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A periodic boundary value problem with vanishing Green's function

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Abstract

In this work, the authors consider the boundary value problem

$$\begin{cases} y'' + a(t)y = g(t)f(y), & 0 \le t \le 2\pi, \\ y(0) = y(2\pi), & y'(0) = y'(2\pi), \end{cases}$$

and establish the existence of nonnegative solutions in the case where the associated Green's function may have zeros. The results are illustrated with an example.
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1. Introduction

Recently, Krasnosel'skii's theorem of cone expansion/compression type has been used to study the existence of positive solutions of periodic boundary value problems in several papers; see, for example, Atici and Guseinov [1], Jiang et al. [4,5], O'Regan and Wang [7], Torres [8], Zhang and Wang [10], and the references contained therein. In these papers, the major assumption is that their associated Green's functions are of one sign. In Section 2 of this work, we generalize the related results to the case where the associated Green's functions have zeros. More specifically, we study the existence of nonnegative solutions to the periodic boundary value problem

$$\begin{cases} y'' + a(t)y = g(t)f(y), & 0 \le t \le 2\pi, \\ y(0) = y(2\pi), & y'(0) = y'(2\pi), \end{cases} \tag{1.1}$$

without the assumption that the associated Green's function is strictly positive, i.e., it only needs to be nonnegative. One of the key features in our proof is that a new cone is defined in which to apply Krasnosel'skii's fixed point theorem. While we do not assume that the Green's function $G(t,s)$ for (1.1) is positive for all t and s, we do ask that

$$\beta = \min_{0\le s\le 2\pi} \int_0^{2\pi} G(t,s)\mathrm{d}t > 0. \tag{1.2}$$

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For example, consider the problem

$$\begin{cases} y'' + m^2 y = g(t) f(y), & 0 \le t \le 2\pi, \\ y(0) = y(2\pi), & y'(0) = y'(2\pi), \end{cases} \tag{1.3}$$

where $m > 0$ is a constant. It is well known that if $m \ne 1, 2, \ldots$, then the Green's function for (1.3) is given by

$$G(t,s) = \begin{cases} \dfrac{\sin m(t-s) + \sin m(2\pi - t + s)}{2m(1 - \cos 2m\pi)}, & 0 \le s \le t \le 2\pi, \\ \dfrac{\sin m(s-t) + \sin m(2\pi - s + t)}{2m(1 - \cos 2m\pi)}, & 0 \le t \le s \le 2\pi. \end{cases}$$

Let

$$\hat{G}(x) = \frac{\sin(mx) + \sin m(2\pi - x)}{2m(1 - \cos 2m\pi)} \quad \text{for } x \in [0, 2\pi].$$

Then, it is easy to check that $\hat{G}$ is increasing on $[0, \pi]$, decreasing on $[\pi, 2\pi]$, and $G(t,s) = \hat{G}(|t-s|)$. Thus,

$$\frac{\sin 2m\pi}{2m(1 - \cos 2m\pi)} = \hat{G}(0) \le G(t,s) \le \hat{G}(\pi) = \frac{\sin m\pi}{m(1 - \cos 2m\pi)}$$

for $s, t \in [0, 2\pi]$. Moreover, $G(t,s)$ is positive on $[0, 2\pi] \times [0, 2\pi]$ for $0 < m < 1/2$. When the Green's function is positive, we can always find its positive minimum A and maximum B. Define a cone as follows:

$$\left\{ u \in C[0, 2\pi] : \min_{0 \le t \le 2\pi} u(t) \ge \frac{A}{B} \|u\| \right\}.$$

Then, Krasnosel'skii's fixed point theorem can be used to prove the existence and multiplicity of positive solutions (see [1,4,5,7,8,10]). However, if $m = 1/2$, then the Green's function is zero at $t = s$. The minimum value of the Green's function is zero and the above cone cannot be used to apply Krasnosel'skii's theorem. However, (1.2) holds.

2. Existence results when Green's functions have zeros

The assumptions to be used in this work are as follows:

(H1) $f : [0, \infty) \to [0, \infty)$ is continuous;
(H2) $g : [0, 2\pi] \to [0, \infty)$ is continuous and $\eta = \min_{t \in [0, 2\pi]} g(t) > 0$;
(H3) $f : [0, \infty) \to [0, \infty)$ is convex and nondecreasing.

For convenience, we introduce the notation

$$f_0 = \lim_{u \to 0} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

We now state our main results in this work. Analogous results for the Dirichlet/Neumann boundary value problems were established in [2].

Theorem 2.1. *Assume that* (H1) *and* (H2) *hold.*

(a) *If* $f_0 = \infty$ *and* $f_\infty = 0$, *then* (1.1) *has a nontrivial solution* $u(t) \ge 0$.
(b) *If* $f_0 = 0$, $f_\infty = \infty$, *and* (H3) *holds, then* (1.1) *has a nontrivial solution* $u(t) \ge 0$.

To prove Theorem 2.1, we define a new function

$$f^*(u) = \max_{0 \le t \le u} \{f(t)\}$$

and let $f_0^* = \lim_{u \to 0} f^*(u)/u$ and $f_\infty^* = \lim_{u \to \infty} f^*(u)/u$. The following two lemmas are needed in the proof of Theorem 2.1.

