0^\tau \ , \end{aligned}$$

where the second and fifth equalities are due to $\dfrac{db(t)}{dt} = \beta_0 m$. The third equality is due to $\dfrac{d\lambda(b(t)p_1)}{dt} = p_1 \dfrac{d\lambda}{d(b(t)p_1)} \dfrac{db(t)}{dt}$.

Suppose that $p_1^*(\tau)$ satisfies $\frac{\partial R_1(p_1, \tau)}{\partial p_1} = 0$. Using the above equality and condition (ii), we can verify that $\frac{\partial^2 R_1(p_1,\tau)}{\partial p_1^2}|_{p_1 = p_1^*(\tau)} \leq 0$ (the details are skipped for brevity). Hence R_1 is unimodal in p_1 for a given switching time τ. Therefore, by setting the rightmost expression to zero, the optimal first-period price $p_1^*(\tau)$ should satisfy:

$$b(\tau)\lambda(b(\tau)p_1^*(\tau)) = b(0)\lambda(b(0)p_1^*(\tau)) \ . \qquad \text{(A.18)}$$

Similarly, optimizing R_2 over p_2 for a given τ, we obtain

$$b(\tau)\lambda(b(\tau)p_2^*(\tau)) \quad = \quad b(T)\lambda(b(T)p_2^*(\tau)) \ . \tag{A.19}$$

Therefore the total profit as a function of the switching time τ is given by

$$R(\tau) = R_1(p_1^*(\tau), \tau) + R_2(p_2^*(\tau), \tau) \ .$$

From the proof of the equation (A.2) in Appendix A.3, we know that the optimal τ shall satisfy

$$p_1^*(\tau)\lambda(b(\tau)p_1^*(\tau)) = p_2^*(\tau)\lambda(b(\tau)p_2^*(\tau)) \ . \tag{A.20}$$

From equations (A.18) to (A.20), we obtain

$$b(0)p_1^*\lambda(b(0)p_1^*) = b(\tau)p_1^*\lambda(b(\tau)p_1^*) = b(\tau)p_2^*\lambda(b(\tau)p_2^*) = b(T)p_2^*\lambda(b(T)p_2^*) \ . \tag{A.21}$$

Let the value of the above equalities be z. The above equalities imply that $b(0)p_1^*$, $b(\tau)p_1^*$, $b(\tau)p_2^*$ and $b(T)p_2^*$ are roots of the equation $x\lambda(x) = z$. Because of the monotonicity of $b(t)$ and because $p_1^* \geq p_2^*$ (see the monotonicity result from Appendix A.1),

$$b(0)p_1^* \ < \ b(\tau)p_1^* \ \geq \ b(\tau)p_2^* \ < \ b(T)p_2^*.$$

However, conditions (i) and (ii) imply that $x\lambda(x)$ is concave in x; thus there could be at most two distinct roots for equation $x\lambda(x) = z$. Therefore, we must have

$$b(0)p_1^* \quad = \quad b(\tau)p_2^* \tag{A.22}$$

$$b(\tau)p_1^* \quad = \quad b(T)p_2^* \ , \tag{A.23}$$

which leads to $\frac{b(\tau)}{b(0)} = \frac{b(T)}{b(\tau)}$ and thus $b(\tau) = (b(0)b(T))^{1/2}$. Because $b(t) = \beta_0(1 + mt)$, we have

$$\tau^* = [(1 + mT)^{1/2} - 1]/m \ ,$$

which is equal to the optimal switching time for the two-price problem under linear demand and constant $h(t)$ as derived from Proposition 3.1. □