Is the Feynman path integral complex enough?

Gökçe Başar

University of North Carolina, Chapel Hill

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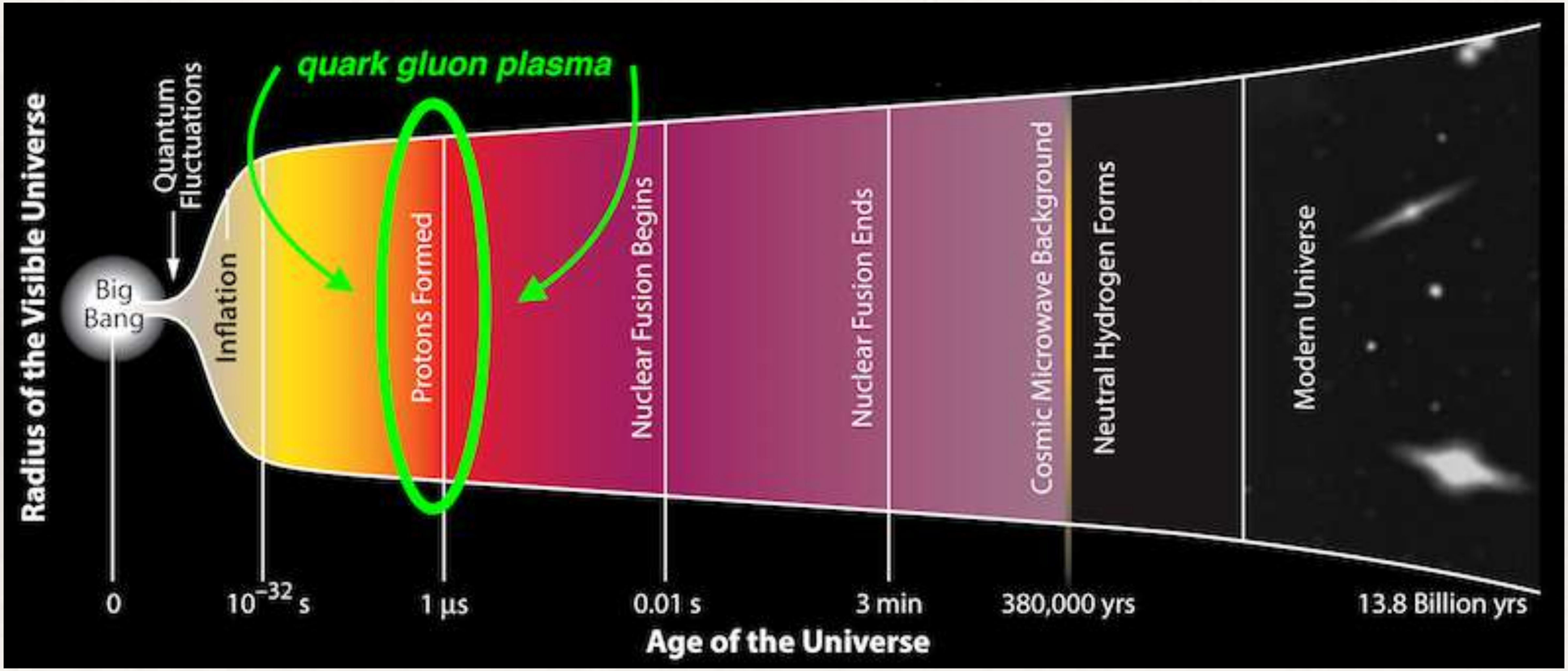
[with A. Alexandru, P. Bedaque, N. Warrington, G. Ridgway]

Motivations

first-principles studies of strongly interacting systems



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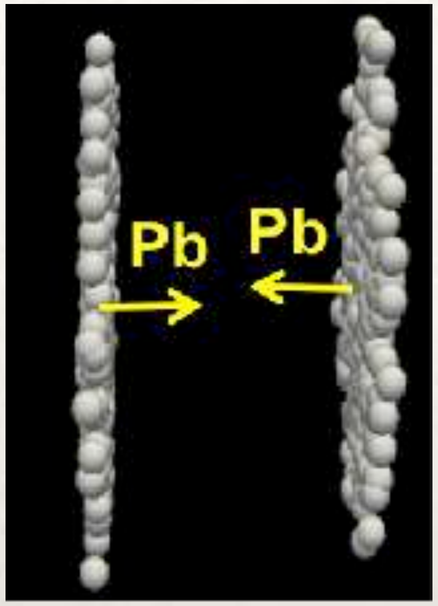

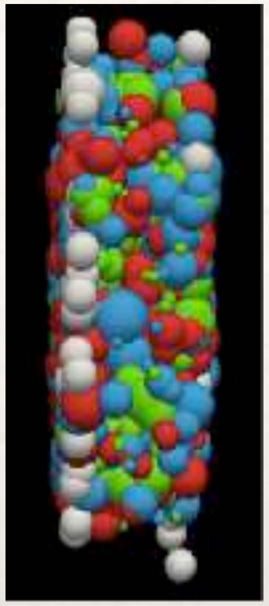

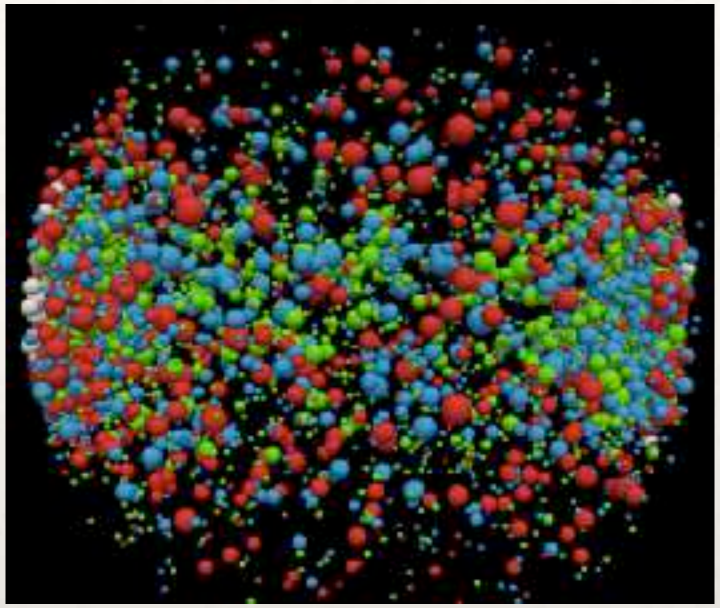

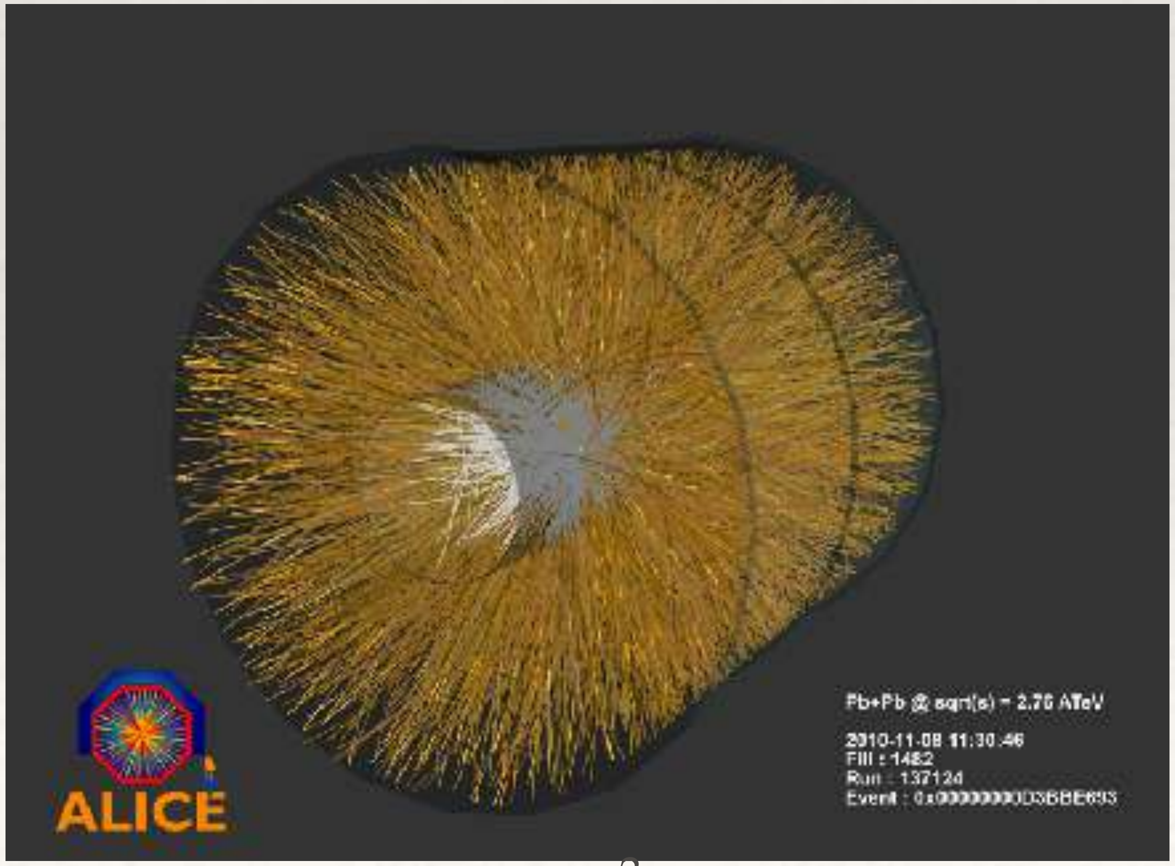

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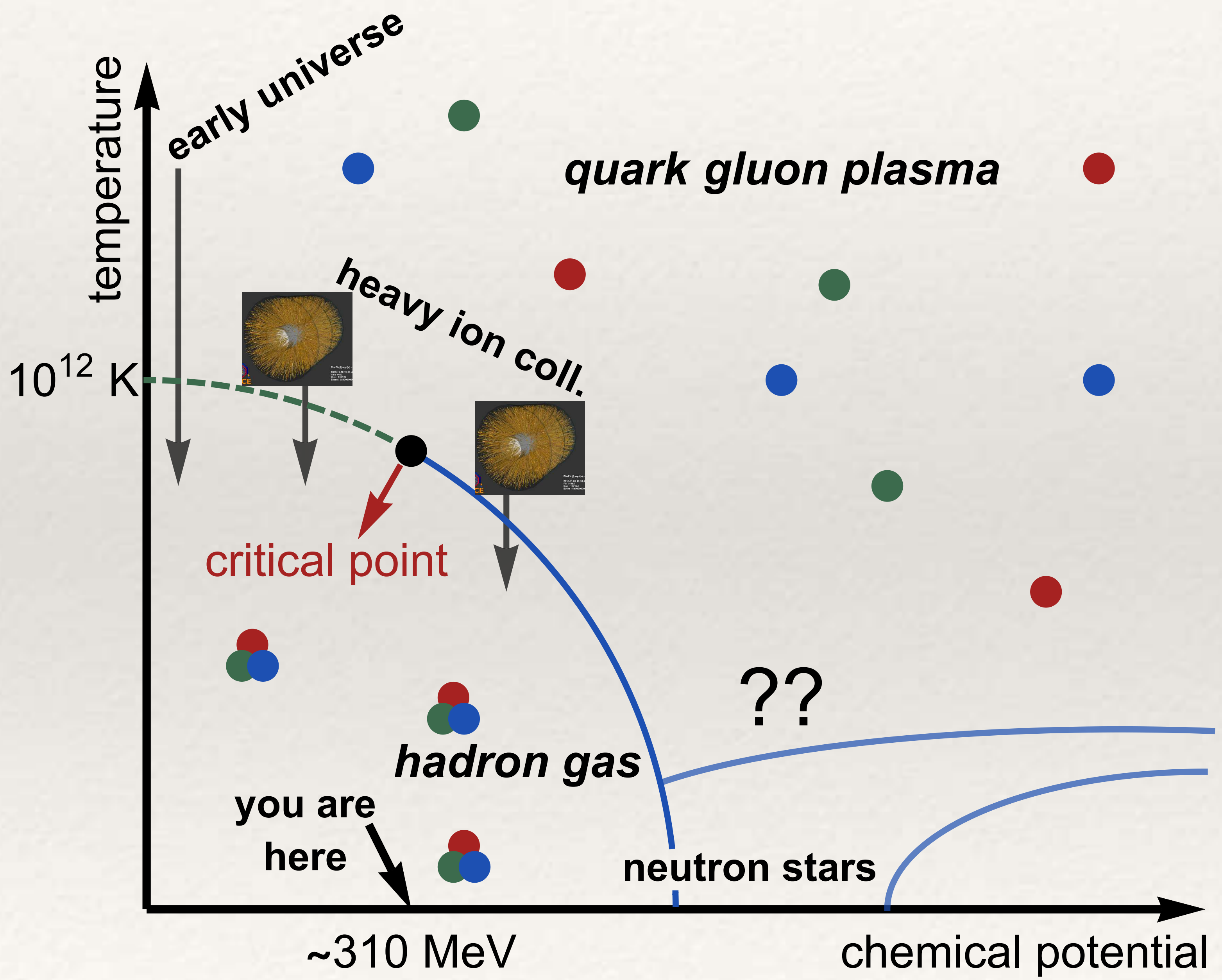
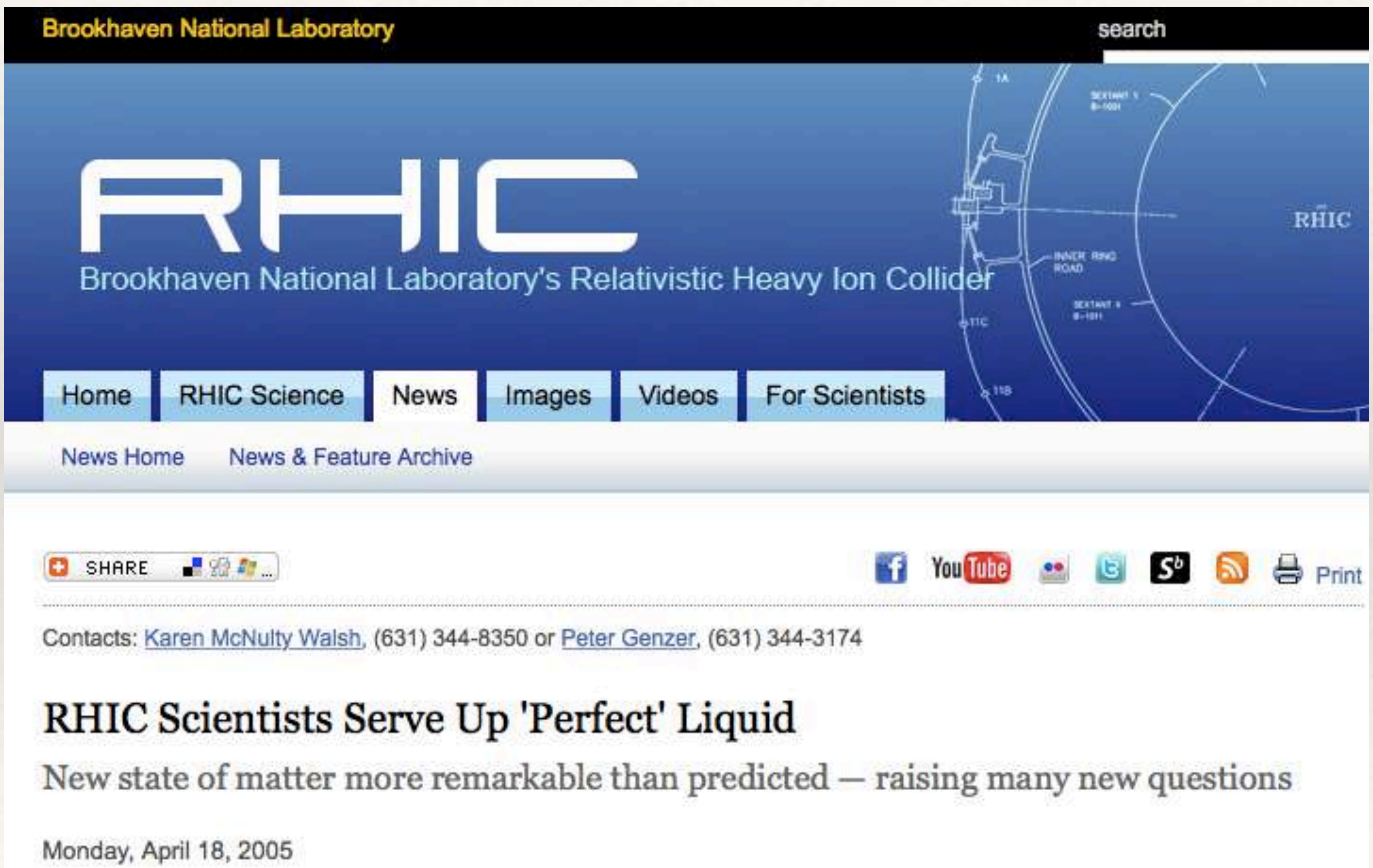