Lemma 2.1 ([9]). *Assume* (H1) *holds. Then* $f_0^* = f_0$ *and* $f_\infty^* = f_\infty$.

Lemma 2.2 ([3,6]). *Let X be a Banach space and let $K \subset X$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$F : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) *$Fu \not\leq u$ for any $u \in K \cap \partial\Omega_1$ and $Fu \not\geq u$ for any $u \in K \cap \partial\Omega_2$,*
or
(ii) *$Fu \not\geq u$ for any $u \in K \cap \partial\Omega_1$ and $Fu \not\leq u$ for any $u \in K \cap \partial\Omega_2$.*

Then F has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let X be the Banach space $C[0, 2\pi]$ endowed with the norm

$$\|u\| = \max_{0 \leq t \leq 2\pi} |u(t)|.$$

Define the cone E in X by

$$E = \left\{ u \in X : u(t) \geq 0 \text{ on } [0, 2\pi] \text{ and } \int_0^{2\pi} u(t)\mathrm{d}t \geq \frac{\beta}{M} \max_{t \in [0,2\pi]} u(t) \right\},$$

where β is defined by (1.2) and $M = \max_{t,s \in [0,2\pi]} |G(t, s)|$. For any $r > 0$, let

$$\Omega_r = \{u \in E : \|u\| < r\}.$$

Define the map $T : E \to X$ by

$$Tu(t) = \int_0^{2\pi} G(t, s)g(s) f(u(s))\mathrm{d}s, \quad 0 \leq t \leq 2\pi.$$

We claim that $T : E \to E$. In fact, note that

$$\begin{aligned} \int_0^{2\pi} Tu(t)\mathrm{d}t &= \int_0^{2\pi} \int_0^{2\pi} G(t, s)g(s) f(u(s))\mathrm{d}s\mathrm{d}t \\ &= \int_0^{2\pi} g(s) f(u(s)) \int_0^{2\pi} G(t, s)\mathrm{d}t\mathrm{d}s. \end{aligned}$$

Then, from (1.2), we see that

$$\int_0^{2\pi} Tu(t)\mathrm{d}t \geq \beta \int_0^{2\pi} g(s) f(u(s))\mathrm{d}s.$$

On the other hand,

$$Tu(t) = \int_0^{2\pi} G(t, s)g(s) f(u(s))\mathrm{d}s \leq M \int_0^{2\pi} g(s) f(u(s))\mathrm{d}s$$

for $t \in [0, 1]$. Thus,

$$\int_0^{2\pi} Tu(t)\mathrm{d}t \geq \frac{\beta}{M} \max_{t \in [0,2\pi]} Tu(t),$$

i.e., $TE \to E$.

Proof of Theorem 2.1. Part (a). Since $f_0 = \infty$, we can choose $r_1 > 0$ sufficiently small that

$$f(u) \geq \theta u \quad \text{for } u \leq r_1,$$

where θ satisfies $\beta^2\eta\theta/(2\pi M) > 1$ with η defined in (H2). We now show that

$$Tu \not\leq u \quad \text{for } u \in \partial\Omega_{r_1}.$$

In fact, if there exists $u_1 \in \partial\Omega_{r_1}$ such that $Tu_1 \le u_1$, then, from (1.2) and the definition of η, we have

$$\begin{aligned}
\|u_1\| \ge \|Tu_1\| &\ge \frac{1}{2\pi}\int_0^{2\pi} Tu_1(t)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(s)f(u_1(s))\int_0^{2\pi} G(t,s)\mathrm{d}t\mathrm{d}s \\
&\ge \frac{1}{2\pi}\beta\eta\int_0^{2\pi} f(u_1(s))\mathrm{d}s \ge \frac{1}{2\pi}\beta\eta\theta\int_0^{2\pi} u_1(s)\mathrm{d}s \\
&\ge \frac{\beta^2\eta\theta}{2\pi M}\|u_1\| > \|u_1\|,
\end{aligned}$$

which is a contradiction.

Since $f_\infty = 0$, Lemma 2.1 implies $\lim_{u\to\infty} f^*(u)/u = 0$. Thus, there exists $r_2 \in (r_1, \infty)$ such that

$$f^*(r_2) < \frac{1}{2\pi M\|g\|} r_2.$$

We next show that

$$Tu \not\ge u \quad \text{for } u \in \partial\Omega_{r_2}.$$

Now if there exists $u_2 \in \partial\Omega_{r_2}$ such that $Tu_2 \ge u_2$, then

$$r_2 = \|u_2\| \le \|Tu_2\| \le 2\pi M\|g\| f^*(r_2) < r_2,$$

which is a contradiction. Hence, from the first part of Lemma 2.2, T has a fixed point $u \in (\overline{\Omega}_2 \setminus \Omega_1)$. Clearly, $u(t) \ge 0$ is a nontrivial solution of (1.1).

