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On the number of positive solutions of nonlinear systems

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Abstract

We prove that appropriate combinations of superlinearity and sublinearity of $\mathbf{f}(\mathbf{u})$ with respect to $\boldsymbol{\Phi}$ at zero and infinity guarantee the existence, multiplicity, and nonexistence of positive solutions to boundary value problems for the n-dimensional system $(\boldsymbol{\Phi}(\mathbf{u}'))' + \lambda\mathbf{h}(t)\mathbf{f}(\mathbf{u}) = 0$, $0 < t < 1$. The vector-valued function $\boldsymbol{\Phi}$ is defined by $\boldsymbol{\Phi}(\mathbf{u}') = (\varphi(u_1'), \dots, \varphi(u_n'))$, where $\mathbf{u} = (u_1, \dots, u_n)$ and φ covers the two important cases $\varphi(u') = u'$ and $\varphi(u') = |u'|^{p-2}u'$, $p > 1$. Our methods employ fixed point theorems in a cone.

1. Introduction

In this paper we consider the existence, multiplicity, and nonexistence of positive solutions for the system

$$\big(\boldsymbol{\Phi}(\mathbf{u}')\big)' + \lambda\mathbf{h}(t)\mathbf{f}(\mathbf{u}) = 0, \quad 0 < t < 1, \tag{1.1}$$

with one of the following three sets of the boundary conditions:

$$\mathbf{u}(0) = \mathbf{u}(1) = 0, \tag{1.2a}$$

$$\mathbf{u}'(0) = \mathbf{u}(1) = 0, \tag{1.2b}$$

$$\mathbf{u}(0) = \mathbf{u}'(1) = 0, \tag{1.2c}$$

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where $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{\Phi}(\mathbf{u}') = (\varphi(u_1'), \ldots, \varphi(u_n'))$, $\mathbf{h}(t) = \mathrm{diag}[h_1(t), \ldots, h_n(t)]$, and $\mathbf{f}(\mathbf{u}) = (f^1(u_1, \ldots, u_n), \ldots, f^n(u_1, \ldots, u_n))$. We understand that $\mathbf{u}$, $\mathbf{\Phi}$, $\mathbf{f}(\mathbf{u})$ are (column) n-dimensional vector-valued functions. (1.1) means that

$$\begin{cases} (\varphi(u_1'))' + \lambda h_1(t) f^1(u_1, \ldots, u_n) = 0, & 0 < t < 1, \\ \quad \ldots \\ (\varphi(u_n'))' + \lambda h_n(t) f^n(u_1, \ldots, u_n) = 0, & 0 < t < 1. \end{cases} \tag{1.3}$$

By a solution $\mathbf{u}$ to (1.1)–(1.2) we understand a vector-valued function $\mathbf{u} \in C^1([0,1], \mathbb{R}^n)$ with $\mathbf{\Phi}(\mathbf{u}') \in C^1((0,1), \mathbb{R}^n)$, which satisfies (1.1) for $t \in (0,1)$ and one of (1.2). A solution $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))$ is positive if, for each $i = 1, \ldots, n$, $u_i(t) \geqslant 0$ for all $t \in (0,1)$ and there is at least one nontrivial component of $\mathbf{u}$. In fact, we shall show that such a nontrivial component of $\mathbf{u}$ is positive on $(0,1)$.

When $n = 1$, (1.1) reduces to the scalar equation

$$\big(\varphi(u')\big)' + \lambda h(t) f(u) = 0, \quad 0 < t < 1. \tag{1.4}$$

The investigation of the existence of positive solutions of boundary value problems for (1.4) originates from a variety of different areas of applied mathematics and physics and has received growing attention in connection with positive radial solutions of partial differential equations in annular regions. For the classical case where $\varphi(u') = u'$, several results are available in the literature. Bandle et al. [2] and Lin [11] established the existence of positive solutions of (1.4) with (1.2) ($n = 1$) under the assumption that f is superlinear, i.e., $f_0 = \lim_{u\to 0}(f(u)/u) = 0$ and $f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty$.

On the other hand, we [13] obtained the existence of positive solutions of (1.4) with (1.2) ($n = 1$, $\varphi(u') = u'$) under the assumption that f is sublinear, i.e., $f_0 = \infty$ and $f_\infty = 0$. For the case $\varphi(u') = |u'|^{p-2}u'$, $p > 1$, i.e., the one-dimensional p-Laplacian, we refer to Ben-Naoum and De Coster [3], Manasevich and Mawhin [12], Wang [16], and references therein for some additional details. Related results for scalar equations may also be found in [1,5,8]. For the case $\varphi(u') = u'$ and $n = 2$, Dunninger and Wang [6,7] obtained existence and multiplicity results.

In recent papers [14,15], we introduced a new and general assumption (see A1) on the function $\varphi(u')$, which covers the two important cases $\varphi(u') = u'$ and $\varphi(u') = |u'|^{p-2}u'$, $p > 1$. Under such an assumption, we were able to show that appropriate combinations of superlinearity and sublinearity of $f(u)$ with respect to $\boldsymbol{\varphi}$ at zero and infinity guarantee the existence, multiplicity, and nonexistence of positive solutions of (1.4). Specifically, we proved that results similar to Theorems 1.1 and 1.2 hold for (1.4) with (1.2) ($n = 1$).

The main purpose of this paper is to extend the above results to the n-dimensional system (1.1). For this purpose, we introduce some new notation in (1.5), $\mathbf{f}_0$ and $\mathbf{f}_\infty$, to characterize superlinearity and sublinearity with respect to φ for (1.1). They are natural extensions of f_0 and f_∞ defined above for the scalar equation (1.4). Based on the new notation, we obtain criteria of determining the number of positive solutions of (1.1)–(1.2). Our main results (Theorems 1.1 and 1.2) clearly exhibit the structure of the set of positive solutions of (1.1)–(1.2). These results are new even for the cases $\varphi(u') = u'$ and $\varphi(u') = |u'|^{p-2}u'$, $p > 1$. Our arguments are closely related to those of [13]. In [13] we used a fixed point theorem in a cone due to Krasnoselskii, which is essentially the same as Lemma 2.1.

Let

$$\mathbb{R} = (-\infty, \infty), \qquad \mathbb{R}_+ = [0, \infty), \quad \text{and} \quad \mathbb{R}_+^n = \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n}.$$

Also, for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$, let $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$. We make the following assumptions:

(A1) φ is an odd, increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ and there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0, \infty)$ onto $(0, \infty)$ such that

$$\psi_1(\sigma)\varphi(x) \leqslant \varphi(\sigma x) \leqslant \psi_2(\sigma)\varphi(x) \quad \text{for all } \sigma \text{ and } x > 0.$$

(A2) $f^i : \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous, $i = 1, \dots, n$.

(A3) $h_i(t) : [0, 1] \to \mathbb{R}_+$ is continuous and $h_i(t) \not\equiv 0$ on any subinterval of $[0, 1]$, $i = 1, \dots, n$.

(A4) $f^i(u_1, \dots, u_n) > 0$ for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| > 0$, $i = 1, \dots, n$.

In order to state our results we introduce the new notation

$$f_0^i = \lim_{\|\mathbf{u}\| \to 0} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}, \quad f_\infty^i = \lim_{\|\mathbf{u}\| \to \infty} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}, \quad \mathbf{u} \in \mathbb{R}_+^n,\ i = 1, \dots, n,$$
$$\mathbf{f}_0 = \sum_{i=1}^n f_0^i, \qquad \mathbf{f}_\infty = \sum_{i=1}^n f_\infty^i. \tag{1.5}$$

Our main results are:

Theorem 1.1. *Assume* (A1)–(A3) *hold.*

(a) *If* $\mathbf{f}_0 = 0$ *and* $\mathbf{f}_\infty = \infty$*, then for all* $\lambda > 0$ (1.1)–(1.2) *has a positive solution.*

(b) *If* $\mathbf{f}_0 = \infty$ *and* $\mathbf{f}_\infty = 0$*, then for all* $\lambda > 0$ (1.1)–(1.2) *has a positive solution.*

Theorem 1.2. *Assume* (A1)–(A4) *hold.*

(a) *If* $\mathbf{f}_0 = 0$ *or* $\mathbf{f}_\infty = 0$*, then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (1.1)–(1.2) *has a positive solution.*

(b) *If* $\mathbf{f}_0 = \infty$ *or* $\mathbf{f}_\infty = \infty$*, then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (1.1)–(1.2) *has a positive solution.*