Motivations: out-of-equilibrium, transport

Heavy ion collisions: Quark gluon plasma is a *liquid !*



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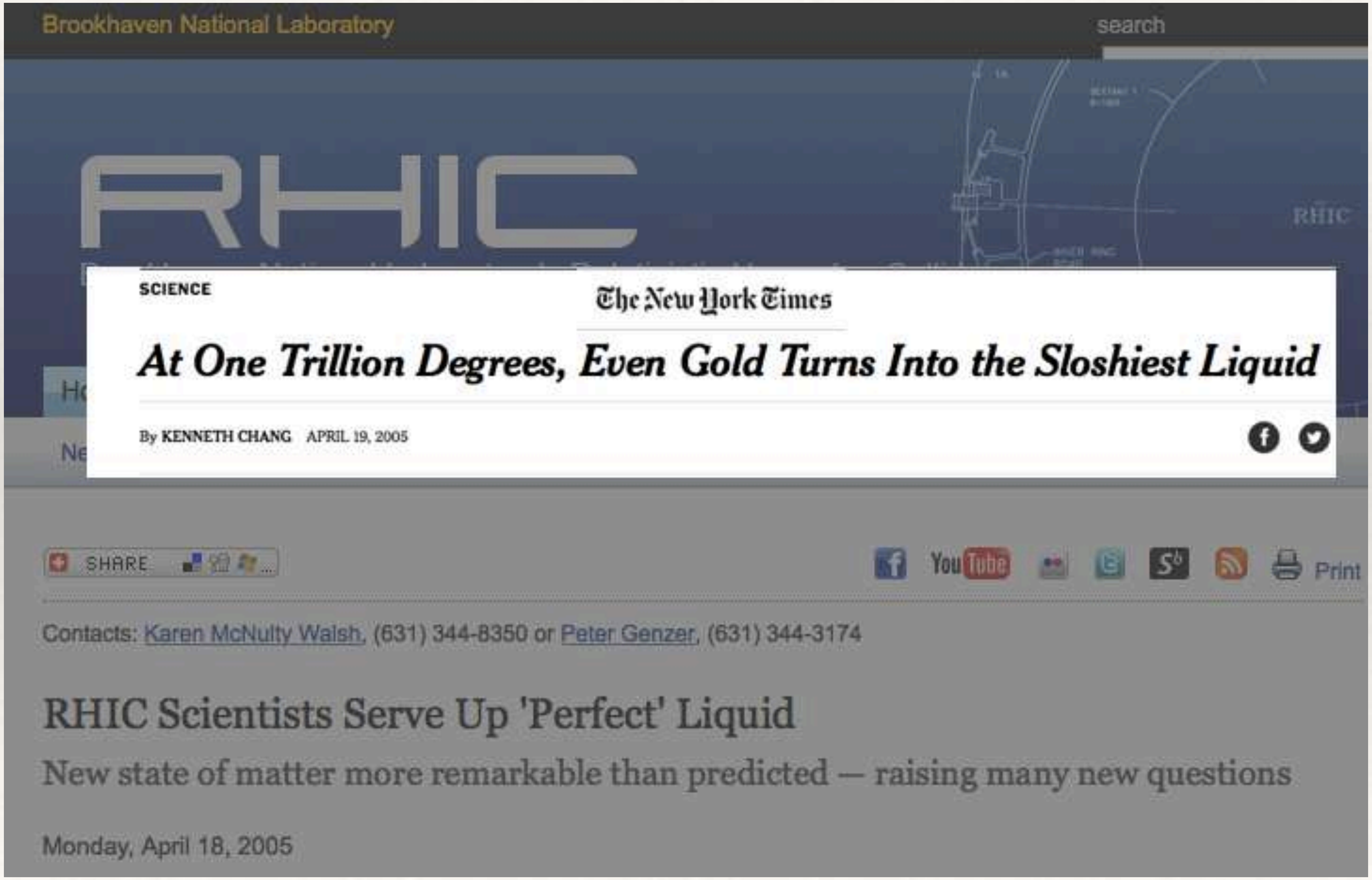

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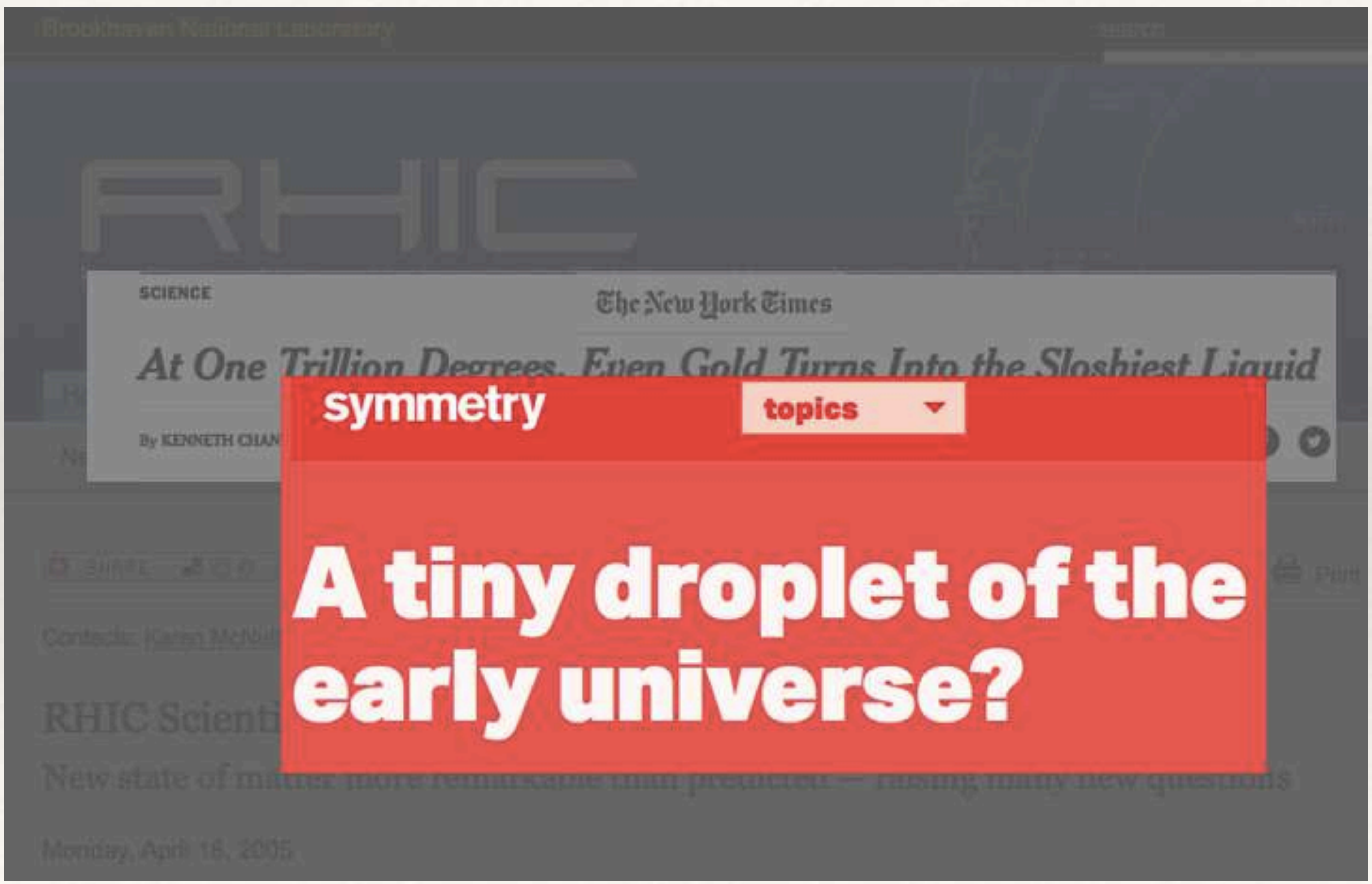

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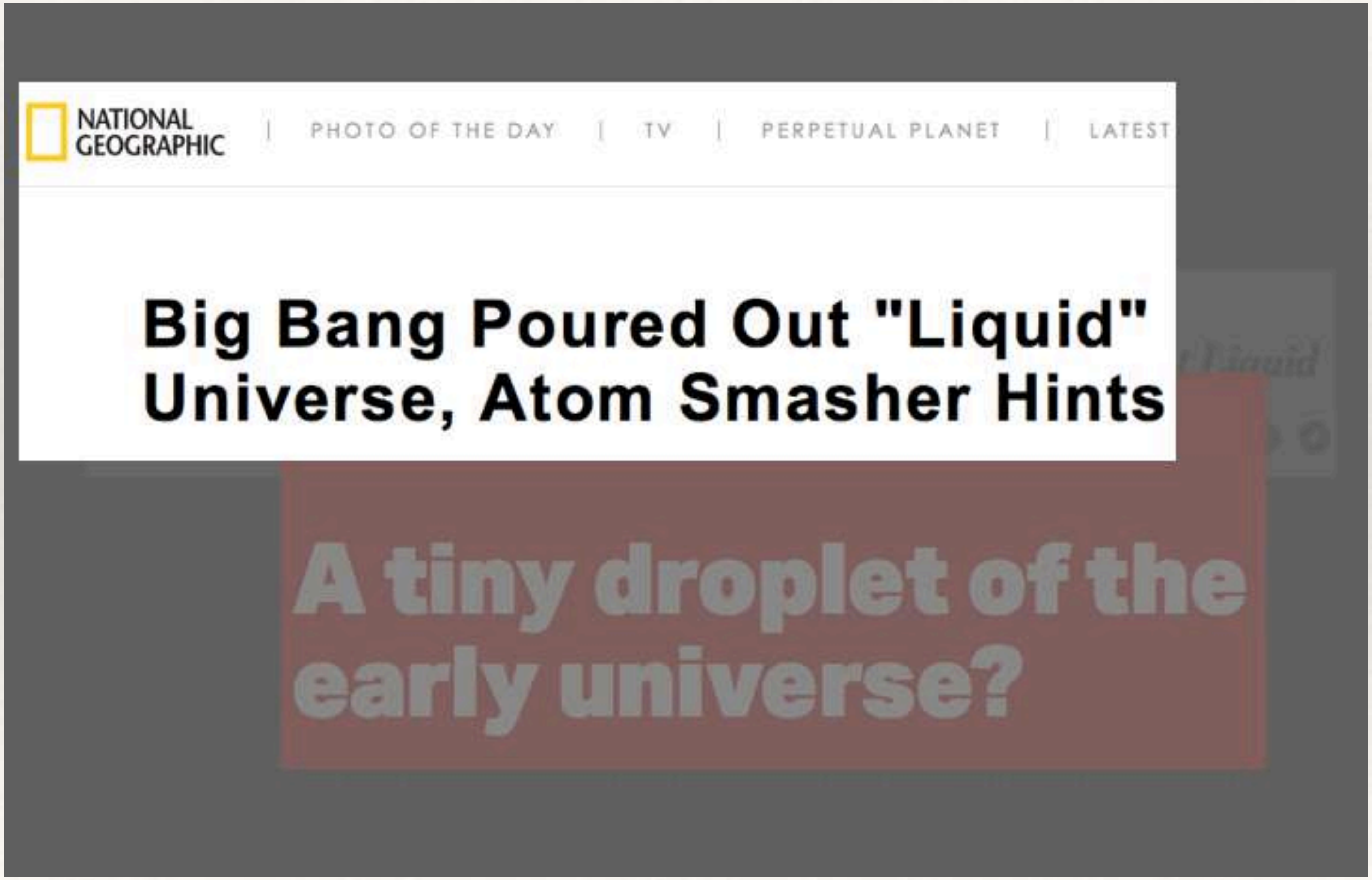
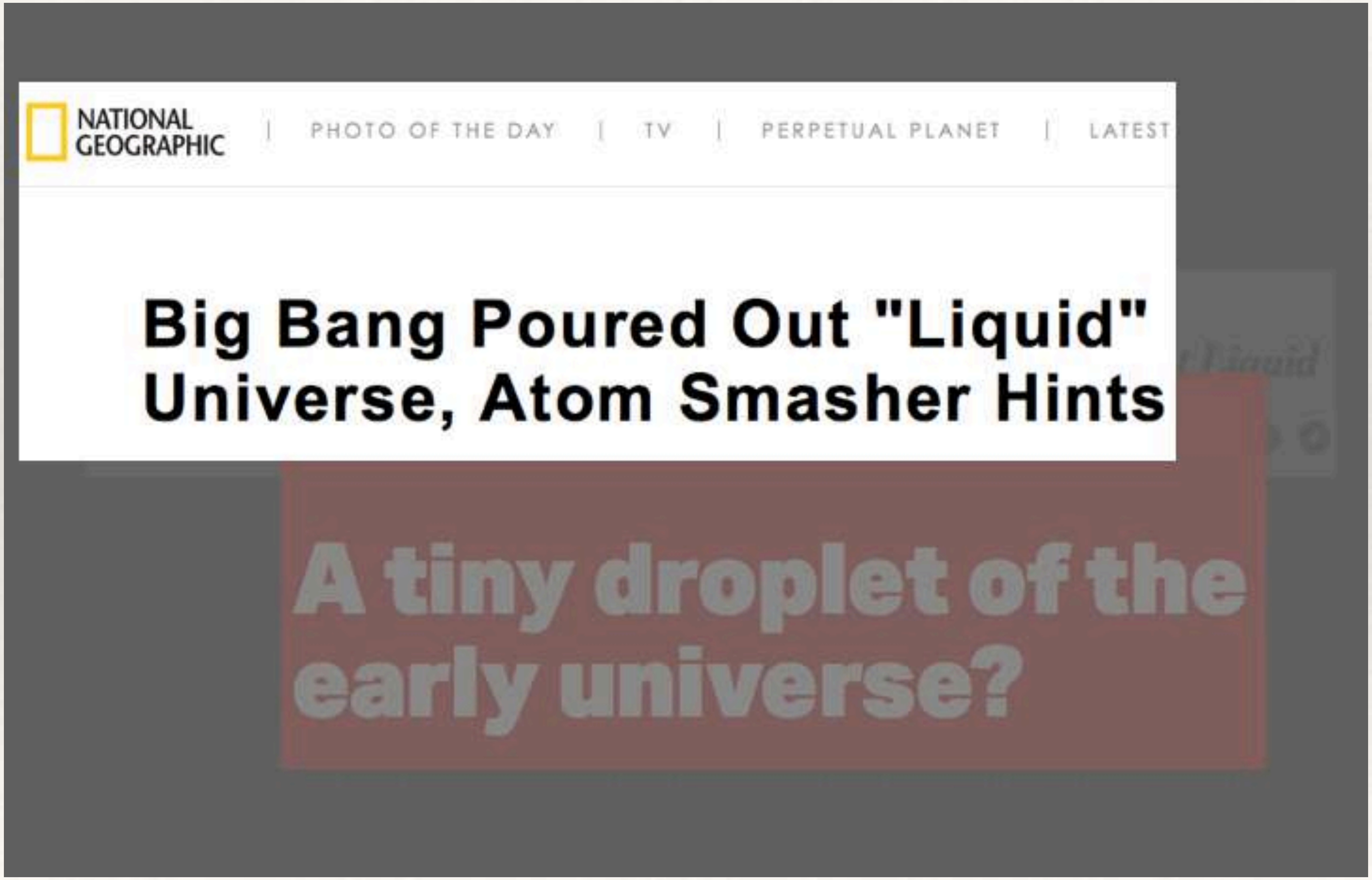


Motivations: out-of-equilibrium, transport

Quark gluon plasma is a *liquid*

what is the viscosity, conductivity …?



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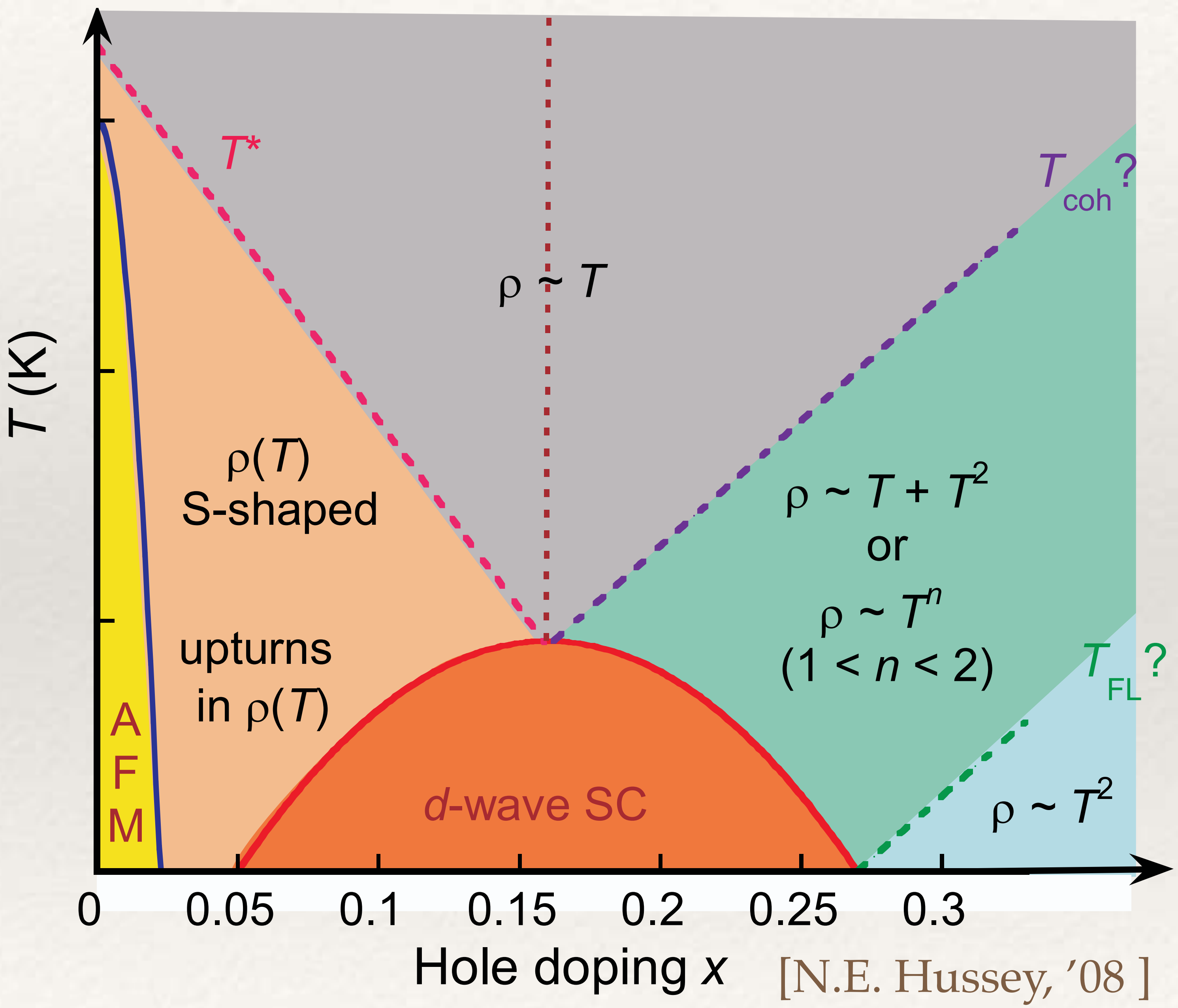
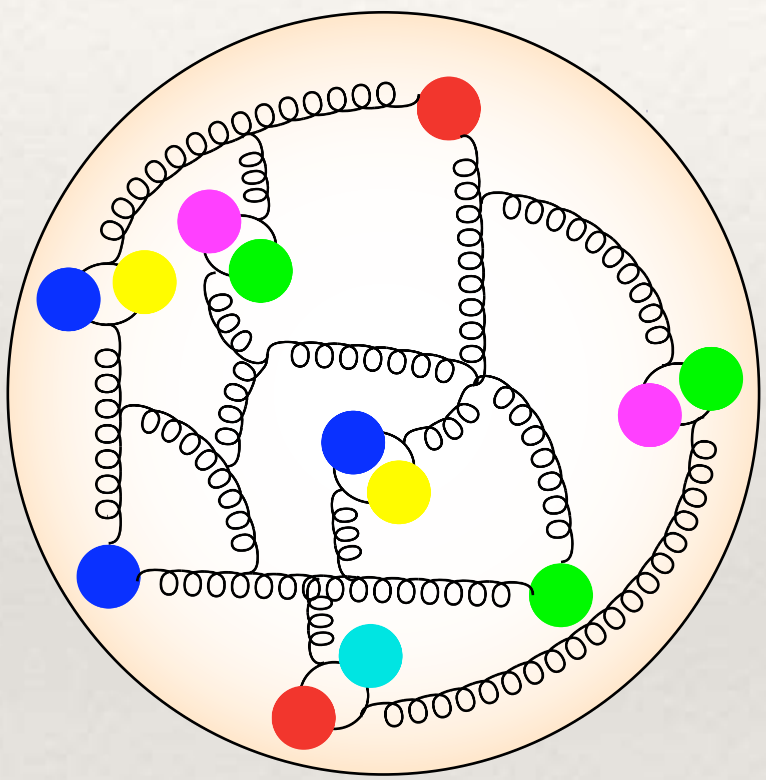
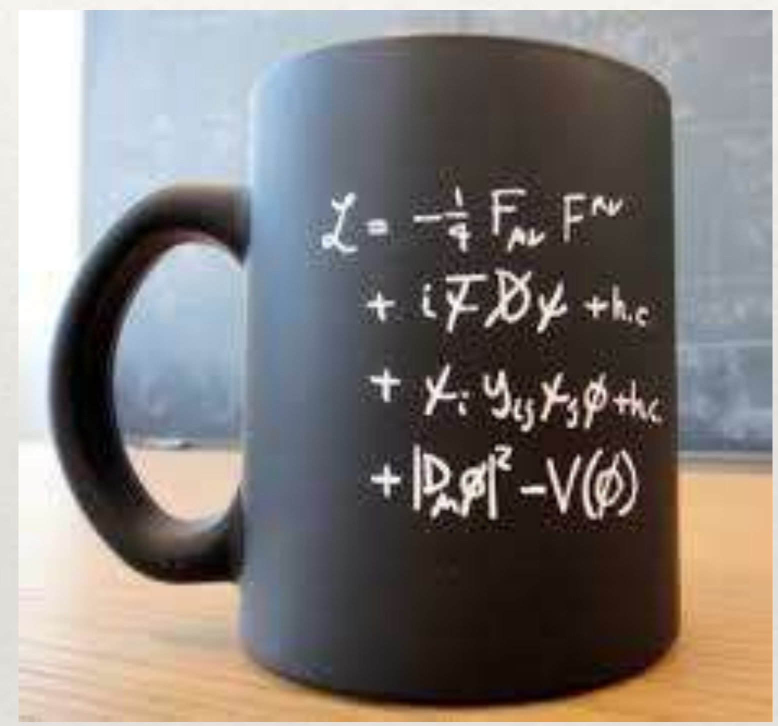