Part (b). Since $f_\infty = \infty$, we can choose $r_2 > 0$ sufficiently large that

$$f\left(\frac{\beta}{M} r_2\right) \ge \frac{\beta}{M}\theta r_2,$$

where θ satisfies that $\beta^2\eta\theta/(2\pi M) > 1$ with η defined in (H2).

We will now show that

$$Tu \not\le u \quad \text{for } u \in \partial\Omega_{r_2}.$$

If there exists $u_2 \in \partial\Omega_{r_2}$ such that $Tu_2 \le u_2$, then, from (1.2) and the definition of η, it is clear that

$$\begin{aligned}
2\pi\|Tu_2\| &\ge \int_0^{2\pi} Tu_2(t)\mathrm{d}t \\
&= \int_0^{2\pi} g(s)f(u_2(s))\int_0^{2\pi} G(t,s)\mathrm{d}t\mathrm{d}s \\
&\ge \beta\eta\int_0^{2\pi} f(u_2(s))\mathrm{d}s.
\end{aligned}$$

Hence, in view of (H3) and Jensen's Inequality, we have

$$\begin{aligned}
2\pi\|Tu_2\| &\ge \beta\eta f\left(\int_0^{2\pi} u_2(s)\mathrm{d}s\right) \\
&\ge \beta\eta f\left(\frac{\beta}{M}\|u_2\|\right).
\end{aligned}$$

Thus,

$$r_2 = \|u_2\| \ge \|Tu_2\| \ge \frac{\beta\eta}{2\pi} f\left(\frac{\beta}{M} r_2\right) \ge \frac{\beta^2\eta\theta}{2\pi M} r_2 > r_2,$$

which is a contradiction.

Since $f_0 = 0$, by Lemma 2.1, $\lim_{u\to 0} f^*(u)/u = 0$. Thus, there exists $r_1 \in (0, r_2)$ such that

$$f^*(r_1) < \frac{1}{2\pi M\|g\|} r_1.$$

To show that

$$Tu \not\geq u \quad \text{for } u \in \partial\Omega_{r_1},$$

suppose there exists $u_1 \in \partial\Omega_{r_1}$ such that $Tu_1 \geq u_1$. Then,

$$r_1 = \|u_1\| \leq \|Tu_1\| \leq 2\pi M\|g\| f^*(r_1) < r_1,$$

which is a contradiction. Hence, from the second part of Lemma 2.2, T has a fixed point $u \in (\overline{\Omega}_2 \setminus \Omega_1)$. Clearly, $u(t) \geq 0$ is a nontrivial solution of (1.1). This completes the proof of the theorem. □

We conclude this work with the following example.

Example 2.1. Consider the boundary value problem (1.3) where $0 < m \leq 1/2$, $g(t)$ is any positive continuous function on $[0, 2\pi]$, and $f(u) = u^\alpha$ with $\alpha \in (0, 1) \cup (1, \infty)$. We claim that (1.3) has a nontrivial solution $u(t) \geq 0$.

In this case,

$$\beta = \frac{2\sin^2 m\pi}{m^2(1 - \cos 2m\pi)},$$

and so (1.2) holds. With the above functions g and f, we see that (H1) and (H2) hold, and, in addition, (H3) holds if $\alpha \in (1, \infty)$. Moreover, it is easy to see that

$$f_0 = \infty \quad \text{and} \quad f_\infty = 0 \quad \text{if } \alpha \in (0, 1)$$

and

$$f_0 = 0 \quad \text{and} \quad f_\infty = \infty \quad \text{if } \alpha \in (1, \infty).$$

Then the conclusion follows from Theorem 2.1(a) if $\alpha \in (0, 1)$ and Theorem 2.1(b) if $\alpha \in (1, \infty)$.

Remark 2.1. As we noted earlier, if $m = 1/2$, the Green's function $G(t, s)$ for (1.3) is zero at $t = s$. Now the papers [1,7,8,10] all consider the same type of boundary conditions as the ones in this work, but none of those results apply since they all require that the Green's function be strictly positive.

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References

[1] F.M. Atici, G.Sh. Guseinov, On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions, J. Comput. Appl. Math. 132 (2001) 341–356.
[2] L.H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (1994) 743–748.
[3] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Orlando, FL, 1988.
[4] D. Jiang, On the existence of positive solutions to second order periodic BVPs, Acta Math. Sci. 18 (1998) 31–35.
[5] D. Jiang, J. Chu, D. O'Regan, R. Agarwal, Multiple positive solutions to superlinear periodic boundary value problems with repulsive singular forces, J. Math. Anal. Appl. 286 (2003) 563–576.
[6] M. Krasnosel'skii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.
[7] D. O'Regan, H. Wang, Positive periodic solutions of systems of second order ordinary differential equations, Positivity 10 (2006) 285–298.
[8] P. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnosel'skii fixed point theorem, J. Differential Equations 190 (2003) 643–662.
[9] H. Wang, On the number of positive solutions of nonlinear systems, J. Math. Anal. Appl. 281 (2003) 287–306.
[10] Z. Zhang, J. Wang, On existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order differential equations, J. Math. Anal. Appl. 281 (2003) 99–107.