(c) *If* $\mathbf{f}_0 = \mathbf{f}_\infty = 0$*, then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (1.1)–(1.2) *has two positive solutions.*

(d) *If* $\mathbf{f}_0 = \mathbf{f}_\infty = \infty$*, then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (1.1)–(1.2) *has two positive solutions.*

(e) *If* $\mathbf{f}_0 < \infty$ *and* $\mathbf{f}_\infty < \infty$*, then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ (1.1)–(1.2) *has no positive solution.*

(f) *If* $\mathbf{f}_0 > 0$ *and* $\mathbf{f}_\infty > 0$*, then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ (1.1)–(1.2) *has no positive solution.*

2. Preliminaries

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.1 [4,9,10]. *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K\colon \ \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K\colon \ \|x\| = r\}$.*

(i) *If $\|Tx\| \geqslant \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tx\| \leqslant \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (1.1)–(1.2), let X be the Banach space

$$\underbrace{C[0,1] \times \cdots \times C[0,1]}_{n}$$

and, for $\mathbf{u} = (u_1, \ldots, u_n) \in X$,

$$\|\mathbf{u}\| = \sum_{i=1}^{n} \sup_{t \in [0,1]} \bigl|u_i(t)\bigr|.$$

For $\mathbf{u} \in X$ or $\mathbb{R}_+^n$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbb{R}_+^n$, respectively.

Define K to be a cone in X by

$$K = \left\{ \mathbf{u} = (u_1, \ldots, u_n) \in X\colon \ u_i(t) \geqslant 0, \ t \in [0,1], \ i = 1, \ldots, n, \ \text{and} \right.$$
$$\left. \min_{1/4 \leqslant t \leqslant 3/4} \sum_{i=1}^{n} u_i(t) \geqslant \frac{1}{4}\|\mathbf{u}\| \right\}.$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \bigl\{\mathbf{u} \in K\colon \ \|\mathbf{u}\| < r\bigr\}.$$

Note that $\partial\Omega_r = \{\mathbf{u} \in K\colon \ \|\mathbf{u}\| = r\}$.

Let $\mathbf{T}_\lambda : K \to X$ be a map with components $(T_\lambda^1, \ldots, T_\lambda^n)$. We define T_λ^i, $i = 1, \ldots, n$, by

$$T_\lambda^i \mathbf{u}(t) = \begin{cases} \int_0^t \varphi^{-1}\bigl(\int_s^{\sigma_i} \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\, d\tau\bigr)\, ds, & 0 \leqslant t \leqslant \sigma_i, \\ \int_t^1 \varphi^{-1}\bigl(\int_{\sigma_i}^s \lambda h_i(\tau) f^i(\mathbf{u}(\tau))\, d\tau\bigr)\, ds, & \sigma_i \leqslant t \leqslant 1, \end{cases} \tag{2.1}$$

where $\sigma_i = 0$ for (1.1), (1.2b) and $\sigma_i = 1$ for (1.1), (1.2c). For (1.1), (1.2a), $\sigma_i \in (0,1)$ is a solution of the equation

$$\Theta^i \mathbf{u}(t) = 0, \quad 0 \leqslant t \leqslant 1, \tag{2.2}$$

where the map $\Theta^i : K \to C[0,1]$ is defined by

$$\Theta^i \mathbf{u}(t) = \int_0^t \varphi^{-1}\left(\int_s^t \lambda h_i(\tau) f^i\big(\mathbf{u}(\tau)\big)\,d\tau\right) ds - \int_t^1 \varphi^{-1}\left(\int_t^s \lambda h_i(\tau) f^i\big(\mathbf{u}(\tau)\big)\,d\tau\right) ds, \quad 0 \leqslant t \leqslant 1. \tag{2.3}$$

By virtue of Lemma 2.2, the operator $\mathbf{T}_\lambda$ is well defined.

Lemma 2.2. *Assume* (A1)–(A3) *hold. Then, for any* $\mathbf{u} \in K$ *and* $i = 1, \ldots, n$, $\Theta^i \mathbf{u}(t) = 0$ *has at least one solution in* $(0,1)$. *In addition, if* $\sigma_i^1 < \sigma_i^2 \in (0,1)$, $i = 1, \ldots, n$, *are two solutions of* $\Theta^i \mathbf{u}(t) = 0$, *then* $h_i(t) f^i(\mathbf{u}(t)) \equiv 0$ *for* $t \in [\sigma_i^1, \sigma_i^2]$ *and any* $\sigma_i \in [\sigma_i^1, \sigma_i^2]$ *is also a solution of* $\Theta^i \mathbf{u}(t) = 0$. *Furthermore,* $\mathbf{T}_\lambda^i \mathbf{u}(t)$, $i = 1, \ldots, n$, *is independent of the choice of* $\sigma_i \in [\sigma_i^1, \sigma_i^2]$.

Proof. Let $\alpha^i(\tau) = \lambda h_i(\tau) f^i(\mathbf{u}(\tau))$. If $\alpha^i \equiv 0$ on $[0,1]$, we may choose any $\sigma_i \in (0,1)$. Let us assume that there is $\tau \in (0,1)$ such that $\alpha^i(\tau) > 0$. Therefore, $\Theta^i \mathbf{u}(0) < 0$ and $\Theta^i \mathbf{u}(1) > 0$. It follows from the continuity of $\Theta^i \mathbf{u}(t)$ that $\Theta^i \mathbf{u}(t) = 0$ has at least one solution in $(0,1)$. Moreover, it is not difficult to check that while $\int_0^t \varphi^{-1}(\int_s^t \alpha^i(\tau)\,d\tau)\,ds$ is nondecreasing, $\int_t^1 \varphi^{-1}(\int_t^s \alpha^i(\tau)\,d\tau)\,ds$ is nonincreasing. Therefore, $\Theta^i \mathbf{u}(t)$ is nondecreasing function on $[0,1]$.

If $\sigma_i^1 < \sigma_i^2 \in (0,1)$ are two solutions of $\Theta^i \mathbf{u}(t) = 0$, we consider

$$\begin{aligned}\int_{\sigma_i^1}^{\sigma_i^2} \varphi^{-1}\left(\int_s^{\sigma_i^2} \alpha^i(\tau)\,d\tau\right) ds &= \int_0^{\sigma_i^2} \varphi^{-1}\left(\int_s^{\sigma_i^2} \alpha^i(\tau)\,d\tau\right) ds - \int_0^{\sigma_i^1} \varphi^{-1}\left(\int_s^{\sigma_i^2} \alpha^i(\tau)\,d\tau\right) ds \\ &\leqslant \int_0^{\sigma_i^2} \varphi^{-1}\left(\int_s^{\sigma_i^2} \alpha^i(\tau)\,d\tau\right) ds - \int_0^{\sigma_i^1} \varphi^{-1}\left(\int_s^{\sigma_i^1} \alpha^i(\tau)\,d\tau\right) ds.\end{aligned}$$

Now, because of $\Theta^i \mathbf{u}(\sigma_i^1) = \Theta^i \mathbf{u}(\sigma_i^2) = 0$, we have

$$\begin{aligned}\int_{\sigma_i^1}^{\sigma_i^2} \varphi^{-1}\left(\int_s^{\sigma_i^2} \alpha^i(\tau)\,d\tau\right) ds &\leqslant \int_{\sigma_i^2}^{1} \varphi^{-1}\left(\int_{\sigma_i^2}^{s} \alpha^i(\tau)\,d\tau\right) ds - \int_{\sigma_i^1}^{1} \varphi^{-1}\left(\int_{\sigma_i^1}^{s} \alpha^i(\tau)\,d\tau\right) ds \\ &\leqslant -\int_{\sigma_i^1}^{\sigma_i^2} \varphi^{-1}\left(\int_{\sigma_i^1}^{s} \alpha^i(\tau)\,d\tau\right) ds \leqslant 0,\end{aligned}$$

which implies that $\alpha^i(\tau) \equiv 0$ on $[\sigma_i^1, \sigma_i^2]$. Let $\sigma_i \in [\sigma_i^1, \sigma_i^2]$ and observe that

$$\int_0^{\sigma_i} \varphi^{-1}\left(\int_s^{\sigma_i} \alpha^i(\tau)\,d\tau\right)ds = \int_0^{\sigma_i^1} \varphi^{-1}\left(\int_s^{\sigma_i^1} \alpha^i(\tau)\,d\tau\right)ds$$
$$= \int_{\sigma_i^1}^1 \varphi^{-1}\left(\int_{\sigma_i^1}^s \alpha^i(\tau)\,d\tau\right)ds = \int_{\sigma_i}^1 \varphi^{-1}\left(\int_{\sigma_i}^s \alpha^i(\tau)\,d\tau\right)ds.$$