[N.E. Hussey, ’08]

Quantum Chromo Dynamics (QCD)

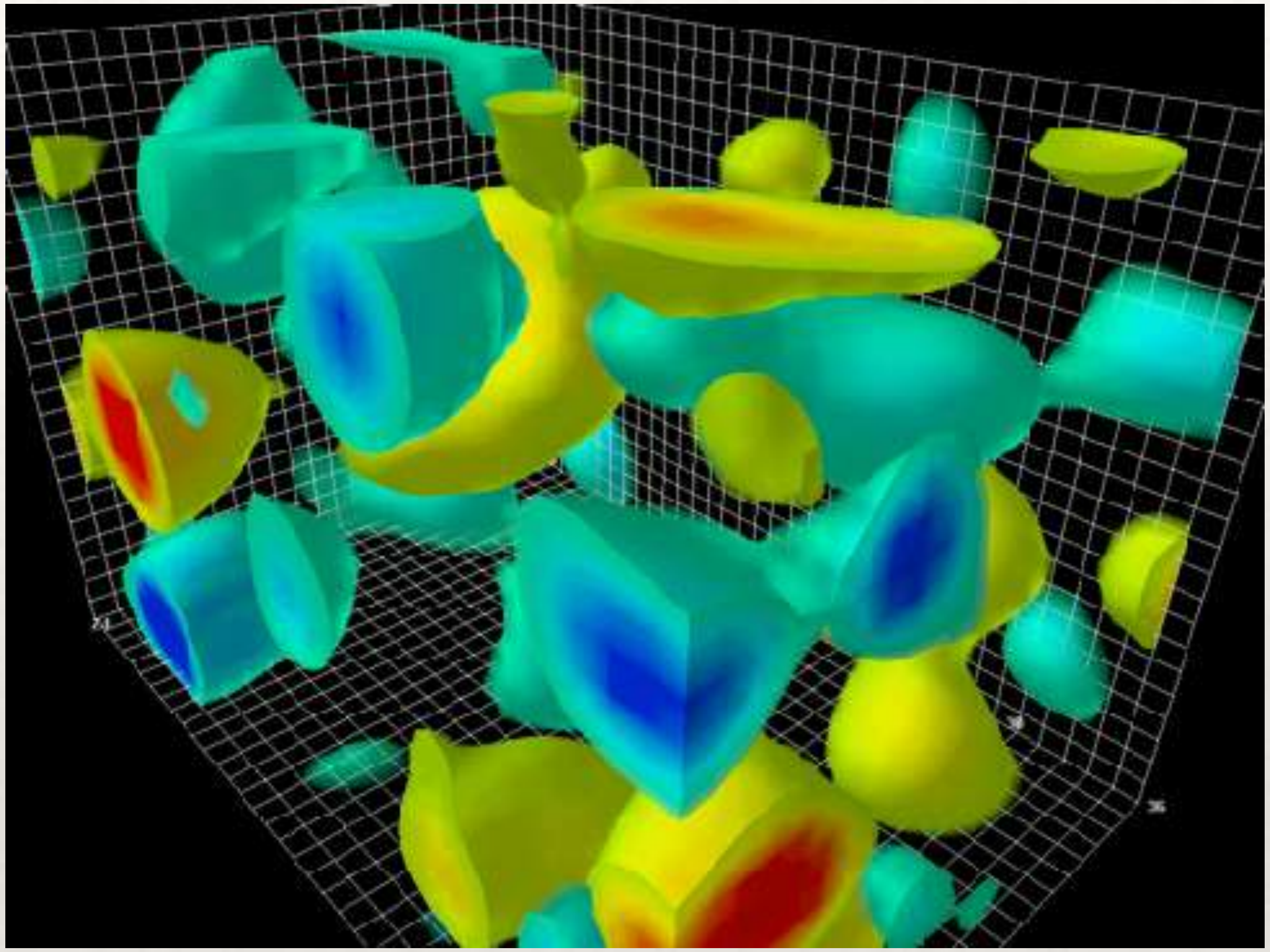
We know how quarks and gluons interact





Why not just compute the phase diagram, viscosity, equation of state, etc…?

Quantum fluctuations



animation: Derek Leinweber, University of Adelaide

we are interested in expectation values

examples: $\langle n \rangle \Leftrightarrow$ equation of state

$\langle \mathbf{J}(t)\mathbf{J}(0) \rangle \Leftrightarrow$ conductivity $\quad \langle T^{ab}(t)T^{cd}(0) \rangle \Leftrightarrow$ viscosity

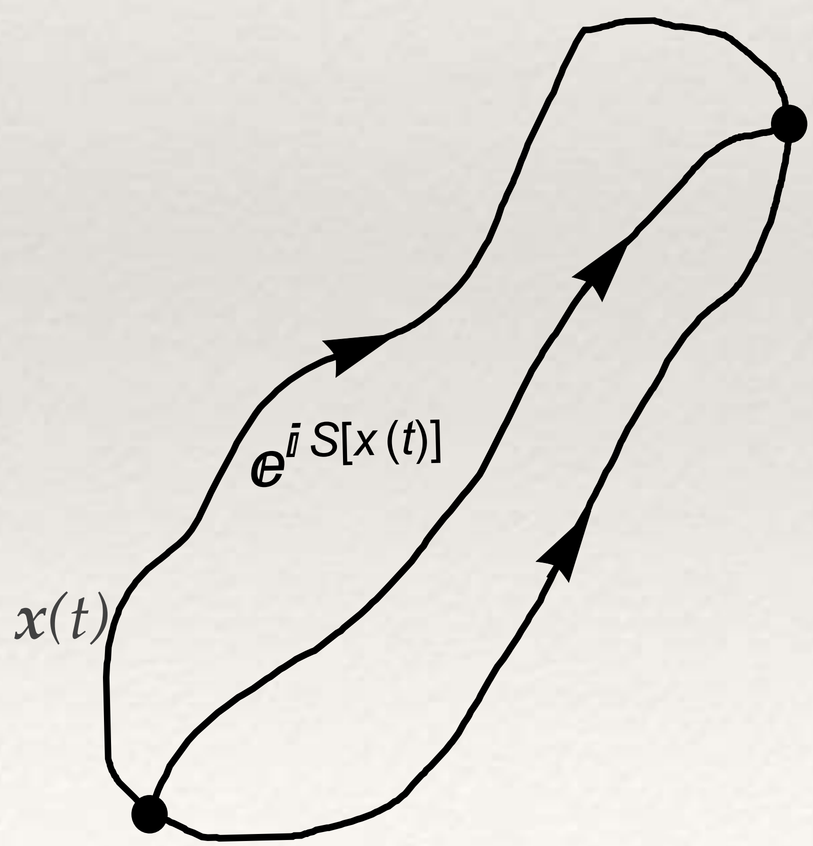
Feynman path integral

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the



The QFT path integral

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to ~~Non~~-Relativistic Quantum ~~Mechanics~~ Fields

R. P. FEYNMAN

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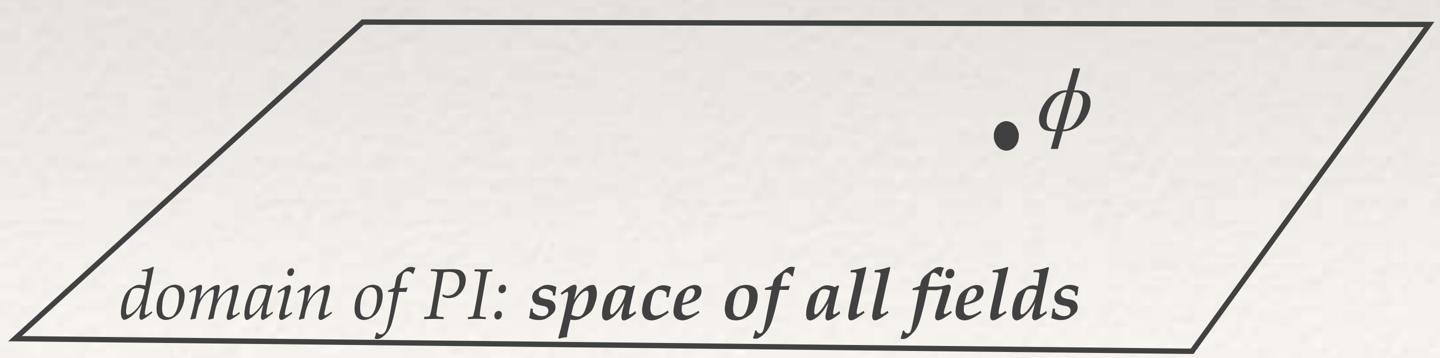
$$\langle \mathcal{O} \rangle = \int [d\phi] e^{iS[\phi]} \mathcal{O}[\phi]$$

all fields



A crash course on Lattice Field Theory

Lattice
~~Space Time~~ Approach to ~~Non~~-Relativistic
Quantum ~~Mechanics~~
Fields

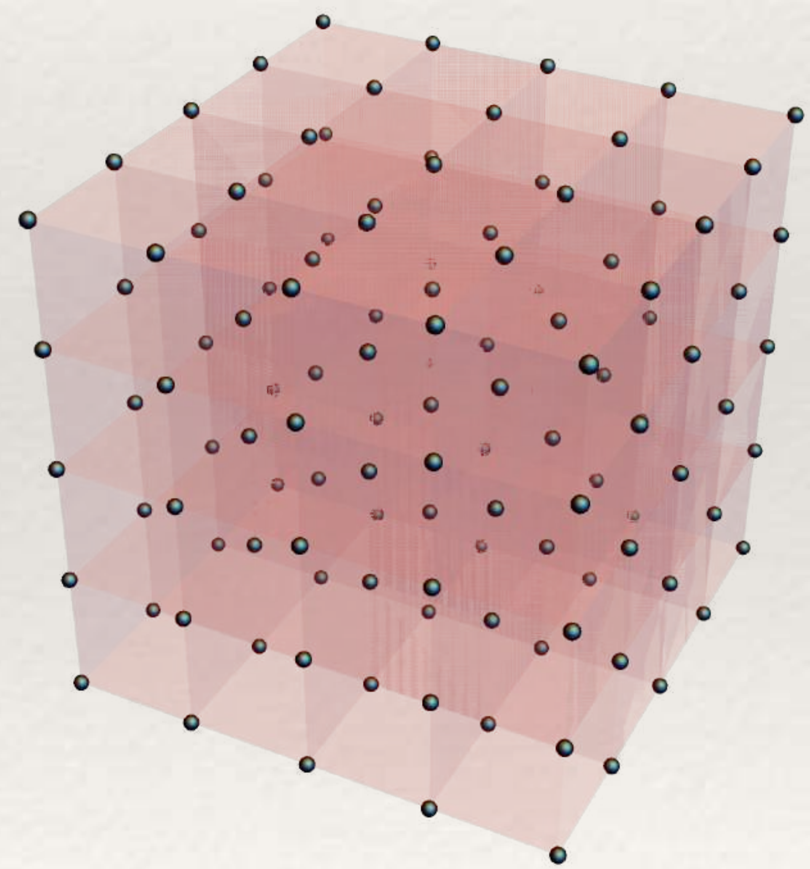
of contributions, one from each path in the region. The contribution from a single field is postulated to be an exponential whose real part is the classical action (in units of $\hbar$) for the field in question. The total contribution from all paths r with imaginary time is the

Main features:

- Discrete space-time
- Imaginary time

$$e^{-i\hat{H}t} \to e^{-\hat{H}\tau}$$

thermal physics!



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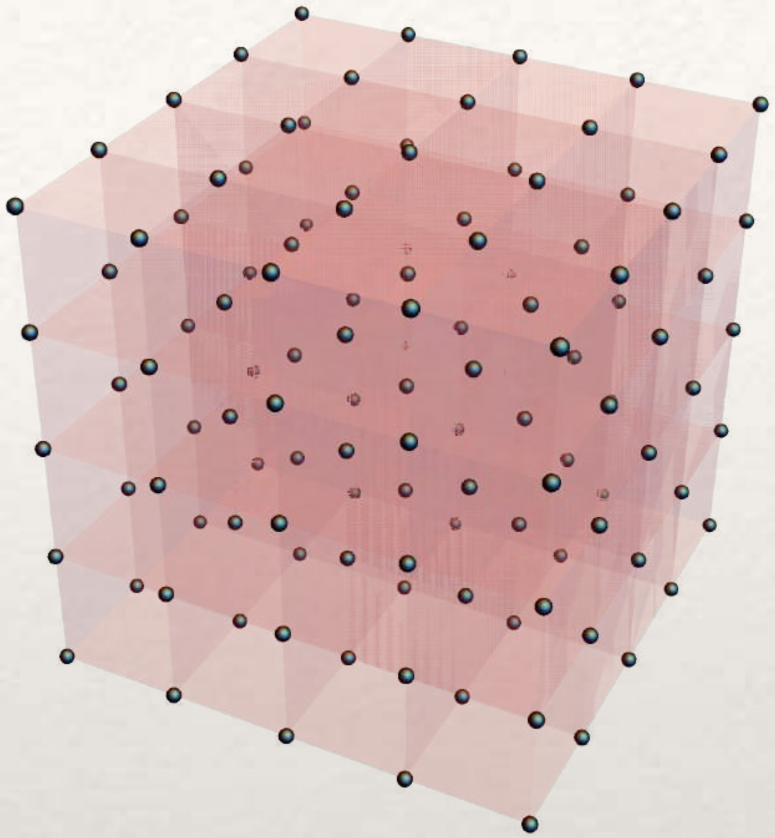


