# Structural Balance and Transitivity

Social Network Analysis, Chapter 6
Wasserman and Faust

# Balance Theory

- Concerned with how an individual's attitudes or opinions coincide with those of others in a network
- balance vs dissonance
  - if two actors that are "friends" have the same "attitude" toward a third entity, there is balance
  - if two friends have different attitudes toward a third entity, there is dissonance

# Structural Balance: Representation

- Signed graph with positive or negative edges
  - positive is "liking"
  - negative is "not liking"
- Edges
  - nondirectional, i.e., mutual
  - directional, i.e., i -> j is distinct from j -> i

# Structural Balance: Signed Nondirectional Graphs

- Characterize a graph by its cycles (see Ch. 4)
- Sign of a cycle is the product of signs of its edges
- Balanced cycle has positive sign
- Simplest cycle is a triple (three edges)
  - zero or two negative edges is balanced
  - one negative edge is unbalanced
- If all triples in a graph have positive signs, it is balanced

# Structural Balance: Signed Nondirectional Graphs

- Sign of *n* -length cycle
  - zero or even number of negative edges is balanced
  - odd number of negative edges is unbalanced
- A signed graph is balanced if and only if all cycles have positive signs (Cartwright and Harary, 1956)
- A graph with no cycles is vacuously balanced: neither balanced nor unbalanced
- A balanced graph can be partitioned into two subsets
  - only positive edges connect nodes within a subset
  - only negative edges connect nodes between subsets

# Structural Balance: Directional Graphs

- Cycles in a directional graph (digraph) require all arcs to "point in the same direction"
- A digraph may not contain any cycles
- Use semicycles, which ignore arc direction
- A signed digraph is balanced if and only if all semicycles have positive signs
-

# Structural Balance: Metric

To measure how unbalanced a graph or digraph is, use the cycle index for balance

- PC = number of positive (semi)cycles
- TC = total number of (semi)cycles
- cycle index for balance = PC/TC

# Clusterability

- Balanced signed graphs can be partitioned into two subgraphs (or clusters), as mentioned previously
- Real-life data tend toward graphs with more than one natural cluster
- A signed graph is clusterable if it can be partitioned into a finite number of subsets s/t each positive edge joins two nodes in the same subset and each negative edge joins two nodes in different subsets
- Clusterability generalizes balance
- Equivalent to graph colorability

# Clustering Theorems

- A signed graph has a clustering if and only if it contains no cycles with exactly one negative edge
- The following characteristics are equivalent with respect to a complete signed graph has
    - is clusterable
    - has a unique clustering
    - has no cycle with exactly one negative edge
    - has no cycle of length 3 with exactly one negative edge

# Clusterability and Incompleteness

- Lack of graph completeness...
  - increases difficulty in deciding clusterability--complete graphs only require inspection of triples
  - prevents guarantee of unique clustering
- Complete graphs are rare in practice
- Important to find ways to find a "good" clustering
- Graphs with no cycles are "vacuously clusterable"

# Clusterability in Practice

- Based on 800 sociomatrices from different sources, Davis and Leinhardt found that
  - Many relations are directional
  - Difficult to implement methods that focus on semicycles
  - Asymmetric (non-mutual) relations are common
  - Signed relations are rare
  - Some digraphs appear ranked or hierarchical, with one set of nodes pointing to another set, and that set pointing to a third, and so on

# Ranked Clusterability

- For complete signed digraphs
- Dyadic relations characterize node relationships
  - [++] connects nodes in the same cluster
  - [--] connects nodes in different clusters
  - [+-] occurs between nodes in different levels of the hierarchy, where the target of the positive arc is in the higher level

# Issues with Ranked Clusterability

- In practice, [++] and [--] relationships tend toward clusterability and [+-] relationships tend toward a hierarchical structure, but the two structures don't integrate well
- Disallowed triples occur too commonly
- Data to construct signed digraphs are not commonly collected

# Transitivity

- A key structural property in social network data
- For unsigned digraphs
- Directed edges are either present or null
- A triad of nodes i, j, and k is transitive if whenever i -> j and j -> k, then i -> k.
- A triad is vacuously transitive if either condition is not met
- A digraph is transitive if every triad it contains is transitive