This yields that σ_i is a solution of $\Theta^i\mathbf{u}(t) = 0$. Hence, (2.1) implies

$$T_\lambda^i\mathbf{u}(t) = \begin{cases} \int_0^t \varphi^{-1}(\int_s^{\sigma_i^1} \alpha^i(\tau)\,d\tau)\,ds, & 0 \leqslant t \leqslant \sigma_i^1, \\ \int_0^{\sigma_i^1} \varphi^{-1}(\int_s^{\sigma_i^1} \alpha^i(\tau)\,d\tau)\,ds, & \sigma_i^1 \leqslant t \leqslant \sigma_i, \\ \int_{\sigma_i^2}^1 \varphi^{-1}(\int_{\sigma_i^2}^s \alpha^i(\tau)\,d\tau)\,ds, & \sigma_i \leqslant t \leqslant \sigma_i^2, \\ \int_t^1 \varphi^{-1}(\int_{\sigma_i^2}^s \alpha^i(\tau)\,d\tau)\,ds, & \sigma_i^2 \leqslant t \leqslant 1, \end{cases} \tag{2.4}$$

which is independent of σ_i. □

Lemma 2.3. *Assume* (A1) *holds. Let u and $v \in C[0,1]$ with $u \geqslant 0$ and $v \leqslant 0$ satisfying $(\varphi(u'))' = v$. Then*

$$u(t) \geqslant \min\{t, 1-t\} \sup_{t\in[0,1]} u(t) \quad \textit{for } t \in [0,1].$$

In particular, $\min_{1/4\leqslant t\leqslant 3/4} u(t) \geqslant (1/4)\sup_{t\in[0,1]} u(t)$.

Proof. Since $\varphi(u')$ is nonincreasing and φ^{-1} is increasing, it follows that u' is nonincreasing. Hence, for $0 \leqslant t_0 < t < t_1 \leqslant 1$,

$$u(t) - u(t_0) = \int_{t_0}^t u'(s)\,ds \geqslant (t-t_0)u'(t)$$

and

$$u(t_1) - u(t) = \int_t^{t_1} u'(s)\,ds \leqslant (t_1-t)u'(t),$$

from which we have

$$u(t) \geqslant \frac{(t-t_0)u(t_1) + (t_1-t)u(t_0)}{t_1-t_0}.$$

Considering the above inequality on $[0,\sigma]$ and $[\sigma,1]$, we obtain

$$u(t) \geqslant t \sup_{t\in[0,1]} u(t) \quad \text{for } t \in [0,\sigma],$$
$$u(t) \geqslant (1-t) \sup_{t\in[0,1]} u(t) \quad \text{for } t \in [\sigma,1],$$

where $\sigma \in [0,1]$ is such that $u(\sigma) = \sup_{t\in[0,1]} u(t)$. Hence, we have

$$u(t) \geqslant \min\{t, 1-t\} \sup_{t\in[0,1]} u(t) \quad \text{for } t \in [0,1]. \qquad \Box$$

We remark that, according to Lemma 2.3, any nontrivial component of non-negative solutions of (1.1)–(1.2) is positive on $(0,1)$.

Lemma 2.4. *Assume* (A1)–(A3) *hold. Then, for* $i = 1, \dots, n$, $\Theta^i : K \to C[0,1]$ *is compact and continuous.*

Proof. Let $R > 0$ and define

$$M_R^i = 1 + \lambda\Big[\sup_{t\in[0,1]} h_i(t)\Big]\Big[\sup\big\{f^i(\mathbf{u})\colon \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \leqslant R\big\}\Big] > 0$$

and $C_R^i = \sup_{s\in[0,M_R^i]} \varphi^{-1}(s) > 0$. We now show that $\Theta^i : K \to C[0,1]$ is compact. Let $(\mathbf{u}_m)_{m\in\mathbb{N}}$ be a bounded sequence in K and let $R > 0$ be such that $\|\mathbf{u}_m\| \leqslant R$ for all $m \in \mathbb{N}$. Set $v_m^i = \Theta^i \mathbf{u}_m$. Thus $|v_m^i(t)| \leqslant 2C^i$, $t \in [0,1]$. In other words, $(v_m^i)_{m\in\mathbb{N}}$ is uniformly bounded in $C[0,1]$. We next show the equicontinuity of $(v_m^i)_{m\in\mathbb{N}}$. Again, let $\alpha_m^i(\tau) = \lambda h_i(\tau) f^i(\mathbf{u}_m(\tau))$. For any $\varepsilon > 0$, from the continuity of φ^{-1} on $[0, M_R^i]$, it follows that there exists a $\delta_1 > 0$ such that $|\varphi^{-1}(t_1) - \varphi^{-1}(t_2)| < \varepsilon/4$ for every $t_1, t_2 \in [0, M_R^i]$ and $|t_1 - t_2| < \delta_1$. Thus, if $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta = \min\{\varepsilon/(4C_R^i), \delta_1/M_R^i\}$, we have (without loss of generality assume that $t_1 < t_2$)

$$\begin{aligned}
\big|v_m^i(t_2) - v_m^i(t_1)\big| &\leqslant \left|\int_{t_1}^{t_2} \varphi^{-1}\left(\int_s^{t_2} \alpha_m^i(\tau)\,d\tau\right) ds\right| \\
&\quad + \left|\int_0^{t_1} \left[\varphi^{-1}\left(\int_s^{t_2} \alpha_m^i(\tau)\,d\tau\right) ds - \varphi^{-1}\left(\int_s^{t_1} \alpha_m^i(\tau)\,d\tau\right)\right] ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} \varphi^{-1}\left(\int_{t_1}^{s} \alpha_m^i(\tau)\,d\tau\right) ds\right| \\
&\quad + \left|\int_{t_2}^{1} \left[\varphi^{-1}\left(\int_{t_1}^{s} \alpha_m^i(\tau)\,d\tau\right) ds - \varphi^{-1}\left(\int_{t_2}^{s} \alpha_m^i(\tau)\,d\tau\right)\right] ds\right| \\
&< 2C_R^i|t_2 - t_1| + \frac{\varepsilon}{2} \leqslant \varepsilon.
\end{aligned}$$

This shows that $(v_m^i)_{m\in\mathbb{N}}$ is equicontinuous on $[0,1]$. Therefore, it follows from the Arzela–Ascoli theorem that there exist a function $v \in C[0,1]$ and a subsequence of $(v_m^i)_{m\in\mathbb{N}}$ converging uniformly to v on $[0,1]$.

Finally, we prove the continuity of Θ^i. Let $(\mathbf{u}_m)_{m\in\mathbb{N}}$ be any sequence converging on K to $\mathbf{u} \in K$ and $R > 0$ be such that $\|\mathbf{u}_m\| \leqslant R$ for all $m \in \mathbb{N}$. Note that φ^{-1} is continuous on on $[0, M_R^i]$ and $f^i(\mathbf{u})$ is continuous on the closed set $\{\mathbf{u} \in \mathbb{R}_+^n\colon \|\mathbf{u}\| \leqslant R\}$. It is not hard to see that the dominated convergence theorem guarantees that

$$\lim_{m\to\infty} \Theta^i \mathbf{u}_m(t) = \Theta^i \mathbf{u}(t) \tag{2.5}$$

for each $t \in [0,1]$. Moreover, the compactness of Θ^i implies that $\Theta^i \mathbf{u}_m(t)$ converges uniformly to $\Theta^i \mathbf{u}(t)$ on $[0,1]$. Suppose this is false. Then there exist $\varepsilon_0 > 0$ and a subsequence $(\mathbf{u}_{m_j})_{j\in\mathbb{N}}$ of $(\mathbf{u}_m)_{m\in\mathbb{N}}$ such that

$$\sup_{t\in[0,1]} \left|\Theta^i \mathbf{u}_{m_j}(t) - \Theta^i \mathbf{u}(t)\right| \geqslant \varepsilon_0, \quad j \in \mathbb{N}. \tag{2.6}$$