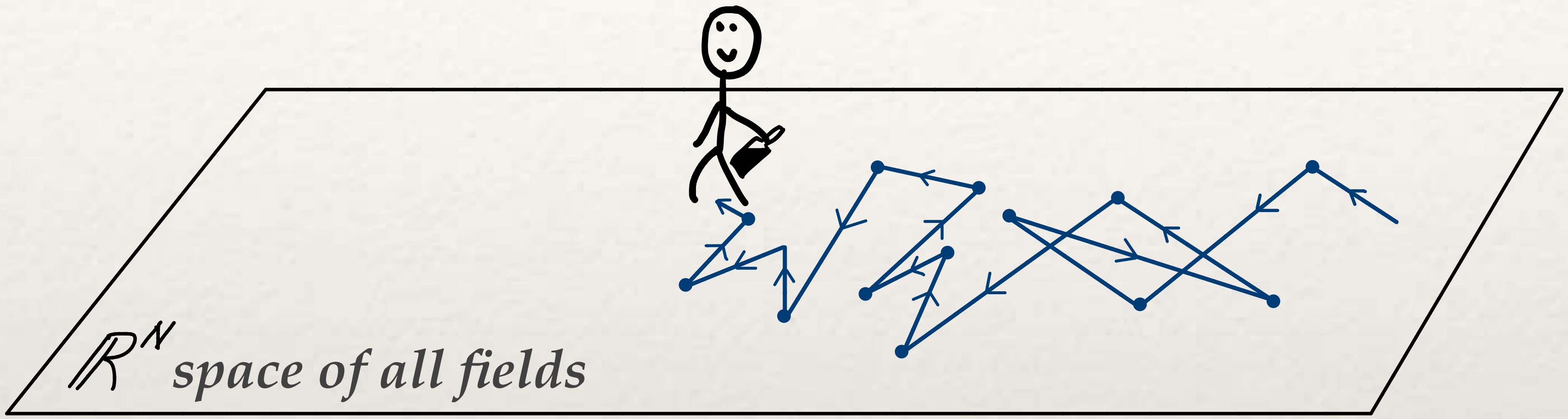
$$\langle \mathcal{O} \rangle = \int d\phi_1 \ldots d\phi_N \, e^{-S[\phi]} \mathcal{O}[\phi] = \mathrm{Tr}[e^{-\hat{H}/T} \hat{\mathcal{O}}]$$

finite *positive*

- importance of the field configuration ϕ: $e^{-S[\phi]}$

Importance sampling ("Monte-Carlo" method)

importance of the field configuration ϕ: $e^{-S[\phi]}$

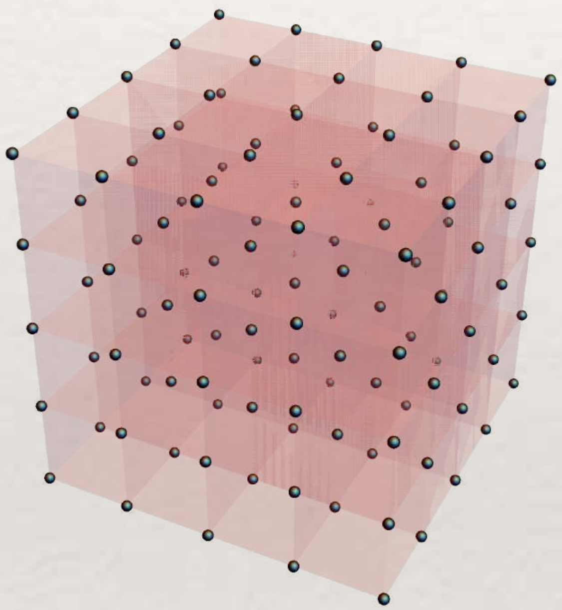


pick out the important (small action) configurations

path integral ~ statistical average with $P(\phi) \propto e^{-S[\phi]}$

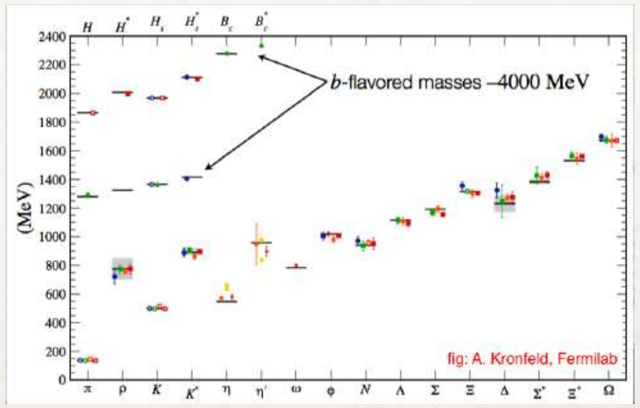
$$\langle \mathcal{O} \rangle \approx \frac{1}{\mathcal{N}} \sum_{a=1}^{\mathcal{N}} \mathcal{O}[\phi_a]$$

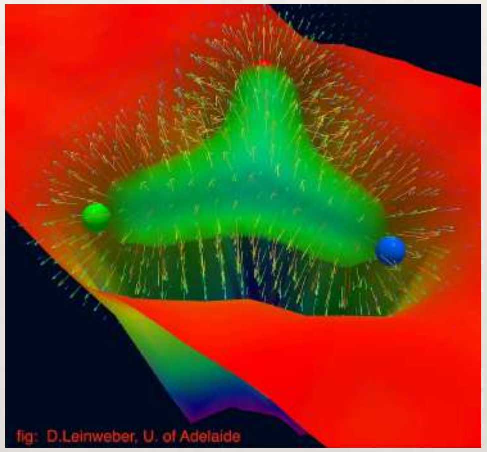
Lattice QCD

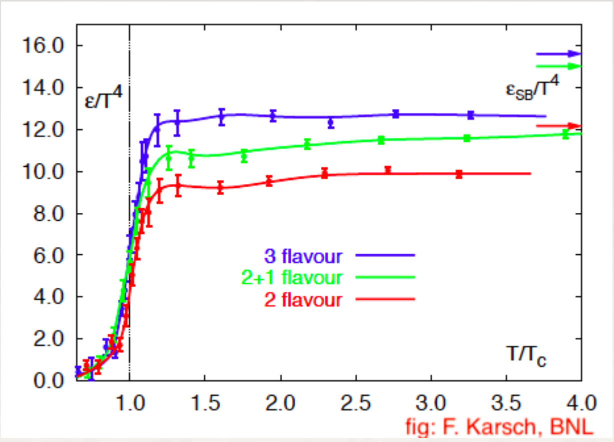


lattice

importance sampling (Monte-Carlo)







The sign problem

In a variety of problems of interest S is *complex*

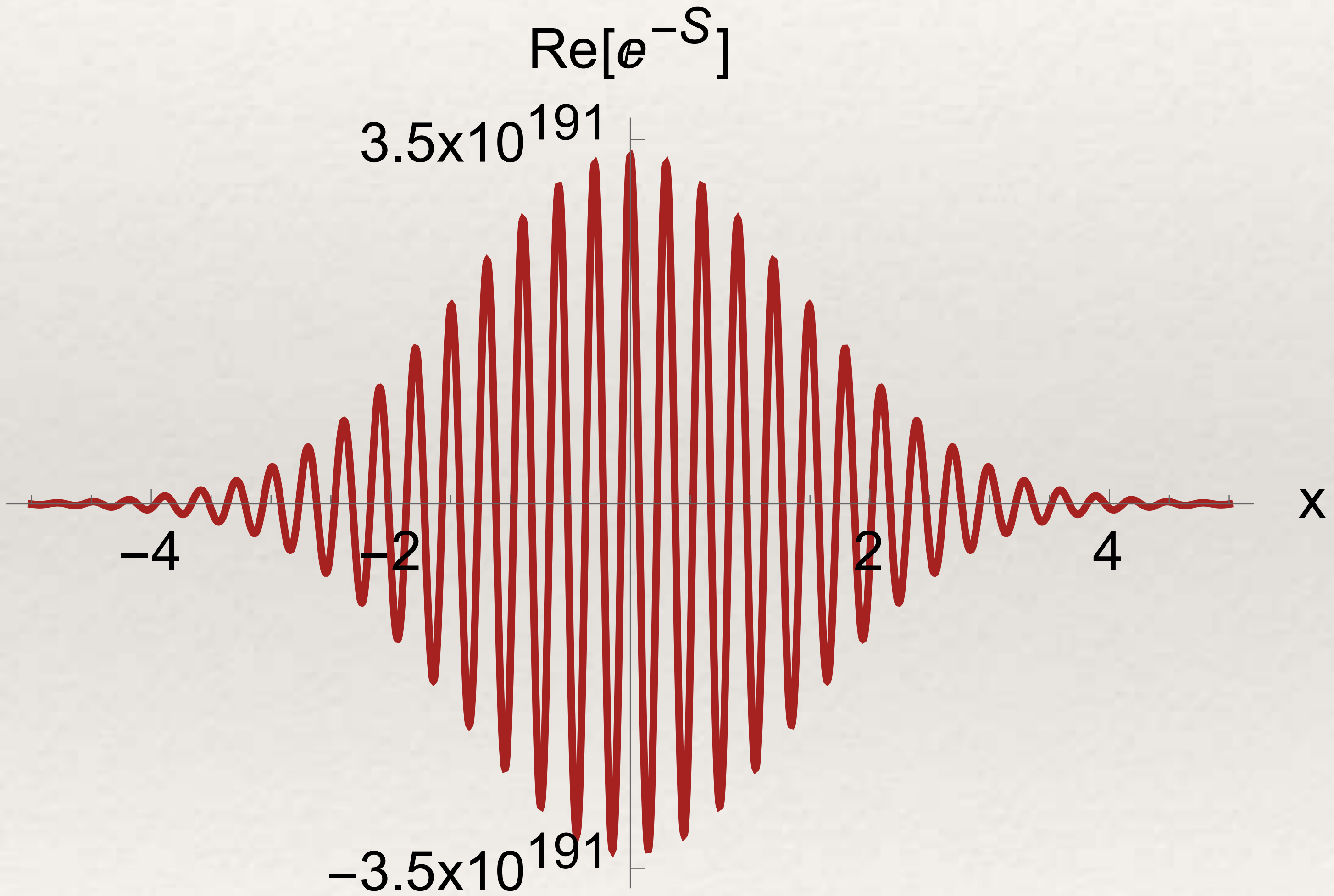
$e^{-S[\phi]}$ is not a probability distribution

- Most theories with finite density
- Hubbard model away from half filling
- Dynamical problems *(transport, out-of-equilibrium physics…)*
- QCD with nonzero $\boldsymbol{\theta}$ angle

$\vdots$

The sign problem

$$\int_{-\infty}^{\infty} e^{-(x+42i)^2} dx = 2\sqrt{\pi}$$

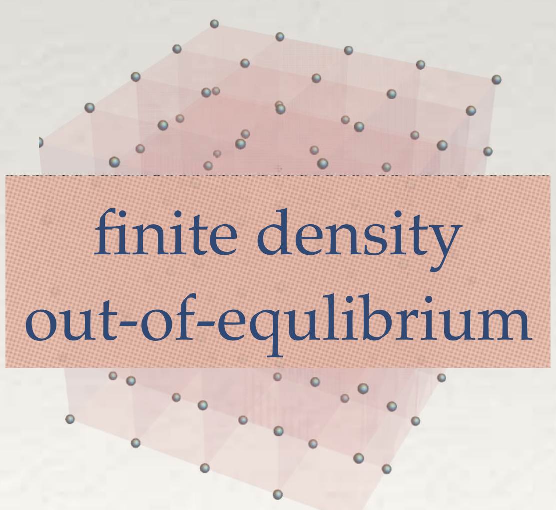


The sign problem

importance $\propto e^{-S_R[\phi]}$ *"reweighting"*

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{-iS_I[\phi]} \rangle_{S_R}}{\langle e^{-iS_I[\phi]} \rangle_{S_R}}$$

$\langle e^{-iS_I[\phi]} \rangle_{S_R} \propto e^{-volume/T}$ → need *exponentially large* resources



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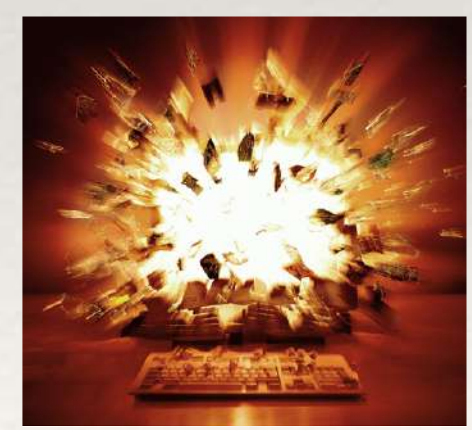

The sign problem

Ways around the sign problem

- Imaginary chemical potential
- Taylor series in μ
- Dual variables
- Fermion bags
- Complex Langevin
- Canonical partition function

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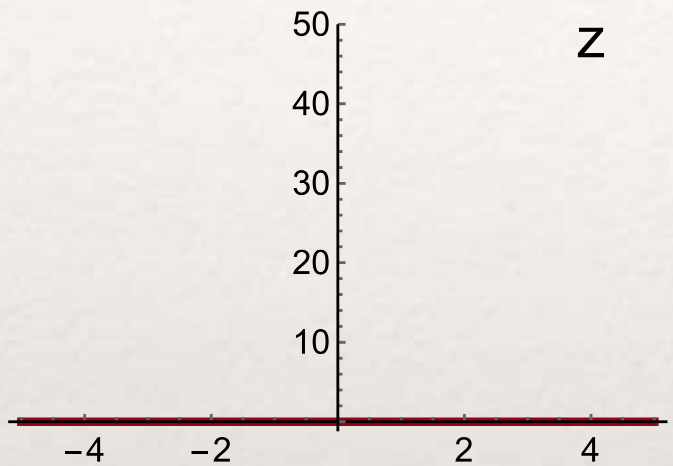
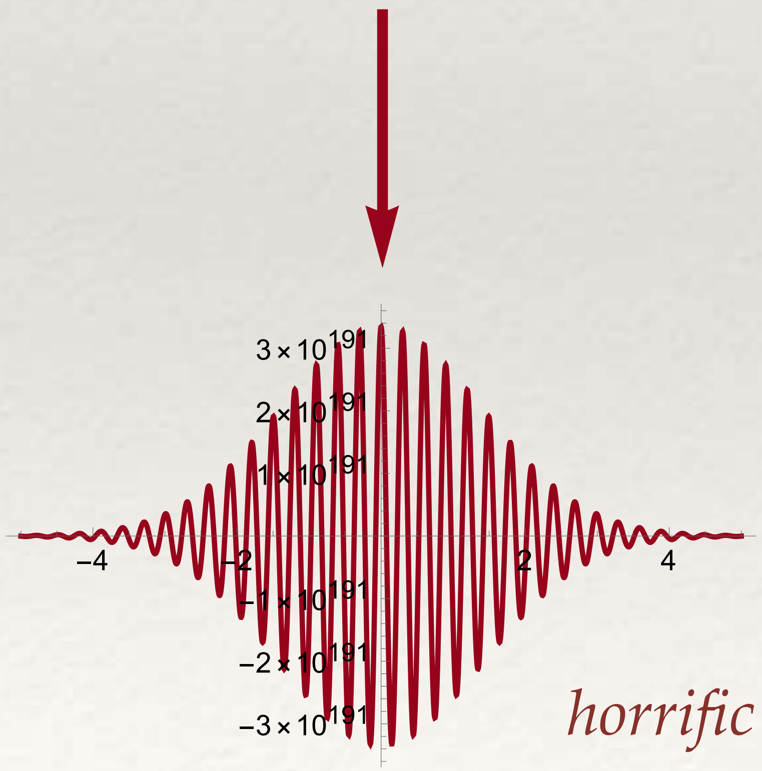
.

A complex way around the sign problem

$$\int_{-\infty}^{\infty} e^{-(x+42i)^2} dx = 2\sqrt{\pi}$$

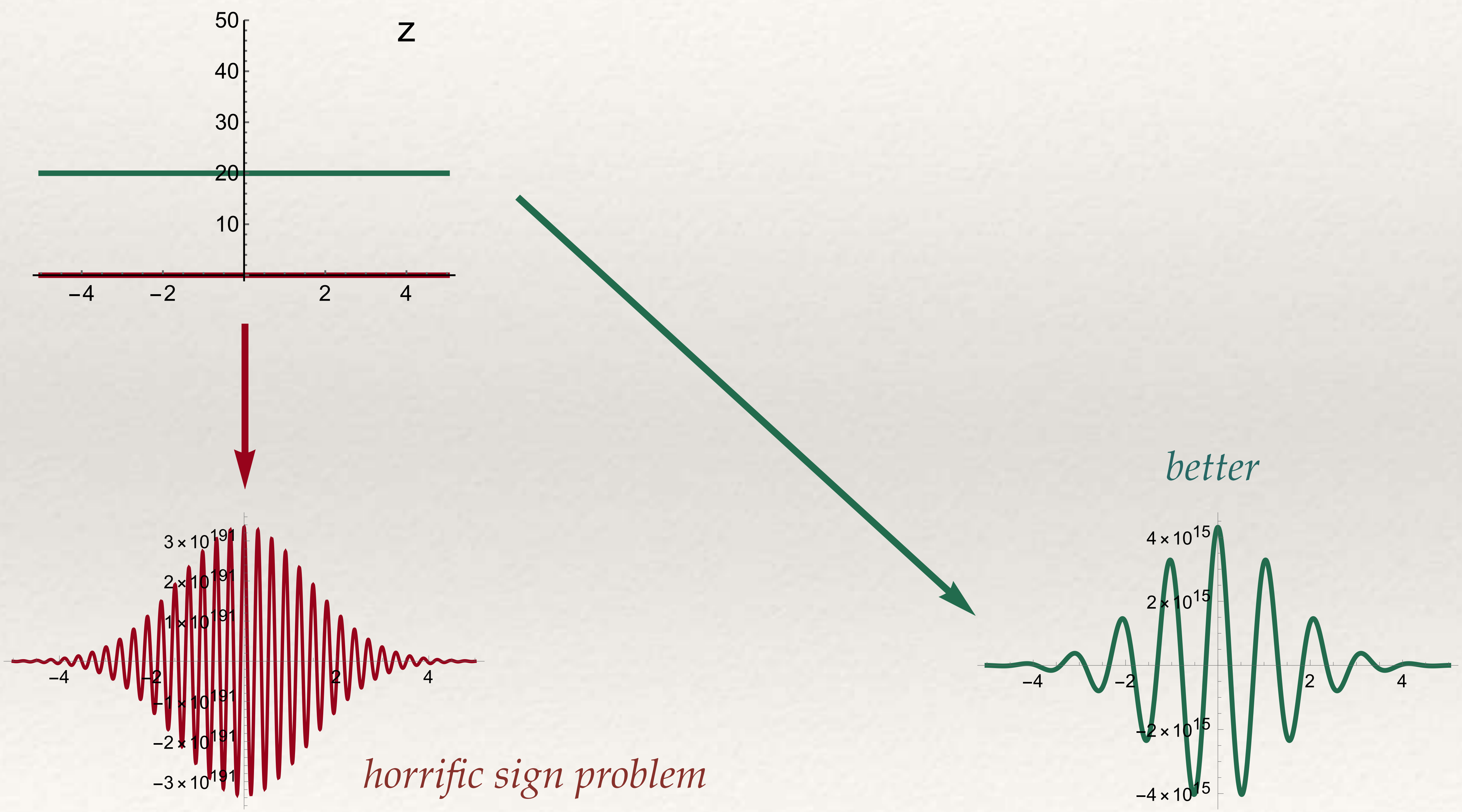




horrific sign problem

A complex way around sign problem

$$\int_{\mathscr{C}} e^{-(z+42i)^2} dz = 2\sqrt{\pi}$$



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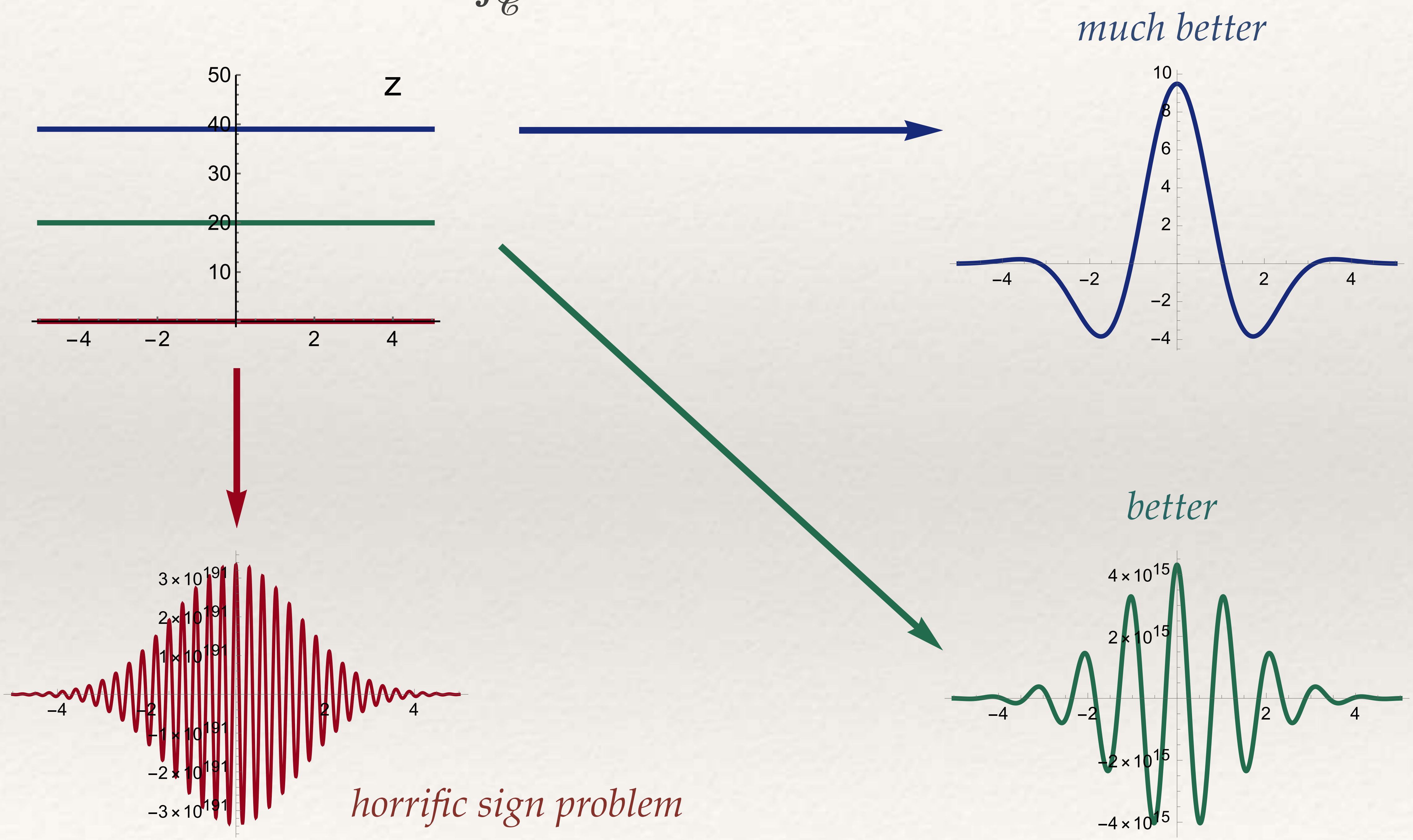


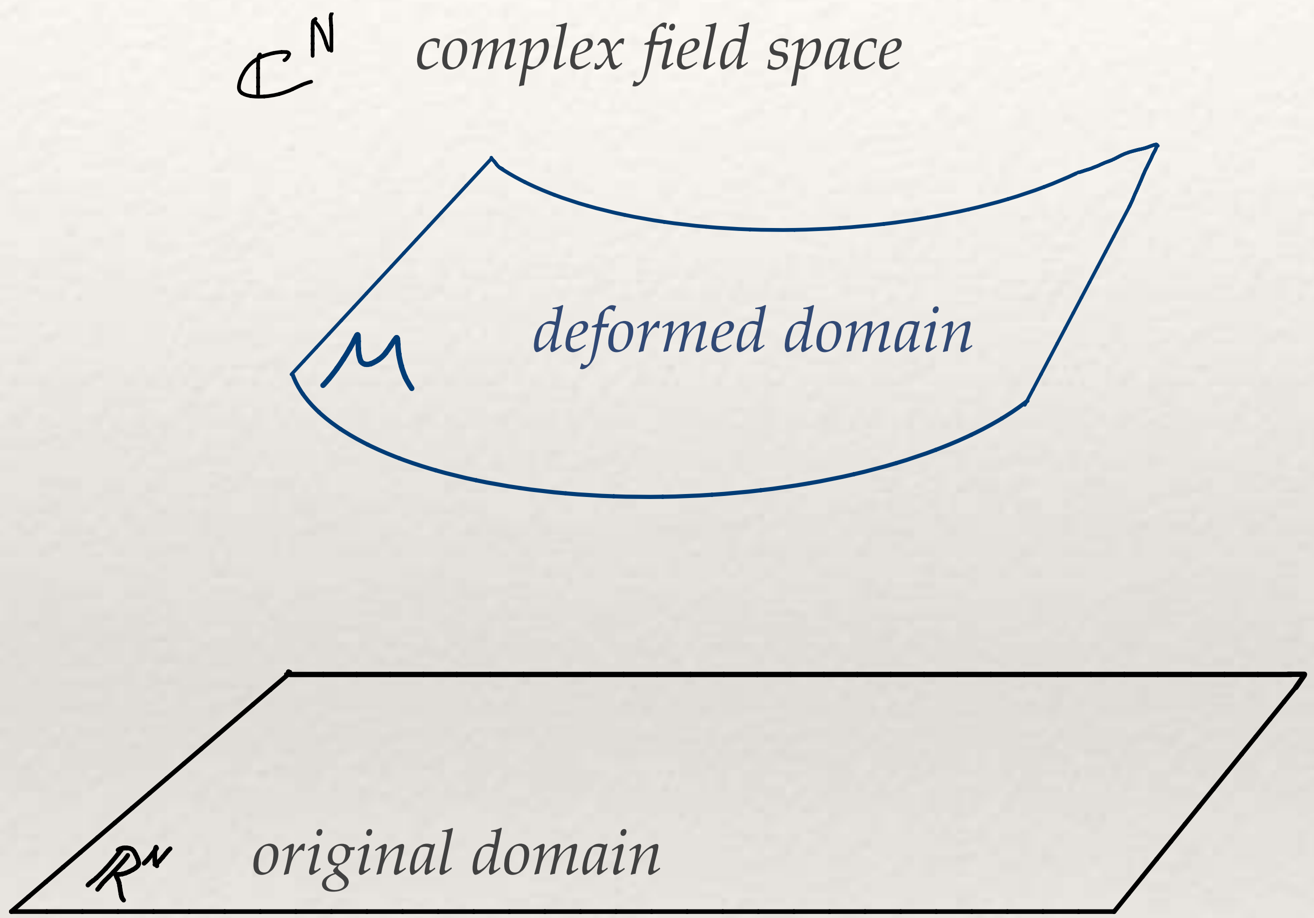
The main idea: deform the QFT path integral domain to a better one in complex field space where the sign problem is mild.

Review article : "Complex paths around the sign problem"
[Alexandru, GB, Bedaque, Warrington]
coming soon…

[also work by Cristoforetti, Di Renzo et al, Fujii et al., Tanizaki et al.,…]

Mathematical origins: Picard-Lefschetz theory
[Pham, Fedoryuk, Witten, ….]

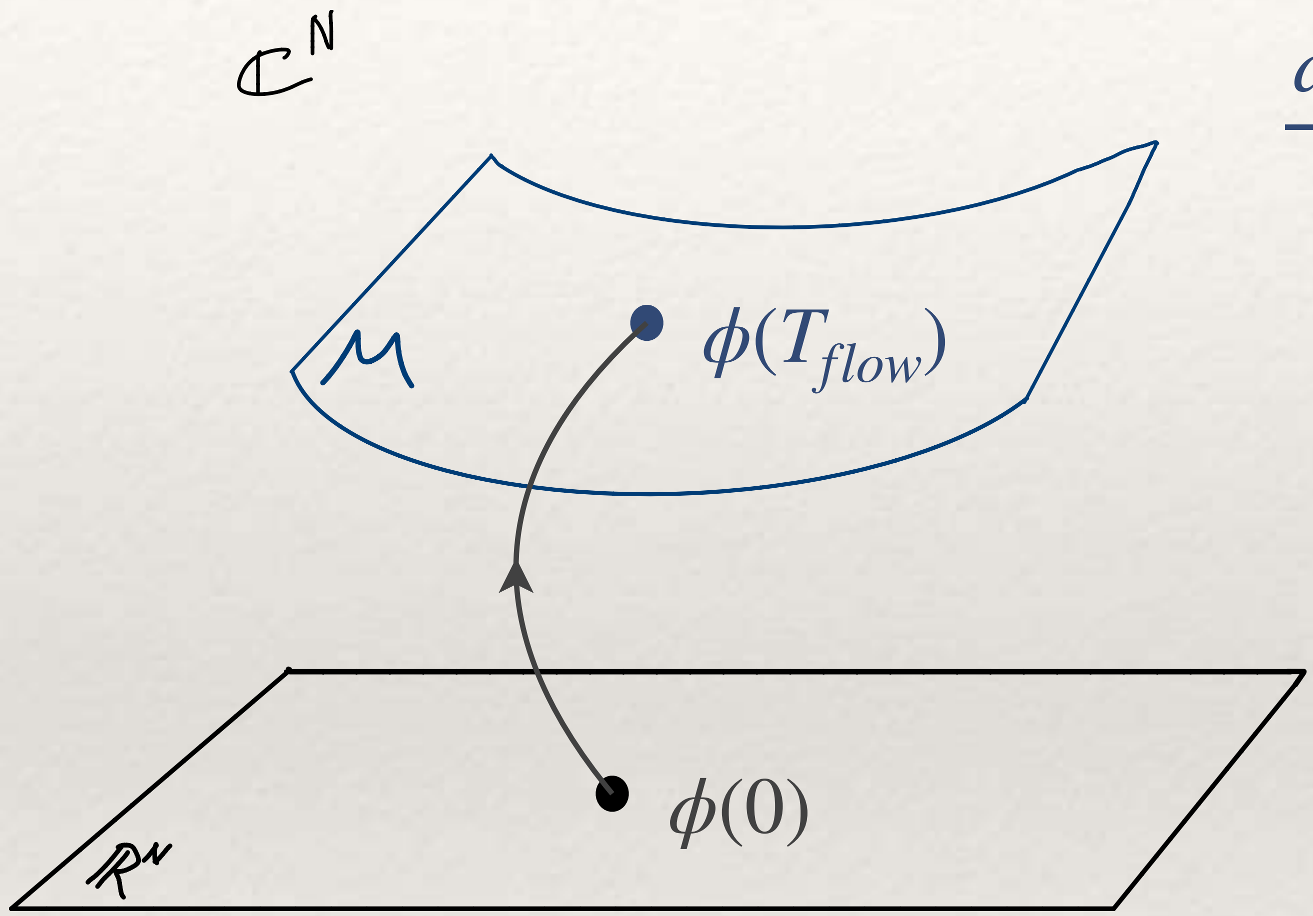
Good deformations



- path integral on $\mathcal{M}$ = path integral on $\mathbb{R}^N$ ("allowed")
- sign problem on $\mathcal{M}$ << sign problem on $\mathbb{R}^N$ ("good")

Good deformations

follow an equation of motion, ``holomorphic gradient flow"



$$\frac{d\phi(\tau)}{d\tau} = \overline{\frac{\partial S[\phi]}{\partial \phi}}$$

- path integral on $\mathscr{M}$ = path integral on $\mathbb{R}^N$ ("allowed") ✓
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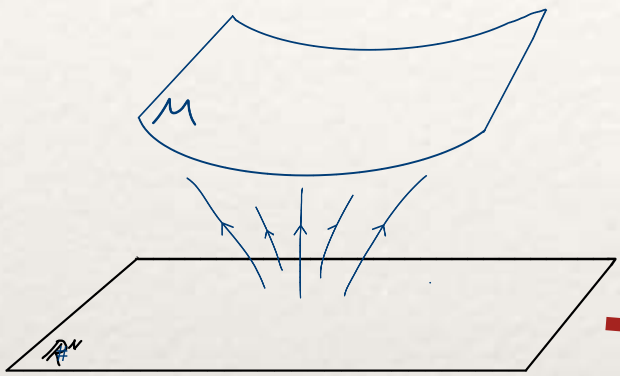
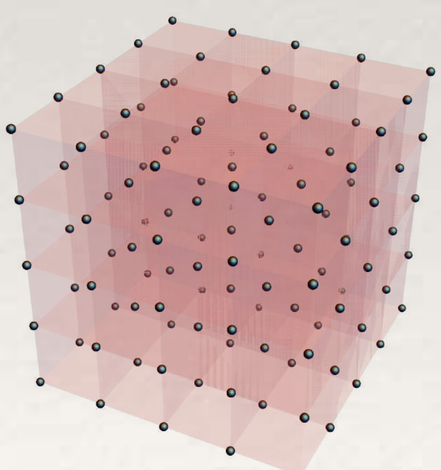
Strategy

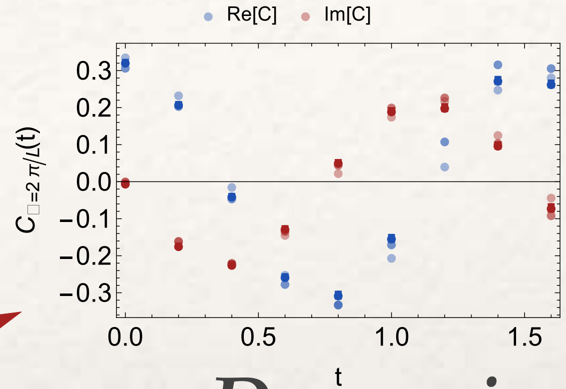
deformation



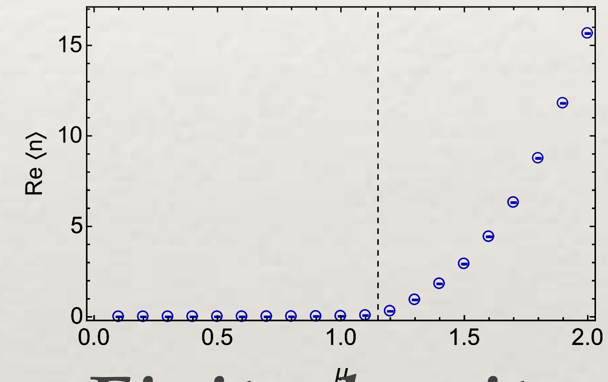


discretization

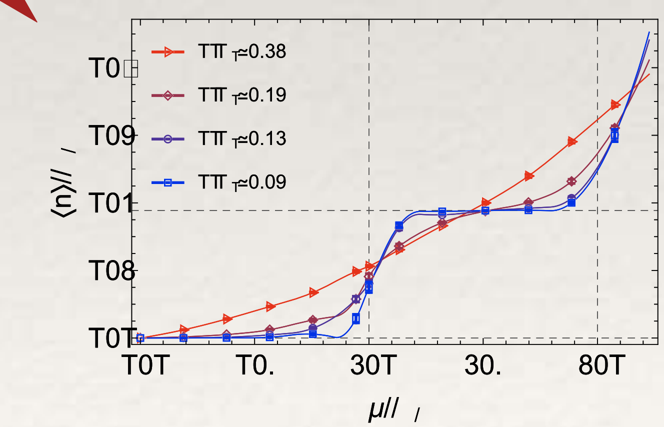
importance sampling



Dynamics ✓



Finite density ✓



Real time dynamics

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \frac{1}{Z}\int [d\phi] e^{\frac{i}{\hbar}S[\phi]}\mathcal{O}(t)\mathcal{O}(0)$$

transport (viscosity, conductivity),
out-of equilibrium physics…

$e^{\frac{i}{\hbar}S[\phi]}$ leads to quantum interference

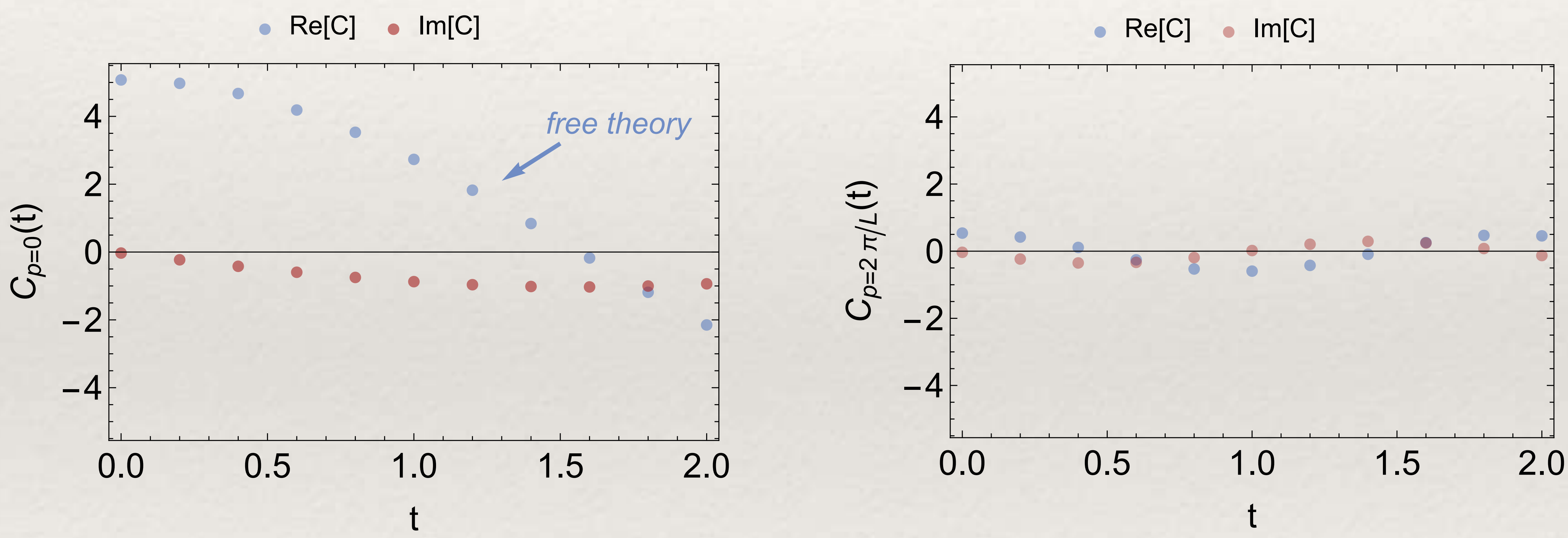
…and the ultimate sign problem

$$\langle e^{-iS_I[\phi]}\rangle_{S_R} = 0$$

Real time dynamics - 1+1d QFT

interacting Bose gas: $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

free theory $\lambda=0$



$$C_p(t) = \langle \phi(t,p)\phi(0,p) \rangle_\beta$$

[Alexandru, GB, Bedaque, Ridgway, Vartak, Warrington, PRL 117081602, PRD 95 114501]