Now, it follows from the compactness of Θ^i that there exists a subsequence of $(\mathbf{u}_{m_j})_{j\in\mathbb{N}}$ (without loss of generality assume that the subsequence is $(\mathbf{u}_{m_j})_{j\in\mathbb{N}}$) such that $(\Theta^i \mathbf{u}_{m_j})_{j\in\mathbb{N}}$ converges uniformly to $y_0 \in C[0,1]$. Thus, from (2.6), we easily see that

$$\sup_{t\in[0,1]} \left|y_0(t) - \Theta^i \mathbf{u}(t)\right| \geqslant \varepsilon_0. \tag{2.7}$$

On the other hand, from the pointwise convergence (2.5) we obtain

$$y_0(t) = \Theta^i \mathbf{u}(t), \quad t \in [0,1].$$

This is a contradiction to (2.7). Therefore Θ^i is continuous. □

Lemma 2.5. *Assume* (A1)–(A3) *hold. Then* $\mathbf{T}_\lambda(K) \subset K$ *and* $\mathbf{T}_\lambda : K \to K$ *is compact and continuous.*

Proof. Lemma 2.3 implies that $\mathbf{T}_\lambda(K) \subset K$. We now show that $\mathbf{T}_\lambda$ is compact. Let $(\mathbf{u}_m)_{m\in\mathbb{N}}$ be a bounded sequence in K and let $R > 0$ be such that $\|\mathbf{u}_m\| \leqslant R$ for all $m \in \mathbb{N}$. Hence by the definition of $\mathbf{T}_\lambda$, we have, for $i = 1, \ldots, n$,

$$\left(T_\lambda^i \mathbf{u}_m\right)'(t) = \begin{cases} \varphi^{-1}\left(\int_t^{\sigma_i} \lambda h_i(\tau) f^i(\mathbf{u}_m(\tau))\, d\tau\right), & 0 \leqslant t \leqslant \sigma_i, \\ -\varphi^{-1}\left(\int_{\sigma_i}^t \lambda h_i(\tau) f^i(\mathbf{u}_m(\tau))\, d\tau\right), & \sigma_i \leqslant t \leqslant 1, \end{cases}$$

where σ_i may be dependent on $\mathbf{u}_m$. Then it is easy to see that both $(\mathbf{T}_\lambda \mathbf{u}_m)_{m\in\mathbb{N}}$ and $((\mathbf{T}_\lambda \mathbf{u}_m)')_{m\in\mathbb{N}}$ are uniformly bounded sequences. It follows from the Arzela–Ascoli theorem that there exists a $\mathbf{v} \in K$ and a subsequence of $\mathbf{T}_\lambda \mathbf{u}_m$ converging uniformly to $\mathbf{v}$ on $[0,1]$.

It remains to show the continuity of $\mathbf{T}_\lambda$. Let us take a sequence $(\mathbf{u}_m)_{m\in\mathbb{N}}$ in K converging uniformly on $[0,1]$ to $\mathbf{u} \in K$ and fix i, $i = 1, \ldots, n$. Again, let $\alpha^i(\tau) = \lambda h_i(\tau) f^i(\mathbf{u}(\tau))$ and $\alpha_m^i(\tau) = \lambda h_i(\tau) f^i(\mathbf{u}_m(\tau))$. We know that, for all $\mathbf{u} \in K$, σ_i in (2.1) is 0 or 1 for (1.1), (1.2b) or (1.1), (1.2c), respectively. Clearly, for (1.1), (1.2b) and (1.1), (1.2c), the dominated convergence theorem and the compactness of $\mathbf{T}_\lambda$ guarantee that $\mathbf{T}_\lambda \mathbf{u}_m(t)$ converges uniformly to $\mathbf{T}_\lambda \mathbf{u}(t)$ on $[0,1]$ in a similar manner as in Lemma 2.4. We now consider (1.1), (1.2a). Let σ_i^m and σ_i^* be zeros of $\Theta^i \mathbf{u}_m(t) = 0$ and $\Theta^i \mathbf{u}(t) = 0$ on $(0,1)$, respectively. Thus, it follows from Lemma 2.4 that

$$\lim_{m\to\infty} \Theta^i \mathbf{u}\left(\sigma_i^m\right) = 0. \tag{2.8}$$

Furthermore, it is easy to see that if $(\sigma_i^{m_j})_{j\in\mathbb{N}}$ is a subsequence of $(\sigma_i^m)_{m\in\mathbb{N}}$ such that $\lim_{j\to\infty}\sigma_i^{m_j}=\sigma_i^0$, then $\Theta^i\mathbf{u}(\sigma_i^0)=0$. Now let us consider the following two cases:

(i) $\Theta^i\mathbf{u}(t)=0$ has only one zero σ_i^* on $(0,1)$;
(ii) $\Theta^i\mathbf{u}(t)=0$ has at least two zeros on $(0,1)$.

For case (i), we have $\lim_{m\to\infty}\sigma_i^m=\sigma_i^*$. Therefore, the dominated convergence theorem and the compactness of $\mathbf{T}_\lambda$ guarantee that $\mathbf{T}_\lambda\mathbf{u}_m(t)$ converges uniformly to $\mathbf{T}_\lambda\mathbf{u}(t)$ on $[0,1]$ in a similar manner as in Lemma 2.4.

We consider case (ii) in the remaining part of the proof. By Lemma 2.2, it is easy to see that there exist $\beta_i^1<\beta_i^2\in[0,1]$ such that $\Theta^i\mathbf{u}(t)\neq 0$ for $t\in[0,\beta_i^1)\cup(\beta_i^2,1]$ and $\Theta^i\mathbf{u}(t)\equiv 0$ on $[\beta_i^1,\beta_i^2]$. Then, $\alpha^i(\tau)\equiv 0$ on $[\beta_i^1,\beta_i^2]$. Thus $(\sigma_i^m)_{m\in\mathbb{N}}$ is divided into three possible subsequences

$$\left(\sigma_i^{m_j^1}\right)_{j\in\mathbb{N}}\subset\left[0,\beta_i^1\right),\qquad\left(\sigma_i^{m_j^2}\right)_{j\in\mathbb{N}}\subset\left[\beta_i^1,\beta_i^2\right],\quad\text{and}\quad\left(\sigma_i^{m_j^3}\right)_{j\in\mathbb{N}}\subset\left(\beta_i^2,1\right].$$

It is possible that some of the three subsequences are finite or empty. At least one of the three subsequences is infinite. In what follows, we will show that for any fixed $t\in[0,1]$,

$$\lim_{j\to\infty}T_\lambda^i\mathbf{u}_{m_j^\mu}(t)=T_\lambda^i\mathbf{u}(t)\quad\text{if }\left(\sigma_i^{m_j^\mu}\right)_{j\in\mathbb{N}}\text{ is infinite, }\mu=1,2,3.$$

Thus for any fixed $t\in[0,1]$, $\lim_{m\to\infty}T_\lambda^i\mathbf{u}_m(t)=T_\lambda^i\mathbf{u}(t)$. Again, the compactness of $\mathbf{T}_\lambda$ guarantees that $\mathbf{T}_\lambda^i\mathbf{u}_m(t)$ converges uniformly to $\mathbf{T}_\lambda^i\mathbf{u}(t)$ on $[0,1]$, and then $\mathbf{T}_\lambda$ is continuous.

We now turn to the pointwise convergence of $T_\lambda^i\mathbf{u}_m(t)$ for the three subsequences. For simplicity (without loss of generality), we discuss $(\sigma_i^m)_{m\in\mathbb{N}}$ instead of the notation for its three subsequences.

If $(\sigma_i^m)_{m\in\mathbb{N}}\subset[0,\beta_i^1)$, then $\lim_{m\to\infty}\sigma_i^m=\beta_i^1$. Suppose this is false. Then there exist $\sigma_i^0\in[0,\beta_i^1)$ and a subsequence $(\sigma_i^{m_j})_{j\in\mathbb{N}}$ of $(\sigma_i^m)_{m\in\mathbb{N}}$ such that $\lim_{j\to\infty}\sigma_i^{m_j}=\sigma_i^0$. Therefore, we have that $\Theta^i\mathbf{u}(\sigma_i^0)=0$, which is a contradiction. By the same argument, we have that if $(\sigma_i^m)_{m\in\mathbb{N}}\subset(\beta_i^2,1]$, then $\lim_{m\to\infty}\sigma_i^m=\beta_i^2$. Note that both β_i^1 and β_i^2 are zeros of $\Theta^i\mathbf{u}(t)=0$ on $(0,1)$. As for case (i), the dominated convergence theorem implies that $\mathbf{T}_\lambda^i\mathbf{u}_m(t)$ converges to $\mathbf{T}_\lambda^i\mathbf{u}(t)$ on $[0,1]$ for the two cases $(\sigma_i^m)_{m\in\mathbb{N}}\subset[0,\beta_i^1)$ and $(\sigma_i^m)_{m\in\mathbb{N}}\subset(\beta_i^2,1]$.

If $(\sigma_i^m)_{m\in\mathbb{N}}\subset[\beta_i^1,\beta_i^2]$, (2.4) implies that, for any σ_i^m,

$$T_\lambda^i\mathbf{u}(t)=\begin{cases}\int_0^t\varphi^{-1}\left(\int_s^{\sigma_i^m}\alpha^i(\tau)\,d\tau\right)ds, & 0\leqslant t\leqslant\beta_i^1,\\ \int_0^t\varphi^{-1}\left(\int_s^{\sigma_i^m}\alpha^i(\tau)\,d\tau\right)ds, & t\in(\beta_i^1,\beta_i^2),\ t\leqslant\sigma_i^m,\\ \int_t^1\varphi^{-1}\left(\int_{\sigma_i^m}^s\alpha^i(\tau)\,d\tau\right)ds, & t\in(\beta_i^1,\beta_i^2),\ t>\sigma_i^m,\\ \int_t^1\varphi^{-1}\left(\int_{\sigma_i^m}^s\alpha^i(\tau)\,d\tau\right)ds, & \beta_i^2\leqslant t\leqslant 1.\end{cases}\tag{2.9}$$

On the other hand, since $\sigma_i^m \in [\beta_1^i, \beta_2^i]$, $m \in \mathbb{N}$, we have

$$T_\lambda^i \mathbf{u}_m(t) = \begin{cases} \int_0^t \varphi^{-1}(\int_s^{\sigma_i^m} \alpha_m^i(\tau)\,d\tau)\,ds, & 0 \leqslant t \leqslant \beta_i^1, \\ \int_0^t \varphi^{-1}(\int_s^{\sigma_i^m} \alpha_m^i(\tau)\,d\tau)\,ds, & t \in (\beta_i^1, \beta_i^2),\ t \leqslant \sigma_i^m, \\ \int_t^1 \varphi^{-1}(\int_{\sigma_i^m}^s \alpha_m^i(\tau)\,d\tau)\,ds, & t \in (\beta_i^1, \beta_i^2),\ t > \sigma_i^m, \\ \int_t^1 \varphi^{-1}(\int_{\sigma_i^m}^s \alpha_m^i(\tau)\,d\tau)\,ds, & \beta_i^2 \leqslant t \leqslant 1. \end{cases} \tag{2.10}$$

Consequently, the dominated convergence theorem implies that $\lim_{m\to\infty} T_\lambda^i \mathbf{u}_m(t) = T_\lambda^i \mathbf{u}(t)$ for $t \in [0, 1]$. Thus our proof is complete. □

Now it is not difficult to show that (1.1)–(1.2) is equivalent to the fixed point equation

$$\mathbf{T}_\lambda \mathbf{u} = \mathbf{u} \quad \text{in } K.$$

Lemma 2.6. *Assume* (A1) *holds. Then for all* $\sigma, x \in (0, \infty)$

$$\psi_2^{-1}(\sigma)x \leqslant \varphi^{-1}\big(\sigma\varphi(x)\big) \leqslant \psi_1^{-1}(\sigma)x.$$

Proof. Since $\sigma = \psi_1(\psi_1^{-1}(\sigma)) = \psi_2(\psi_2^{-1}(\sigma))$ and $\varphi(\varphi^{-1}(\sigma\varphi(x))) = \sigma\varphi(x)$, it follows that

$$\psi_2\big(\psi_2^{-1}(\sigma)\big)\varphi(x) = \varphi\big(\varphi^{-1}\big(\sigma\varphi(x)\big)\big) = \psi_1\big(\psi_1^{-1}(\sigma)\big)\varphi(x).$$

On the other hand, we have by (A1) that

$$\psi_1\big(\psi_1^{-1}(\sigma)\big)\varphi(x) \leqslant \varphi\big(\psi_1^{-1}(\sigma)x\big) \quad \text{and} \quad \psi_2\big(\psi_2^{-1}(\sigma)\big)\varphi(x) \geqslant \varphi\big(\psi_2^{-1}(\sigma)x\big).$$

Hence, $\varphi(\psi_2^{-1}(\sigma)x) \leqslant \varphi(\varphi^{-1}(\sigma\varphi(x))) \leqslant \varphi(\psi_1^{-1}(\sigma)x)$. Thus, we obtain that $\psi_2^{-1}(\sigma)x \leqslant \varphi^{-1}(\sigma\varphi(x)) \leqslant \psi_1^{-1}(\sigma)x$. □

Let

$$\gamma_i(t) = \frac{1}{8}\left[\int_{1/4}^{t} \psi_2^{-1}\left(\int_s^t h_i(\tau)\,d\tau\right)ds + \int_t^{3/4} \psi_2^{-1}\left(\int_t^s h_i(\tau)\,d\tau\right)ds\right], \quad i = 1, \ldots, n,$$

where $t \in [1/4, 3/4]$. It follows from (A1)–(A3) that

$$\Gamma = \min\left\{\gamma_i(t)\colon\ \frac{1}{4} \leqslant t \leqslant \frac{3}{4},\ i = 1, \ldots, n\right\} > 0.$$

Lemma 2.7. *Assume* (A1)–(A3) *hold. Let* $\mathbf{u} = (u_1, \ldots, u_n) \in K$ *and* $\eta > 0$. *If there exists a component* f^i *of* $\mathbf{f}$ *such that*

$$f^i\big(\mathbf{u}(t)\big) \geqslant \varphi\left(\eta\sum_{i=1}^n u_i(t)\right) \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \psi_2^{-1}(\lambda)\Gamma\eta\|\mathbf{u}\|.$$

Proof. Note, from the definition of $\mathbf{T}_\lambda \mathbf{u}$, that $T_\lambda^i \mathbf{u}(\sigma_i)$ is the maximum value of $T_\lambda^i \mathbf{u}$ on $[0,1]$. If $\sigma_i \in [1/4, 3/4]$, we have

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &\geqslant \sup_{t\in[0,1]} \left|T_\lambda^i \mathbf{u}(t)\right| \\
&\geqslant \frac{1}{2}\left[\int_{1/4}^{\sigma_i} \varphi^{-1}\left(\int_s^{\sigma_i} \lambda h_i(\tau) f^i\big(\mathbf{u}(\tau)\big)\,d\tau\right) ds \right. \\
&\quad \left. + \int_{\sigma_i}^{3/4} \varphi^{-1}\left(\int_{\sigma_i}^{s} \lambda h_i(\tau) f^i\big(\mathbf{u}(\tau)\big)\,d\tau\right) ds\right] \\
&\geqslant \frac{1}{2}\left[\int_{1/4}^{\sigma_i} \varphi^{-1}\left(\int_s^{\sigma_i} \lambda h_i(\tau)\varphi\left(\eta\sum_{i=1}^n u_i(\tau)\right) d\tau\right) ds \right. \\
&\quad \left. + \int_{\sigma_i}^{3/4} \varphi^{-1}\left(\int_{\sigma_i}^{s} \lambda h_i(\tau)\varphi\left(\eta\sum_{i=1}^n u_i(\tau)\right) d\tau\right) ds\right],
\end{aligned}$$

and in view of Lemma 2.3 and condition (A1), we find that

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &\geqslant \frac{1}{2}\left[\int_{1/4}^{\sigma_i} \varphi^{-1}\left(\int_s^{\sigma_i} \psi_2\big(\psi_2^{-1}(\lambda)\big) h_i(\tau)\varphi\left(\frac{\eta}{4}\|\mathbf{u}\|\right) d\tau\right) ds \right. \\
&\quad \left. + \int_{\sigma_i}^{3/4} \varphi^{-1}\left(\int_{\sigma_i}^{s} \psi_2\big(\psi_2^{-1}(\lambda)\big) h_i(\tau)\varphi\left(\frac{\eta}{4}\|\mathbf{u}\|\right) d\tau\right) ds\right] \\
&\geqslant \frac{1}{2}\left[\int_{1/4}^{\sigma_i} \varphi^{-1}\left(\int_s^{\sigma_i} h_i(\tau)\,d\tau\,\varphi\left(\psi_2^{-1}(\lambda)\frac{\eta}{4}\|\mathbf{u}\|\right)\right) ds \right. \\
&\quad \left. + \int_{\sigma_i}^{3/4} \varphi^{-1}\left(\int_{\sigma_i}^{s} h_i(\tau)\,d\tau\,\varphi\left(\psi_2^{-1}(\lambda)\frac{\eta}{4}\|\mathbf{u}\|\right)\right) ds\right].
\end{aligned}$$