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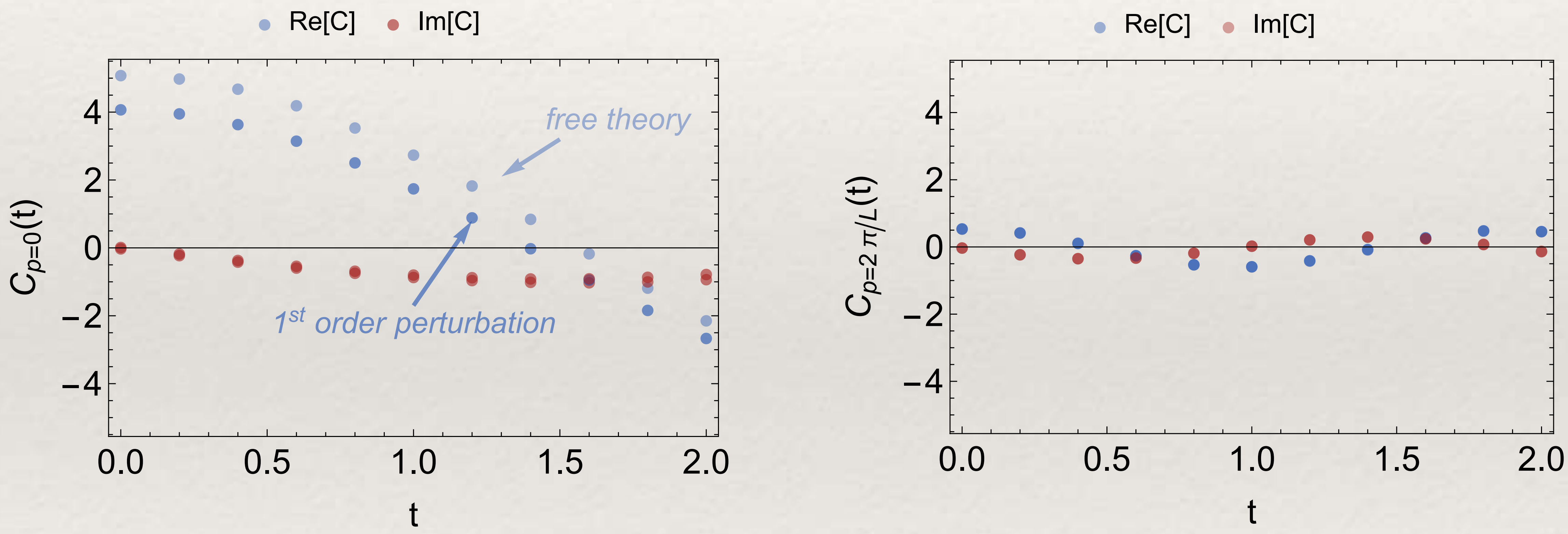
weak coupling λ=0.1



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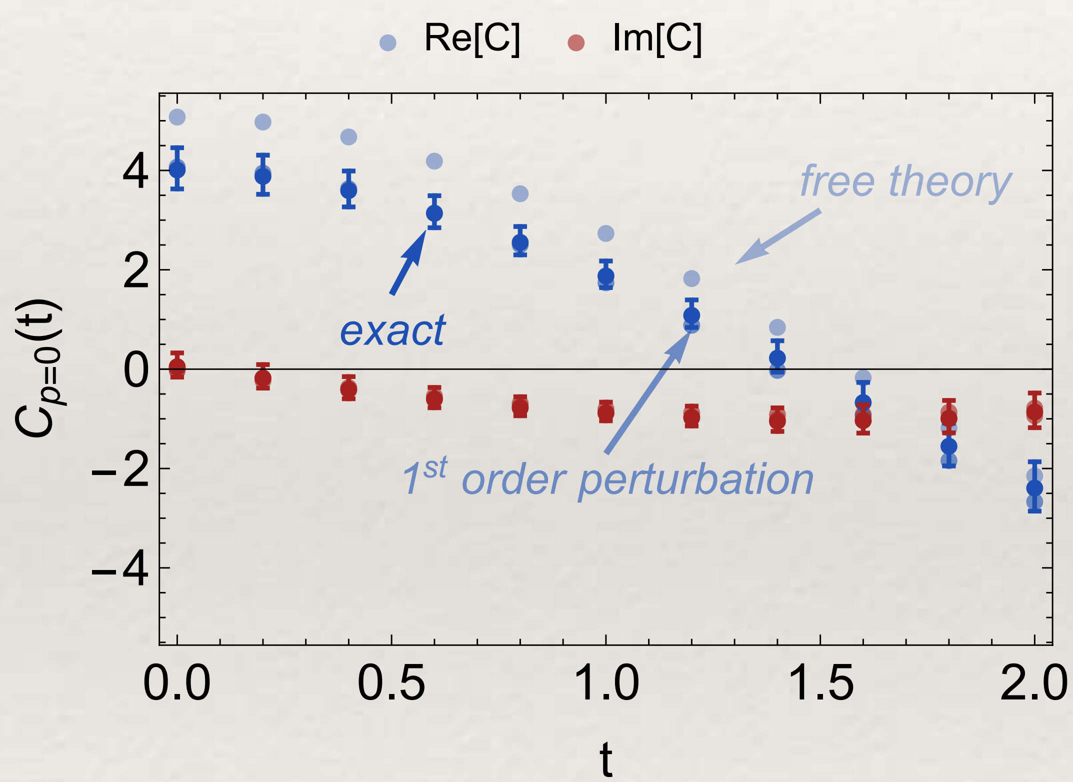

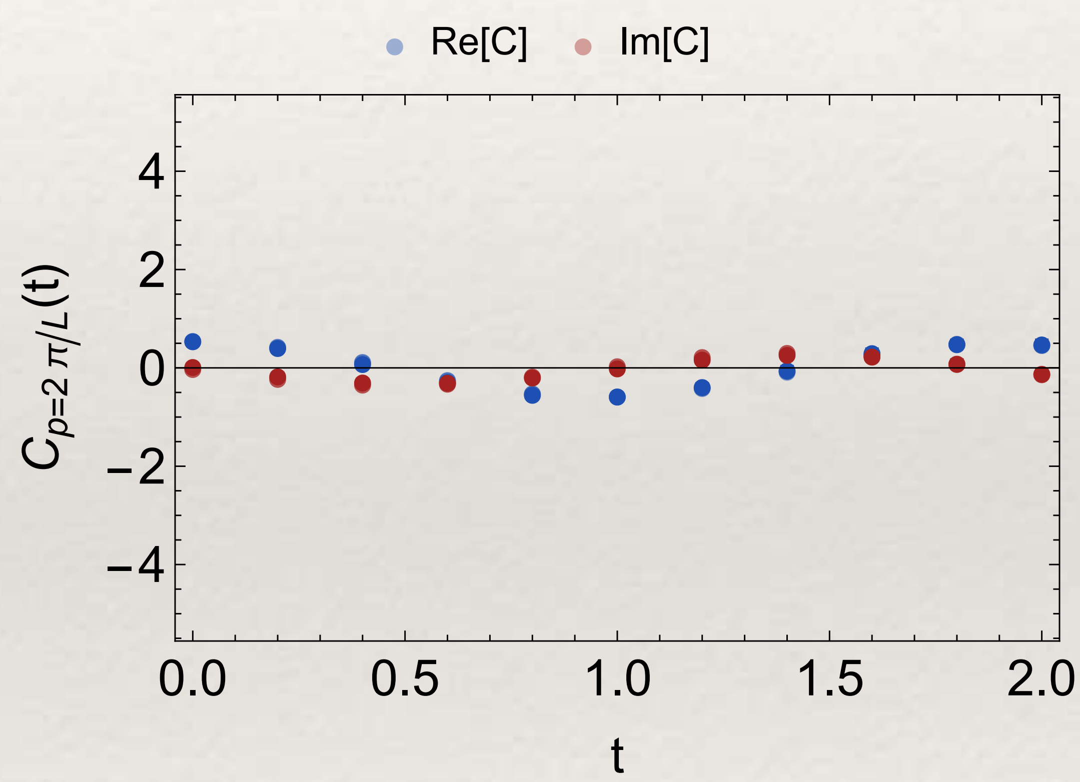

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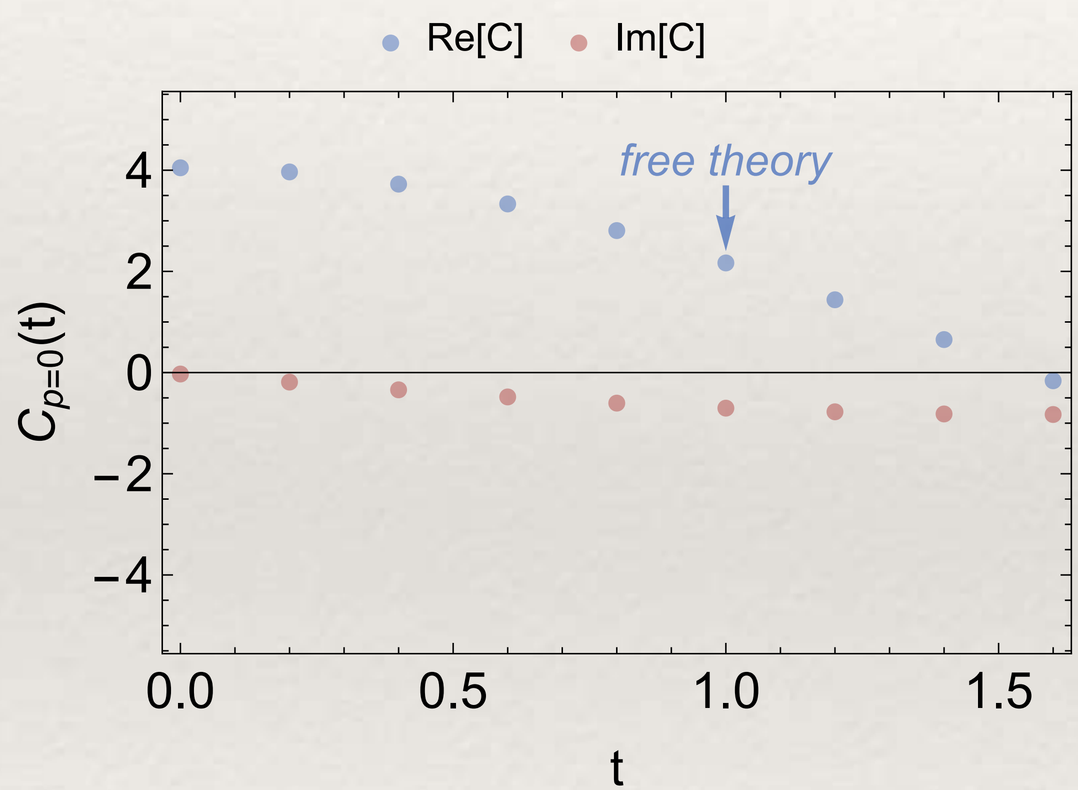

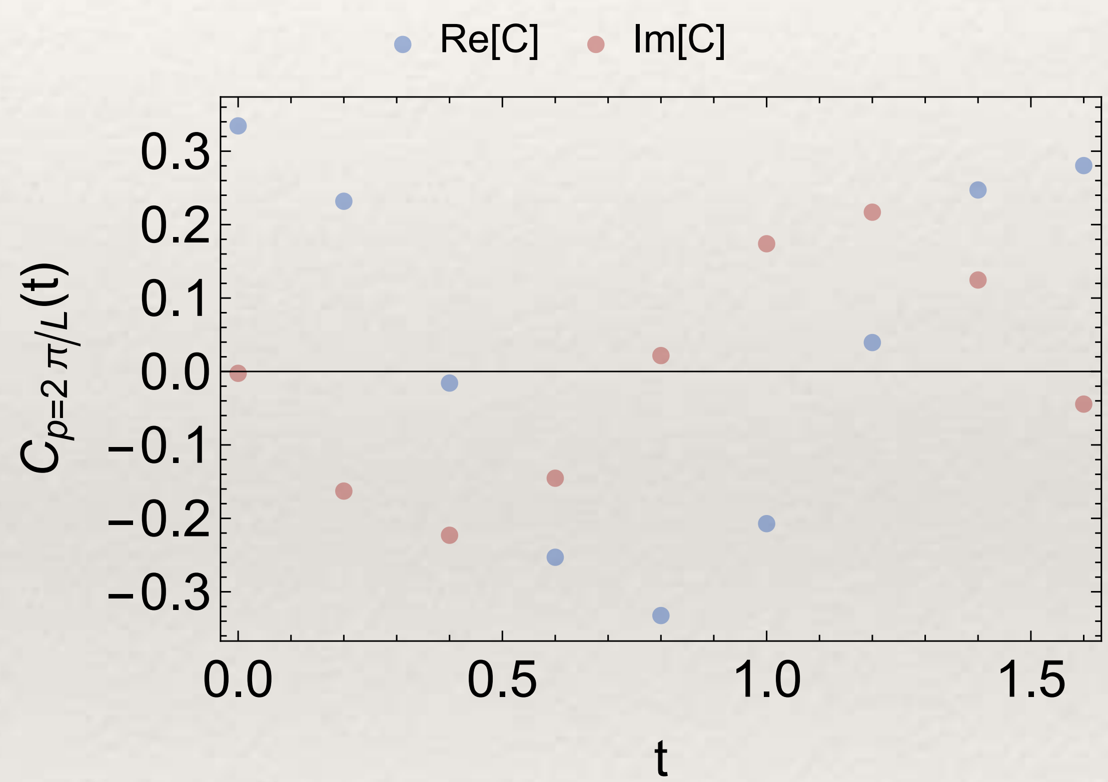

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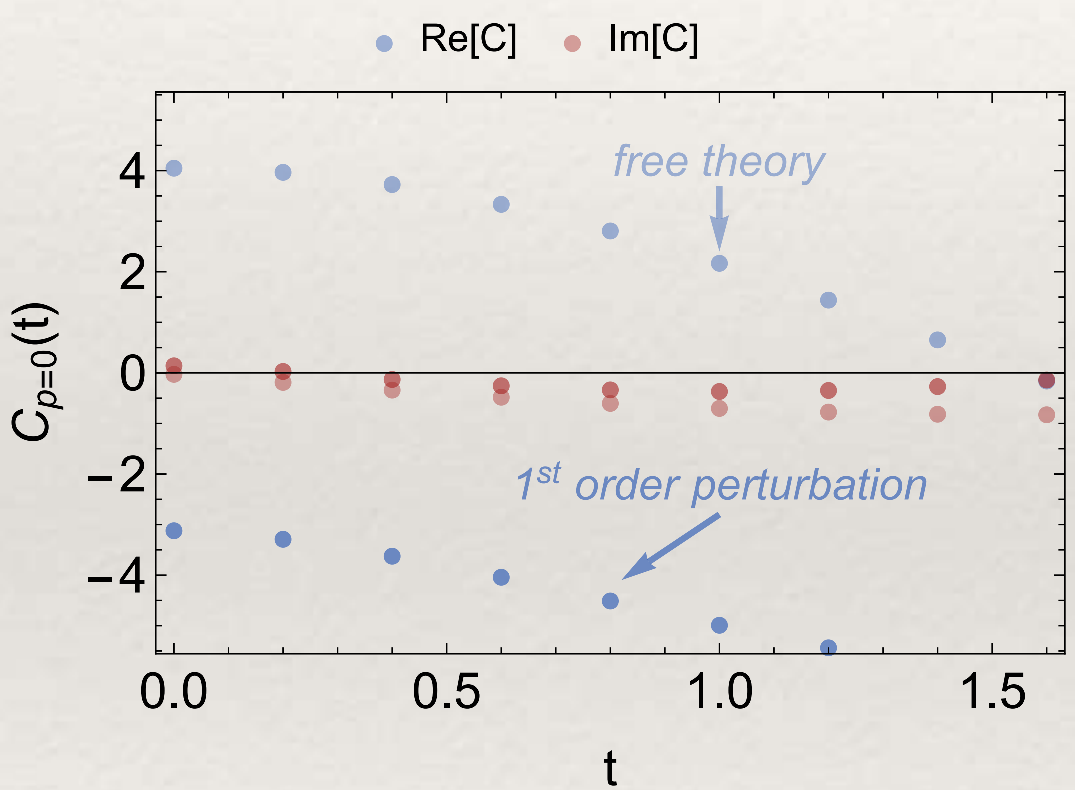