Now, because of Lemma 2.6, we have

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &\geqslant \frac{\psi_2^{-1}(\lambda)\eta\|\mathbf{u}\|}{8}\left[\int_{1/4}^{\sigma_i} \psi_2^{-1}\left(\int_s^{\sigma_i} h_i(\tau)\,d\tau\right) ds + \int_{\sigma_i}^{3/4} \psi_2^{-1}\left(\int_{\sigma_i}^{s} h_i(\tau)\,d\tau\right) ds\right] \\
&\geqslant \psi_2^{-1}(\lambda)\Gamma\eta\|\mathbf{u}\|.
\end{aligned}$$

For $\sigma_i > 3/4$, it is easy to see that

$$\|T_\lambda^i \mathbf{u}\| \geqslant \int_{1/4}^{3/4} \varphi^{-1}\left(\int_s^{3/4} \lambda h_i(\tau) f^i\big(\mathbf{u}(\tau)\big)\,d\tau\right) ds.$$

On the other hand, we have

$$\|T_\lambda^i \mathbf{u}\| \geqslant \int_{1/4}^{3/4} \varphi^{-1}\left(\int_{1/4}^{s} \lambda h_i(\tau) f^i\big(\mathbf{u}(\tau)\big)\,d\tau\right) ds \quad \text{if } \sigma_i < \frac{1}{4}.$$

Therefore, similar arguments show that $\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \psi_2^{-1}(\lambda)\Gamma\eta\|\mathbf{u}\|$ if $\sigma_i > 3/4$ or $\sigma_i < 1/4$. $\square$

For each $i = 1, \ldots, n$, define a new function $\hat{f}^i(t): \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\hat{f}^i(t) = \max\big\{f^i(\mathbf{u}):\ \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leqslant t\big\}.$$

Note that $\hat{f}_0^i = \lim_{t\to 0}(\hat{f}^i(t)/\varphi(t))$ and $\hat{f}_\infty^i = \lim_{t\to\infty}(\hat{f}^i(t)/\varphi(t))$.

Lemma 2.8. *Assume* (A1)–(A2) *hold. Then* $\hat{f}_0^i = f_0^i$ *and* $\hat{f}_\infty^i = f_\infty^i$, $i = 1, \ldots, n$.

Proof. It is easy to see that $\hat{f}_0^i = f_0^i$. For the second part, we consider the following two cases: (a) $f^i(\mathbf{u})$ is bounded and (b) $f^i(\mathbf{u})$ is unbounded. For case (a), it follows, from $\lim_{t\to\infty}\varphi(t) = \infty$, that $\hat{f}_\infty^i = 0 = f_\infty^i$. For case (b), for any $\delta > 0$, let $M^i = \hat{f}^i(\delta)$ and

$$N_\delta^i = \inf\big\{\|\mathbf{u}\|:\ \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \geqslant \delta,\ f^i(\mathbf{u}) \geqslant M^i\big\} \geqslant \delta,$$

then

$$\max\big\{f^i(\mathbf{u}):\ \|\mathbf{u}\| \leqslant N_\delta^i,\ \mathbf{u} \in \mathbb{R}_+^n\big\} = M^i = \max\big\{f^i(\mathbf{u}):\ \|\mathbf{u}\| = N_\delta^i,\ \mathbf{u} \in \mathbb{R}_+^n\big\}.$$

Thus, for any $\delta > 0$, there exists $N_\delta^i \geqslant \delta$ such that

$$\hat{f}^i(t) = \max\big\{f^i(\mathbf{u}):\ N_\delta^i \leqslant \|\mathbf{u}\| \leqslant t,\ \mathbf{u} \in \mathbb{R}_+^n\big\} \quad \text{for } t > N_\delta^i.$$

Now, suppose that $f_\infty^i < \infty$. In other words, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$f_\infty^i - \varepsilon < \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)} < f_\infty^i + \varepsilon \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| > \delta. \tag{2.11}$$

Thus, for $t > N_\delta^i$, there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}_+^n$ such that $\|\mathbf{u}_1\| = t$, $t \geqslant \|\mathbf{u}_2\| \geqslant N_\delta^i$, and $f^i(\mathbf{u}_2) = \hat{f}^i(t)$. Therefore,

$$\frac{f^i(\mathbf{u}_1)}{\varphi(\|\mathbf{u}_1\|)} \leqslant \frac{\hat{f}^i(t)}{\varphi(t)} = \frac{f^i(\mathbf{u}_2)}{\varphi(t)} \leqslant \frac{f^i(\mathbf{u}_2)}{\varphi(\|\mathbf{u}_2\|)}. \tag{2.12}$$

(2.11) and (2.12) yield that

$$f_\infty^i - \varepsilon < \frac{\hat{f}^i(t)}{\varphi(t)} < f_\infty^i + \varepsilon \quad \text{for } t > N_\delta^i. \tag{2.13}$$

Hence $\hat{f}_\infty^i = f_\infty^i$. Similarly, we can show that $\hat{f}_\infty^i = f_\infty^i$ if $f_\infty^i = \infty$. $\square$

Lemma 2.9. *Assume* (A1)–(A3) *hold and let* $r > 0$. *If there exits* $\varepsilon > 0$ *such that*

$$\hat{f}^i(r) \leqslant \psi_1(\varepsilon)\varphi(r), \quad i = 1, \ldots, n,$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \psi_1^{-1}(\lambda)\varepsilon\hat{C}\|\mathbf{u}\| \quad \textit{for } \mathbf{u} \in \partial\Omega_r.$$

where the constant $\hat{C} = \sum_{i=1}^n \psi_1^{-1}(\int_0^1 h_i(\tau)\,d\tau)$.

Proof. From the definition of T_λ, for $\mathbf{u} \in \partial\Omega_r$, we have

$$\|\mathbf{T}_\lambda \mathbf{u}\| = \sum_{i=1}^n \sup_{t\in[0,1]} \left|T_\lambda^i \mathbf{u}(t)\right| \leqslant \sum_{i=1}^n \varphi^{-1}\left(\int_0^1 \lambda h_i(\tau) f^i\big(\mathbf{u}(\tau)\big)\,d\tau\right)$$
$$\leqslant \sum_{i=1}^n \varphi^{-1}\left(\int_0^1 h_i(\tau)\,d\tau\,\lambda\hat{f}^i(r)\right) \leqslant \sum_{i=1}^n \varphi^{-1}\left(\int_0^1 h_i(\tau)\,d\tau\,\lambda\psi_1(\varepsilon)\varphi(r)\right).$$

Note that $\lambda = \psi_1(\psi_1^{-1}(\lambda))$. Then (A1) and Lemma 2.6 imply that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \sum_{i=1}^n \varphi^{-1}\left(\int_0^1 h_i(\tau)\,d\tau\,\varphi\big(\psi_1^{-1}(\lambda)\varepsilon r\big)\right) \leqslant \psi_1^{-1}(\lambda)\varepsilon r \sum_{i=1}^n \psi_1^{-1}\left(\int_0^1 h_i(\tau)\,d\tau\right)$$
$$= \psi_1^{-1}(\lambda)\varepsilon\hat{C}\|\mathbf{u}\|. \qquad \square$$

The following two lemmas are weak forms of Lemmas 2.7 and 2.9.

Lemma 2.10. *Assume* (A1)–(A4) *hold. If* $\mathbf{u} \in \partial\Omega_r$, $r > 0$, *then*

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant 4\psi_2^{-1}(\lambda)\Gamma\varphi^{-1}(\hat{m}_r),$$

where $\hat{m}_r = \min\{f^i(\mathbf{u})\colon\ \mathbf{u} \in \mathbb{R}_+^n \textit{ and } r/4 \leqslant \|\mathbf{u}\| \leqslant r,\ i = 1, \ldots, n\} > 0$.

Proof. Since $f_i(\mathbf{u}(t)) \geqslant \hat{m}_r = \varphi(\varphi^{-1}(\hat{m}_r))$ for $t \in [1/4, 3/4]$, $i = 1, \ldots, n$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.7. $\square$

Lemma 2.11. *Assume* (A1)–(A4) *hold. If* $\mathbf{u} \in \partial\Omega_r$, $r > 0$, *then*

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \psi_1^{-1}(\lambda)\varphi^{-1}(\hat{M}_r)\hat{C},$$

where $\hat{M}_r = \max\{f^i(\mathbf{u})\colon\ \mathbf{u} \in \mathbb{R}_+^n \textit{ and } \|\mathbf{u}\| \leqslant r,\ i = 1, \ldots, n\} > 0$ *and* $\hat{C}$ *is the positive constant defined in Lemma* 2.9.

Proof. Since $f_i(\mathbf{u}(t)) \leqslant \hat{M}_r = \varphi(\varphi^{-1}(\hat{M}_r))$ for $t \in [0, 1]$, $i = 1, \ldots, n$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.9. $\square$

3. Proof of Theorem 1.1

(a) $\mathbf{f}_0 = 0$ implies that $f_0^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.8 that $\hat{f}_0^i = 0$, $i = 1, \ldots, n$. Therefore, we can choose $r_1 > 0$ so that $\hat{f}^i(r_1) \leqslant \psi_1(\varepsilon)\varphi(r_1)$, $i = 1, \ldots, n$, where the constant $\varepsilon > 0$ satisfies

$$\psi_1^{-1}(\lambda)\varepsilon\hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.9. We have by Lemma 2.9 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \psi_1^{-1}(\lambda)\varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_1}.$$

Now, since $\mathbf{f}_\infty = \infty$, there exists a component f^i of $\mathbf{f}$ such that $f_\infty^i = \infty$. Therefore, there is $\hat{H} > 0$ such that

$$f^i(\mathbf{u}) \geqslant \psi_2(\eta)\varphi\big(\|\mathbf{u}\|\big)$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \geqslant \hat{H}$, where $\eta > 0$ is chosen so that

$$\psi_2^{-1}(\lambda)\Gamma\eta > 1.$$

Let $r_2 = \max\{2r_1, 4\hat{H}\}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_2}$, then

$$\min_{1/4 \leqslant t \leqslant 3/4} \sum_{i=1}^n u_i(t) \geqslant \frac{1}{4}\|\mathbf{u}\| = \frac{1}{4}r_2 \geqslant \hat{H},$$

which implies that

$$f^i\big(\mathbf{u}(t)\big) \geqslant \psi_2(\eta)\varphi\left(\sum_{i=1}^n u_i(t)\right) \geqslant \varphi\left(\eta\sum_{i=1}^n u_i(t)\right) \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

It follows from Lemma 2.7 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \psi_2^{-1}(\lambda)\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0.$$

It follows from the additivity of the fixed point index that

$$i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = -1.$$

Thus, $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) \neq 0$, which implies that $\mathbf{T}_\lambda$ has a fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ by the existence property of the fixed point index. The fixed point $\mathbf{u} \in \Omega_{r_2} \setminus \bar{\Omega}_{r_1}$ is the desired positive solution of (1.1)–(1.2).

(b) If $\mathbf{f}_0 = \infty$, there exists a component f^i such that $f_0^i = \infty$. Therefore, there is an $r_1 > 0$ such that

$$f^i(\mathbf{u}) \geqslant \psi_2(\eta)\varphi\big(\|\mathbf{u}\|\big)$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \leqslant r_1$, where $\eta > 0$ is chosen so that

$$\psi_2^{-1}(\lambda)\Gamma\eta > 1.$$

If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_1}$, then

$$f^i\big(\mathbf{u}(t)\big) \geqslant \psi_2(\eta)\varphi\left(\sum_{i=1}^{n} u_i(t)\right) \geqslant \varphi\left(\eta\sum_{i=1}^{n} u_i(t)\right) \quad \text{for } t \in [0,1].$$

Lemma 2.7 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \psi_2^{-1}(\lambda)\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_1}.$$

We now determine Ω_{r_2}. $\mathbf{f}_\infty = 0$ implies that $f_\infty^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.8 that $\hat{f}_\infty^i = 0$, $i = 1, \ldots, n$. Therefore there is an $r_2 > 2r_1$ such that

$$\hat{f}^i(r_2) \leqslant \psi_1(\varepsilon)\varphi(r_2), \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\psi_1^{-1}(\lambda)\varepsilon\hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.9. Thus, we have by Lemma 2.9 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \psi_1^{-1}(\lambda)\varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_2}.$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.1)–(1.2). □

4. Proof of Theorem 1.2

(a) Fix a number $r_1 > 0$. Lemma 2.10 implies that there exists $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_1},\ \lambda > \lambda_0.$$

If $\mathbf{f}_0 = 0$, then $f_0^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.8 that

$$\hat{f}_0^i = 0, \quad i = 1, \ldots, n.$$

Therefore, we can choose $0 < r_2 < r_1$ so that

$$\hat{f}^i(r_2) \leqslant \psi_1(\varepsilon)\varphi(r_2), \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\psi_1^{-1}(\lambda)\varepsilon\hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.9. We have by Lemma 2.9 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \psi_1^{-1}(\lambda)\varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_2}.$$

If $\mathbf{f}_\infty = 0$, then $f_\infty^i = 0$, $i = 1, \ldots, n$. It follows from Lemma 2.8 that $\hat{f}_\infty^i = 0$, $i = 1, \ldots, n$. Therefore there is an $r_3 > 2r_1$ such that

$$\hat{f}^i(r_3) \leqslant \psi_1(\varepsilon)\varphi(r_3), \quad i = 1, \ldots, n,$$

where the constant $\varepsilon > 0$ satisfies

$$\psi_1^{-1}(\lambda)\varepsilon\hat{C} < 1,$$

and $\hat{C}$ is the positive constant defined in Lemma 2.9. Thus, we have by Lemma 2.9 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \leqslant \psi_1^{-1}(\lambda)\varepsilon\hat{C}\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1, \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1.$$

Thus $i(\mathbf{T}_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = -1$ and $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$. Hence, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $\mathbf{f}_0 = 0$ or $\mathbf{f}_\infty = 0$, respectively. Consequently, (1.1)–(1.2) has a positive solution for $\lambda > \lambda_0$.

(b) Fix a number $r_1 > 0$. Lemma 2.11 implies that there exists $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0.$$

If $\mathbf{f}_0 = \infty$, there exists a component f^i of $\mathbf{f}$ such that $f_0^i = \infty$. Therefore, there is a positive number $r_2 < r_1$ such that

$$f^i(\mathbf{u}) \geqslant \psi_2(\eta)\varphi\big(\|\mathbf{u}\|\big)$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \leqslant r_2$, where $\eta > 0$ is chosen so that

$$\psi_2^{-1}(\lambda)\Gamma\eta > 1.$$

Then

$$f^i\big(\mathbf{u}(t)\big) \geqslant \psi_2(\eta)\varphi\left(\sum_{i=1}^n u_i(t)\right) \geqslant \varphi\left(\eta\sum_{i=1}^n u_i(t)\right)$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_2}$, $t \in [0, 1]$. Lemma 2.7 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \psi_2^{-1}(\lambda)\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_2}.$$

If $\mathbf{f}_\infty = \infty$, there exists a component f^i of $\mathbf{f}$ such that $f_\infty^i = \infty$. Therefore, there is $\hat{H} > 0$ such that

$$f^i(\mathbf{u}) \geqslant \psi_2(\eta)\varphi\big(\|\mathbf{u}\|\big)$$

for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \geqslant \hat{H}$, where $\eta > 0$ is chosen so that

$$\psi_2^{-1}(\lambda)\Gamma\eta > 1.$$

Let $r_3 = \max\{2r_1, 4\hat{H}\}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \partial\Omega_{r_3}$, then

$$\min_{1/4 \leqslant t \leqslant 3/4} \sum_{i=1}^n u_i(t) \geqslant \frac{1}{4}\|\mathbf{u}\| = \frac{1}{4}r_3 \geqslant \hat{H},$$

which implies that

$$f^i\big(\mathbf{u}(t)\big) \geqslant \psi_2(\eta)\varphi\left(\sum_{i=1}^n u_i(t)\right) \geqslant \varphi\left(\eta\sum_{i=1}^n u_i(t)\right) \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

It follows from Lemma 2.7 that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geqslant \psi_2^{-1}(\lambda)\Gamma\eta\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0, \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 0.$$

Hence, $i(\mathbf{T}_\lambda, \Omega_{r_1} \setminus \bar{\Omega}_{r_2}, K) = 1$ and $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = -1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_1} \setminus \bar{\Omega}_{r_2}$ or $\Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ according to $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, (1.1)–(1.2) has a positive solution for $0 < \lambda < \lambda_0$.