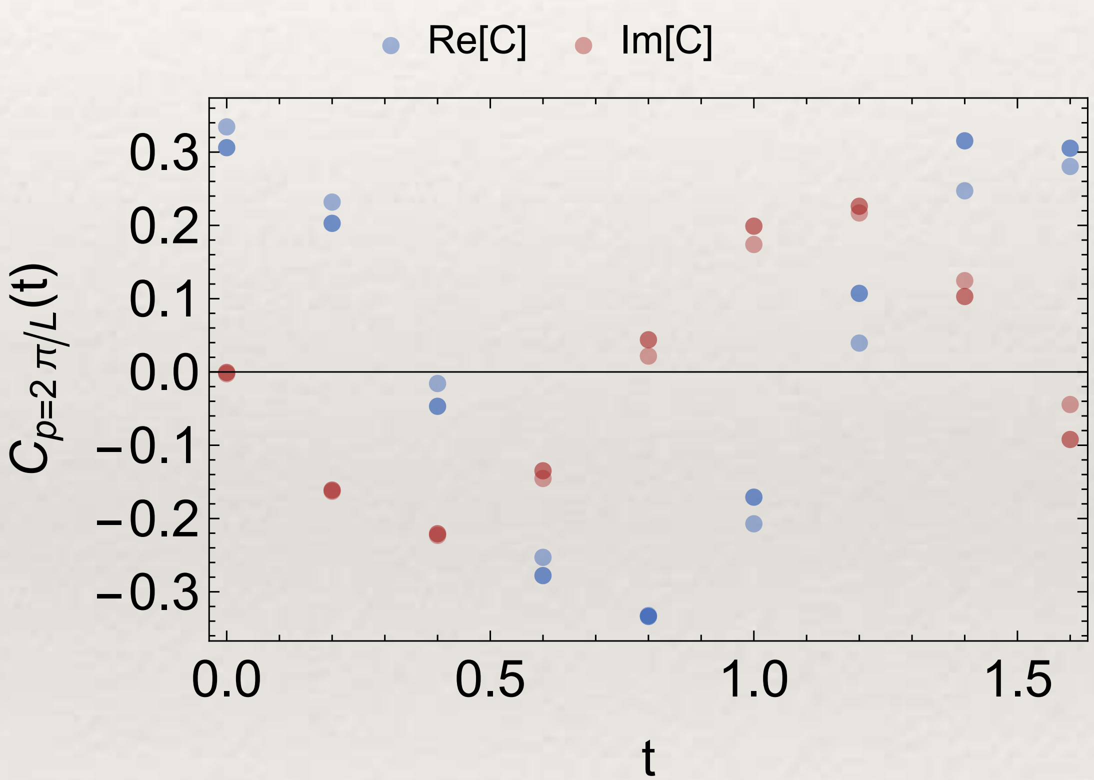

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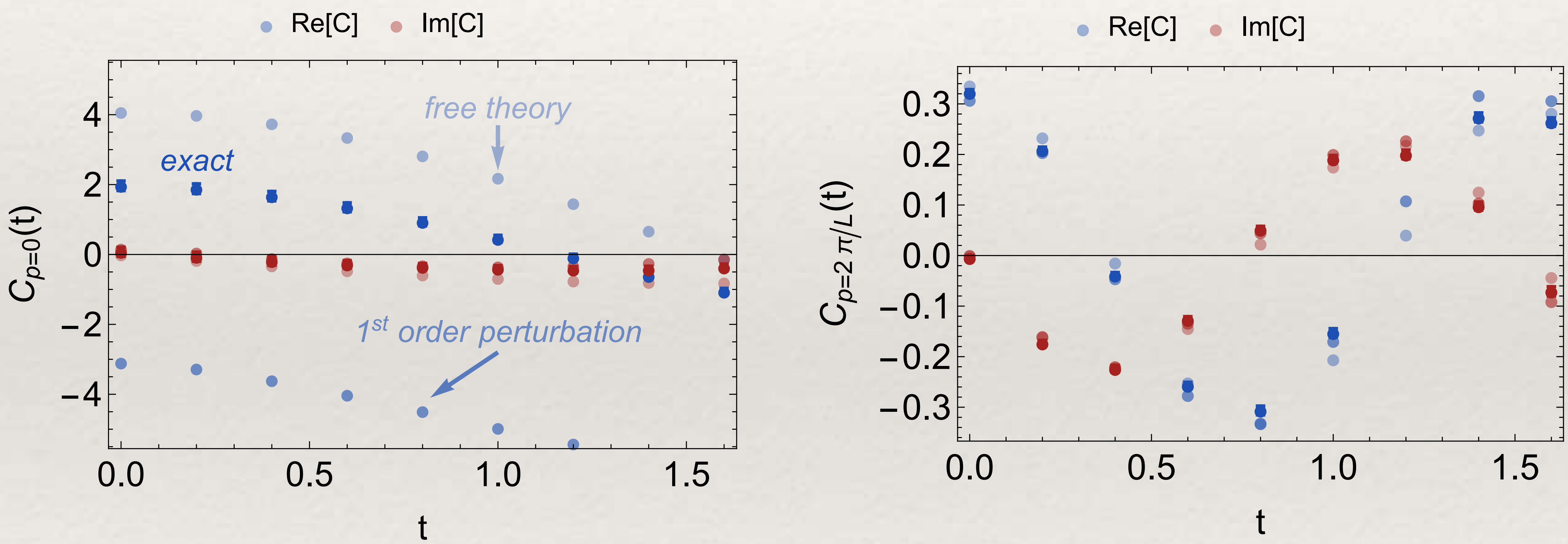

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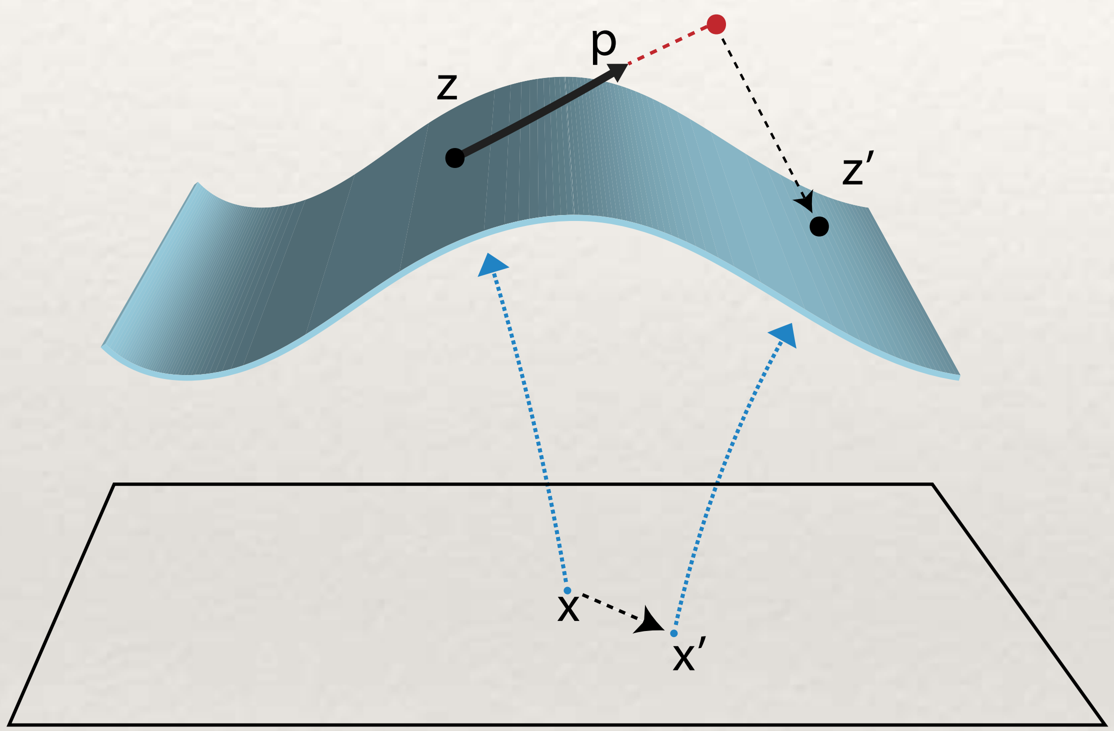
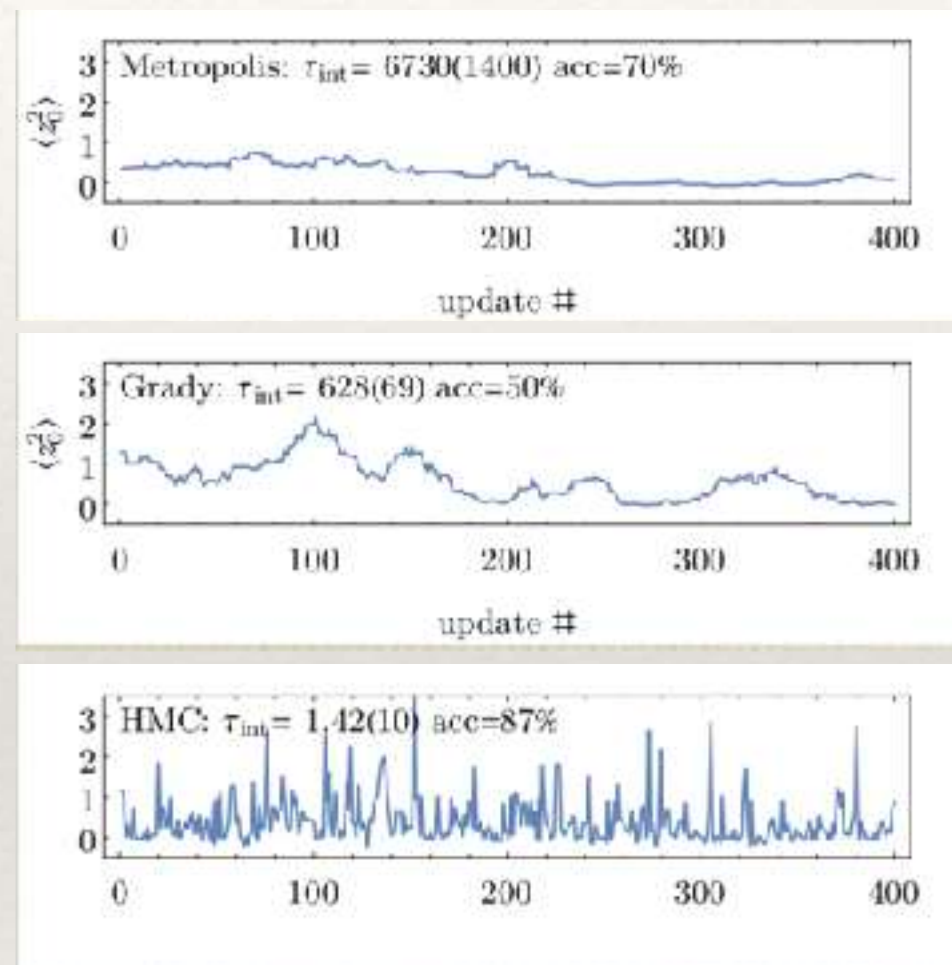
[see also follow-up by Mou, Saffin, Tranberg, '18]

Real time dynamics -Hybrid Monte Carlo

Case Study : 0+1 d anharmonic oscillator $\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$





in progress

[also *(finite density)* Fujii, Honda, Kato, Kikukawa, Komatsu, Sano, JHEP 10 (2013) 147 01]

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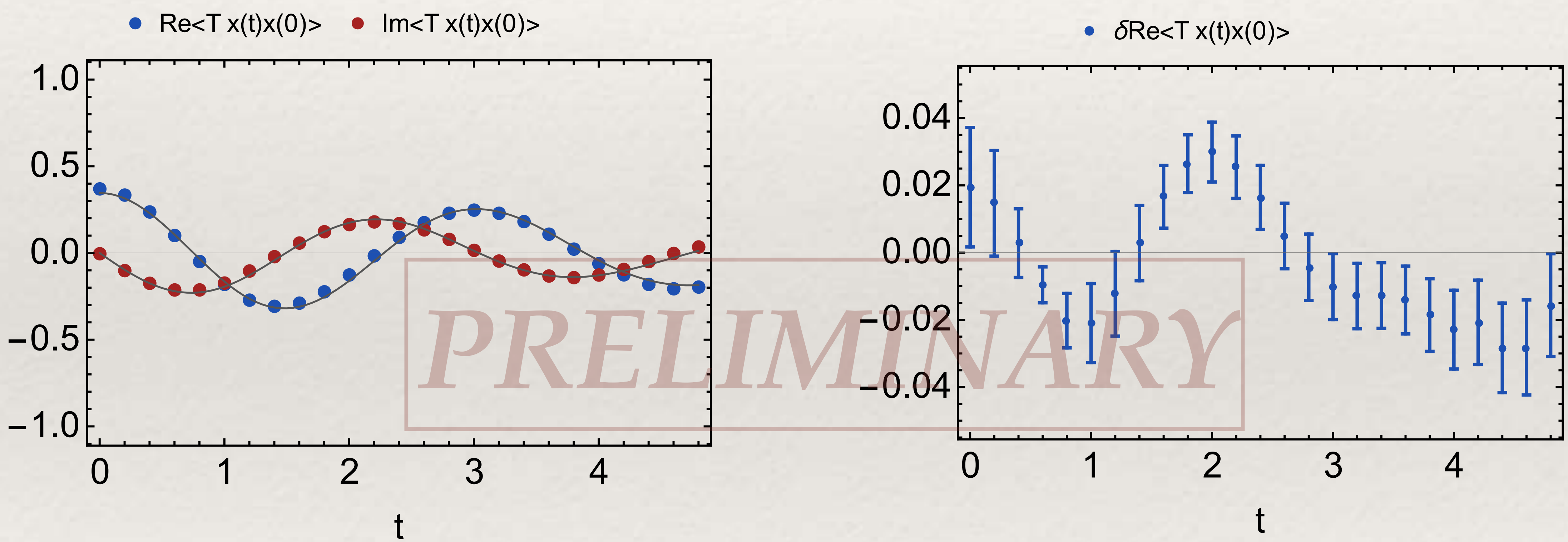
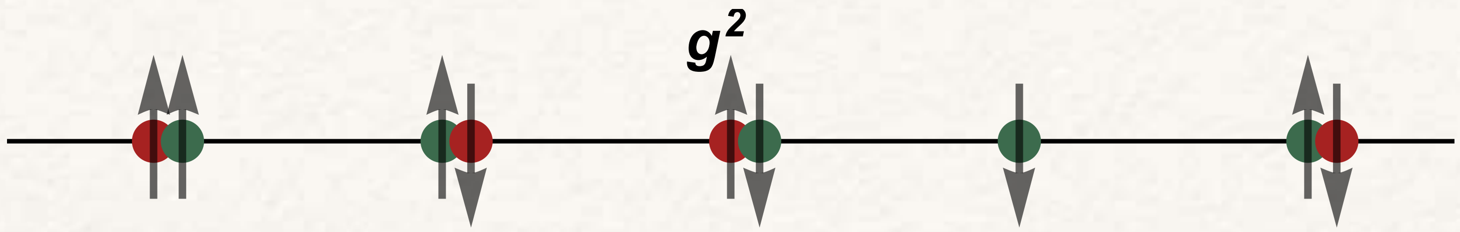
$$N_t = 24, \quad N_\beta = 4, \quad \lambda = 24$$



in progress

Many body physics - 2d Thirring model



chain of interacting fermions

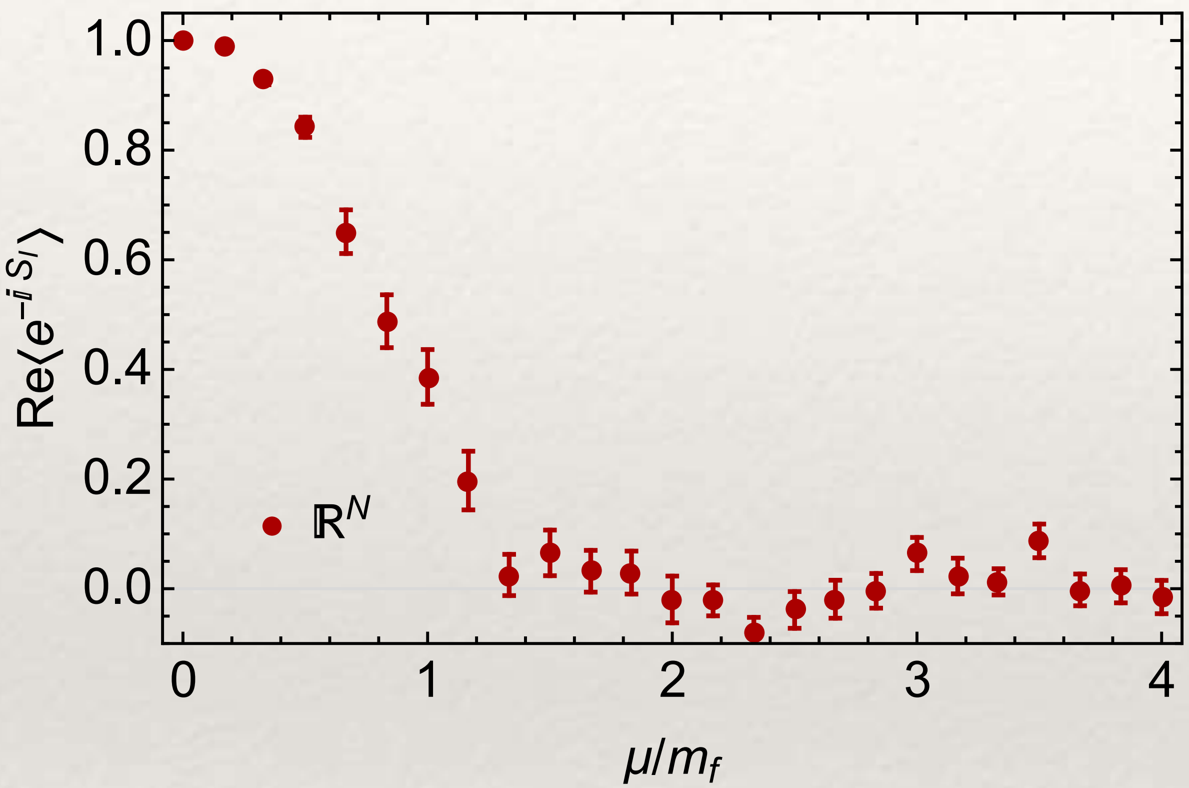
$$S = \int d^2x \bar{\psi}^a \left(\gamma^\mu \partial_\mu + m + \mu\gamma^0\right)\psi^a + \frac{g^2}{2N_f}(\psi^a\gamma^\mu\psi^a)(\psi^b\gamma_\mu\psi^b)$$

$$\rightarrow \frac{N_f}{2g^2}\int d^2x A^\mu A_\mu + \mathrm{tr}\log(\partial\!\!\!/ + A\!\!\!/ + \mu\gamma_0 + m)$$