(c) Fix two numbers $0 < r_3 < r_4$. Lemma 2.10 implies that there exists $\lambda_0 > 0$ such that we have, for $\lambda > \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_i} \ (i = 3, 4).$$

Since $\mathbf{f}_0 = 0$ and $\mathbf{f}_\infty = 0$, it follows from the proof of Theorem 1.2(a) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_i} \ (i = 1, 2).$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 1, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 0, \qquad i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 0.$$

Hence, $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = -1$ and $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_4}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (1.1)–(1.2) for $\lambda > \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

(d) Fix two numbers $0 < r_3 < r_4$. Lemma 2.11 implies that there exists $\lambda_0 > 0$ such that we have, for $0 < \lambda < \lambda_0$,

$$\|\mathbf{T}_\lambda \mathbf{u}\| < \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_i} \ (i = 3, 4).$$

Since $\mathbf{f}_0 = \infty$ and $\mathbf{f}_\infty = \infty$, it follows from the proof of Theorem 1.2(b) that we can choose $0 < r_1 < r_3/2$ and $r_2 > 2r_4$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\| > \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in \partial\Omega_{r_i} \ (i = 1, 2).$$

It follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0, \qquad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 0,$$

and

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1, \qquad i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 1.$$

Hence, $i(\mathbf{T}_\lambda, \Omega_{r_3} \setminus \bar{\Omega}_{r_1}, K) = 1$ and $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_4}, K) = -1$. Thus, $\mathbf{T}_\lambda$ has two fixed points $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $\mathbf{u}_1(t) \in \Omega_{r_3} \setminus \bar{\Omega}_{r_1}$ and $\mathbf{u}_2(t) \in \Omega_{r_2} \setminus \bar{\Omega}_{r_4}$, which are the desired distinct positive solutions of (1.1)–(1.2) for $\lambda < \lambda_0$ satisfying

$$r_1 < \|\mathbf{u}_1\| < r_3 < r_4 < \|\mathbf{u}_2\| < r_2.$$

(e) Since $\mathbf{f}_0 < \infty$ and $\mathbf{f}_\infty < \infty$, then $f_0^i < \infty$ and $f_\infty^i < \infty$, $i = 1, \dots, n$. Therefore, for each $i = 1, \dots, n$, there exist positive numbers ε_1^i, ε_2^i, r_1^i, and r_2^i such that $r_1^i < r_2^i$,

$$f^i(\mathbf{u}) \leqslant \varepsilon_1^i \varphi\big(\|\mathbf{u}\|\big) \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \leqslant r_1^i,$$

and

$$f^i(\mathbf{u}) \leqslant \varepsilon_2^i \varphi\big(\|\mathbf{u}\|\big) \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \geqslant r_2^i.$$

Let

$$\varepsilon^i = \max\left\{\varepsilon_1^i, \varepsilon_2^i, \max\left\{\frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}:\ \mathbf{u} \in \mathbb{R}_+^n,\ r_1^i \leqslant \|\mathbf{u}\| \leqslant r_2^i\right\}\right\} > 0$$

and $\varepsilon = \max_{i=1,\dots,n}\{\varepsilon^i\} > 0$. Thus, we have

$$f^i(\mathbf{u}) \leqslant \varepsilon \varphi\big(\|\mathbf{u}\|\big) \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n,\ i = 1, \dots, n.$$

Assume $\mathbf{v}(t)$ is a positive solution of (1.1)–(1.2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \psi_1\left(\frac{1}{\sum_{i=1}^n \psi_1^{-1}(\varepsilon \int_0^1 h_i(\tau)\,d\tau)}\right).$$

In fact, for $0 < \lambda < \lambda_0$, since $\mathbf{T}_\lambda \mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0, 1]$, we have

$$\begin{aligned}
\|\mathbf{v}\| = \|\mathbf{T}_\lambda \mathbf{v}\| &\leqslant \sum_{i=1}^n \varphi^{-1}\left(\int_0^1 h_i(\tau)\varepsilon\,d\tau\,\lambda\varphi\big(\|\mathbf{v}\|\big)\right) \\
&\leqslant \sum_{i=1}^n \varphi^{-1}\left(\int_0^1 h_i(\tau)\varepsilon\,d\tau\,\varphi\big(\psi_1^{-1}(\lambda)\|\mathbf{v}\|\big)\right) \\
&\leqslant \psi_1^{-1}(\lambda)\sum_{i=1}^n \psi_1^{-1}\left(\varepsilon\int_0^1 h_i(\tau)\,d\tau\right)\|\mathbf{v}\| < \|\mathbf{v}\|,
\end{aligned}$$

which is a contradiction.

(f) Since $\mathbf{f}_0 > 0$ and $\mathbf{f}_\infty > 0$, there exist two components f^i and f^j of $\mathbf{f}$ such that $f_0^i > 0$ and $f_\infty^j > 0$. Therefore, there exist positive numbers η_1, η_2, r_1, and r_2 such that $r_1 < r_2$,

$$f^i(\mathbf{u}) \geqslant \eta_1 \varphi\big(\|\mathbf{u}\|\big) \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \leqslant r_1,$$

and

$$f^j(\mathbf{u}) \geqslant \eta_2 \varphi\big(\|\mathbf{u}\|\big) \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \geqslant r_2.$$

Let

$$\eta_3 = \min\left\{\eta_1, \eta_2, \min\left\{\frac{f^j(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}:\ \mathbf{u} \in \mathbb{R}_+^n,\ \frac{r_1}{4} \leqslant \|\mathbf{u}\| \leqslant r_2\right\}\right\} > 0.$$

Thus, we have

$$f^i(\mathbf{u}) \geqslant \eta_3 \varphi\big(\|\mathbf{u}\|\big) \quad \text{for } \mathbf{u} \in \mathbb{R}_+^n,\ \|\mathbf{u}\| \leqslant r_1,$$

and

$$f^j(\mathbf{u}) \geqslant \eta_3\varphi(\|\mathbf{u}\|) \quad \text{for } \mathbf{u} \in \mathbb{R}^n_+,\ \|\mathbf{u}\| \geqslant \frac{r_1}{4}.$$

Since $\eta_3\varphi(\|\mathbf{u}\|) = \psi_2(\psi_2^{-1}(\eta_3))\varphi(\|\mathbf{u}\|)$, it follows from (A1) that

$$f^i(\mathbf{u}) \geqslant \varphi(\psi_2^{-1}(\eta_3)\|\mathbf{u}\|) \quad \text{for } \mathbf{u} \in \mathbb{R}^n_+,\ \|\mathbf{u}\| \leqslant r_1, \tag{4.1}$$

and

$$f^j(\mathbf{u}) \geqslant \varphi(\psi_2^{-1}(\eta_3)\|\mathbf{u}\|) \quad \text{for } \mathbf{u} \in \mathbb{R}^n_+,\ \|\mathbf{u}\| \geqslant \frac{r_1}{4}. \tag{4.2}$$

Assume $\mathbf{v}(t) = (v_1, \ldots, v_n)$ is a positive solution of (1.1)–(1.2). We will show that this leads to a contradiction for $\lambda > \lambda_0 = \psi_2(1/(\Gamma\psi_2^{-1}(\eta_3)))$. In fact, if $\|\mathbf{v}\| \leqslant r_1$, (4.1) implies that

$$f^i(\mathbf{v}(t)) \geqslant \varphi\left(\psi_2^{-1}(\eta_3)\sum_{i=1}^n v_i(t)\right) \quad \text{for } t \in [0, 1].$$

On the other hand, if $\|\mathbf{v}\| > r_1$, then

$$\min_{1/4\leqslant t\leqslant 3/4} \sum_{i=1}^n v_i(t) \geqslant \frac{1}{4}\|\mathbf{v}\| > \frac{1}{4}r_1,$$

which, together with (4.2), implies that

$$f^j(\mathbf{v}(t)) \geqslant \varphi\left(\psi_2^{-1}(\eta_3)\sum_{i=1}^n v_i(t)\right) \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Since $\mathbf{T}_\lambda\mathbf{v}(t) = \mathbf{v}(t)$ for $t \in [0, 1]$, it follows from Lemma 2.7 that, for $\lambda > \lambda_0$,

$$\|\mathbf{v}\| = \|\mathbf{T}_\lambda\mathbf{v}\| \geqslant \psi_2^{-1}(\lambda)\Gamma\psi_2^{-1}(\eta_3)\|\mathbf{v}\| > \|\mathbf{v}\|,$$

which is a contradiction. □

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