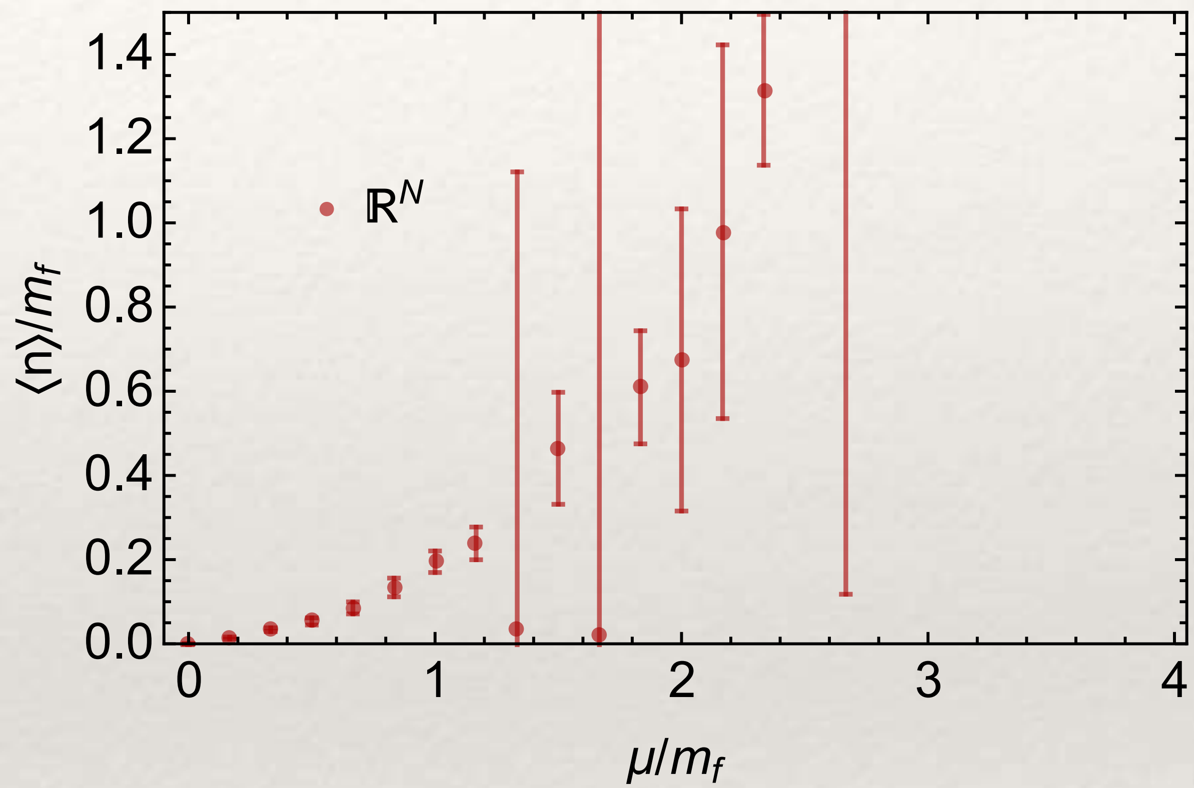
- a prototype of QCD
  asymptotically free, sign problem at finite density
- a 2d cousin of the Hubbard model

[Alexandru, GB, Bedaque, Ridgway, Warrington, Phys. Rev. D95, 014502]

Many body physics - 2d Thirring model



sign problem



equation of state

[Alexandru, GB, Bedaque, Ridgway, Warrington, Phys. Rev. D95, 014502]

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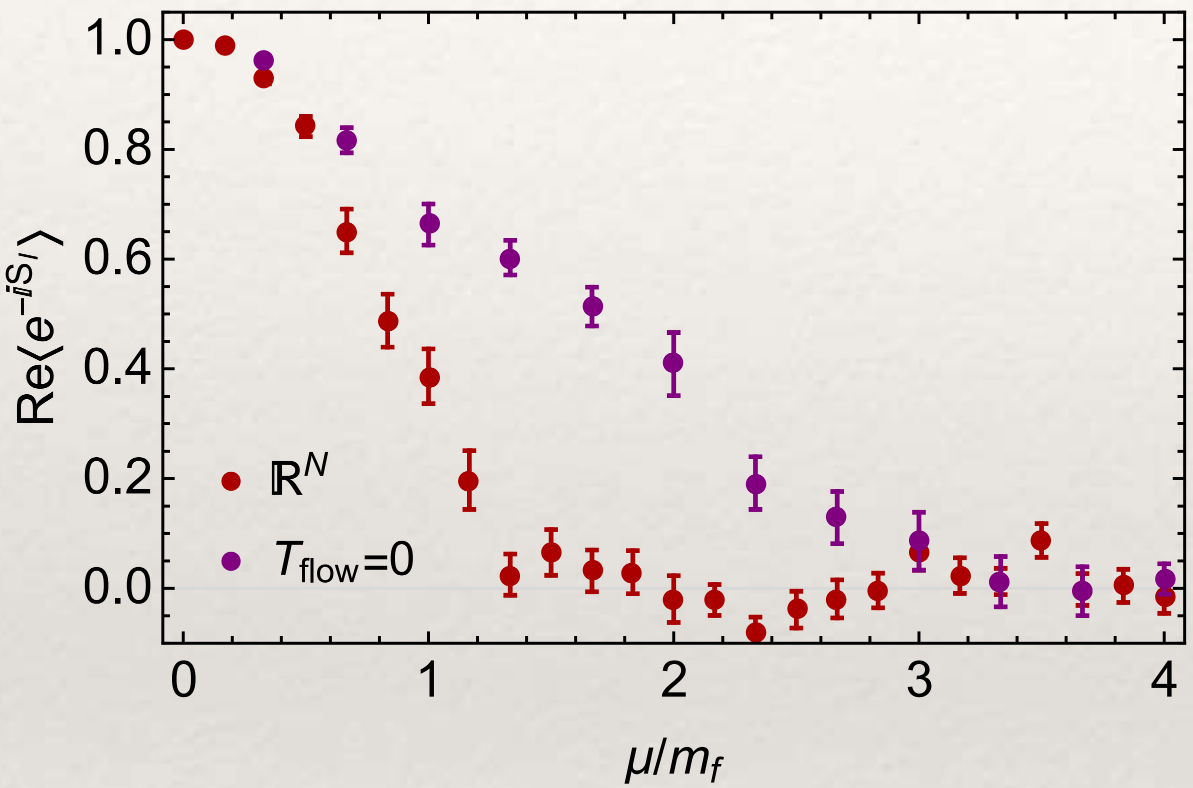
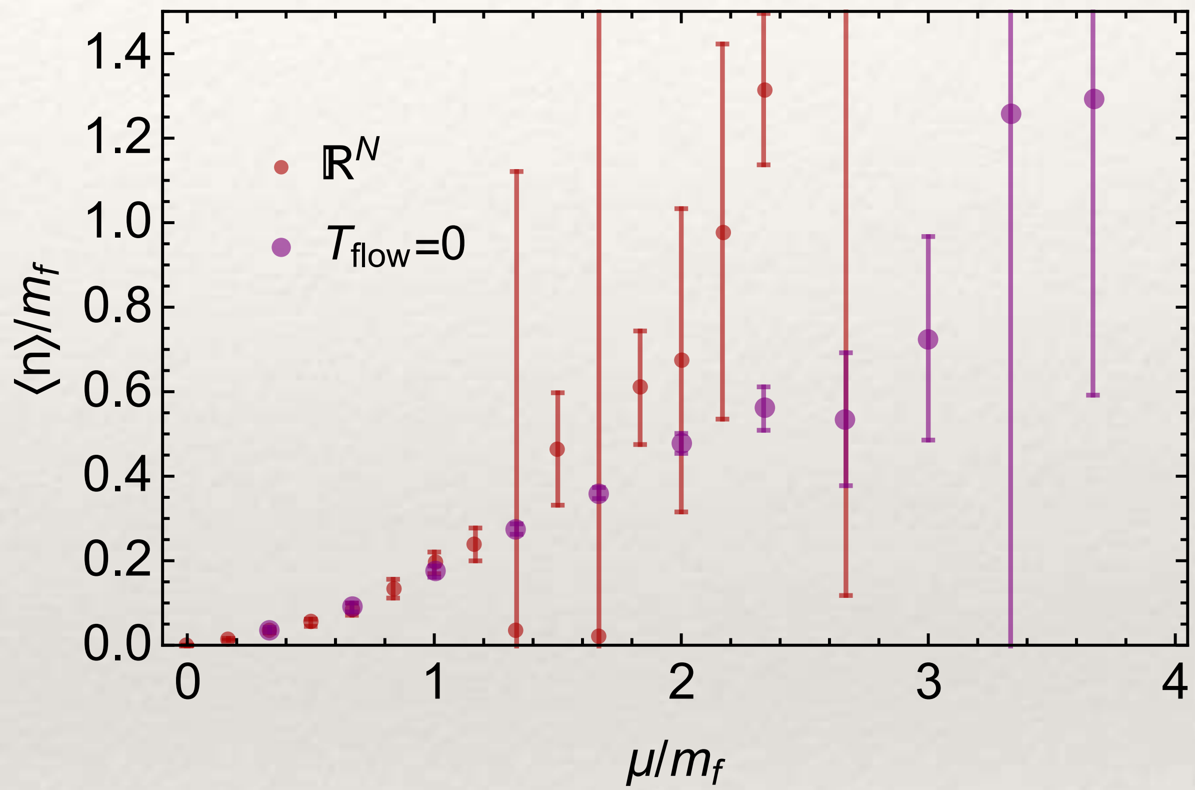


sign problem



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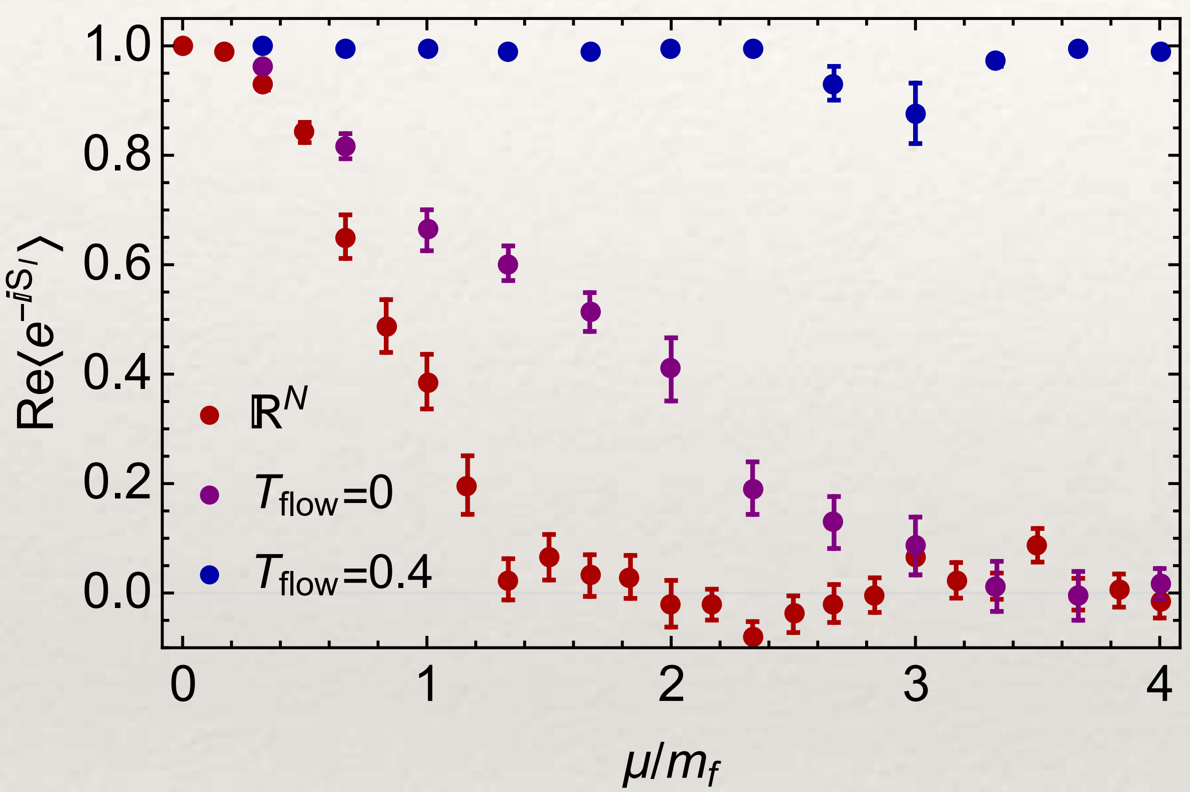
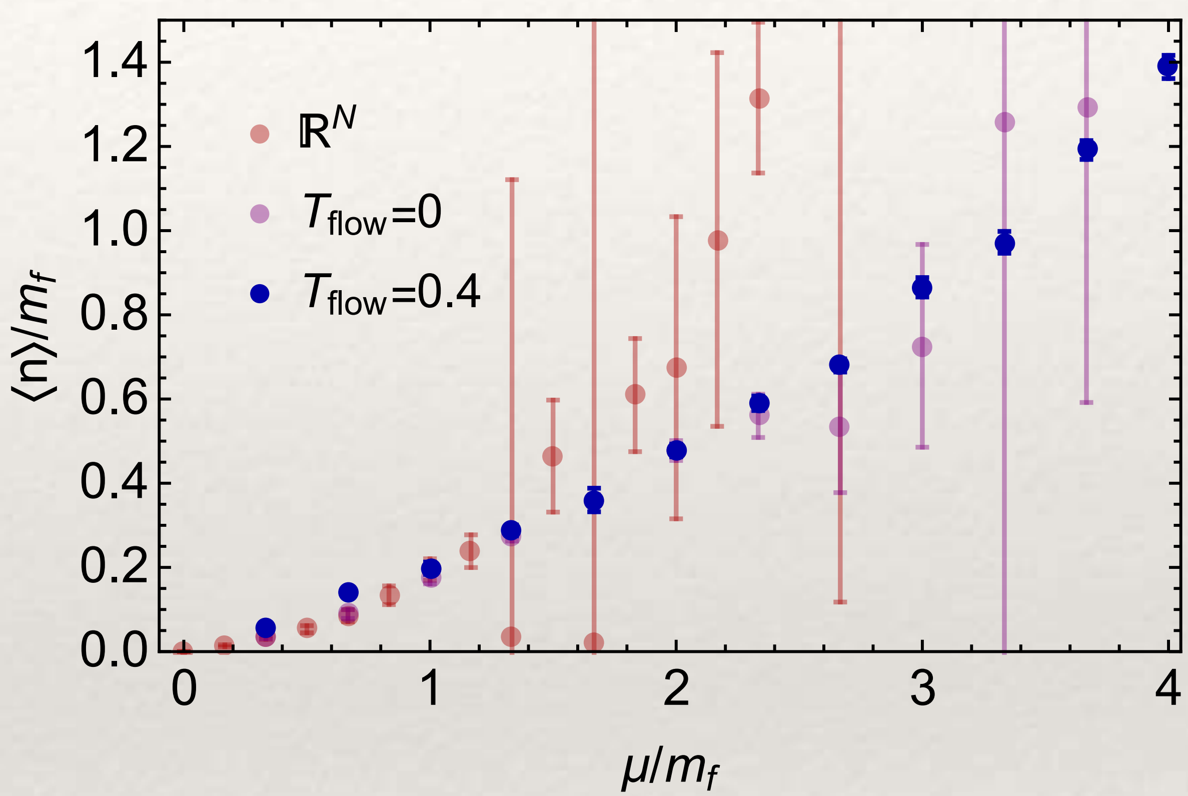


sign problem



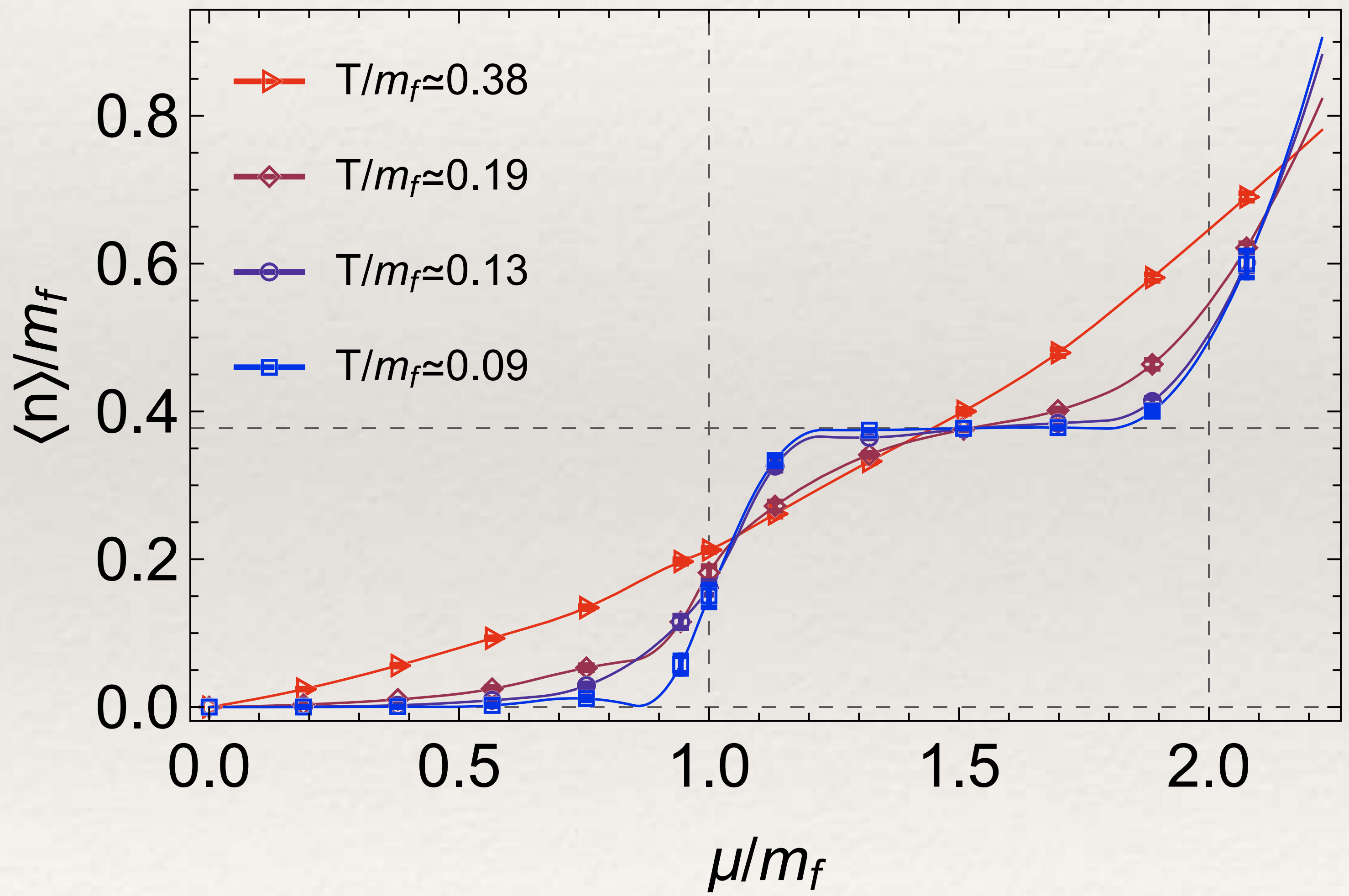
equation of state

[Alexandru, GB, Bedaque, Ridgway, Warrington, Phys. Rev. D95, 014502]

Many body physics - 2d Thirring model

Equation of state: low temperature limit

particularly bad sign problem: $\langle e^{-iS_I[\phi]} \rangle_{S_R} \propto e^{-volume/T}$

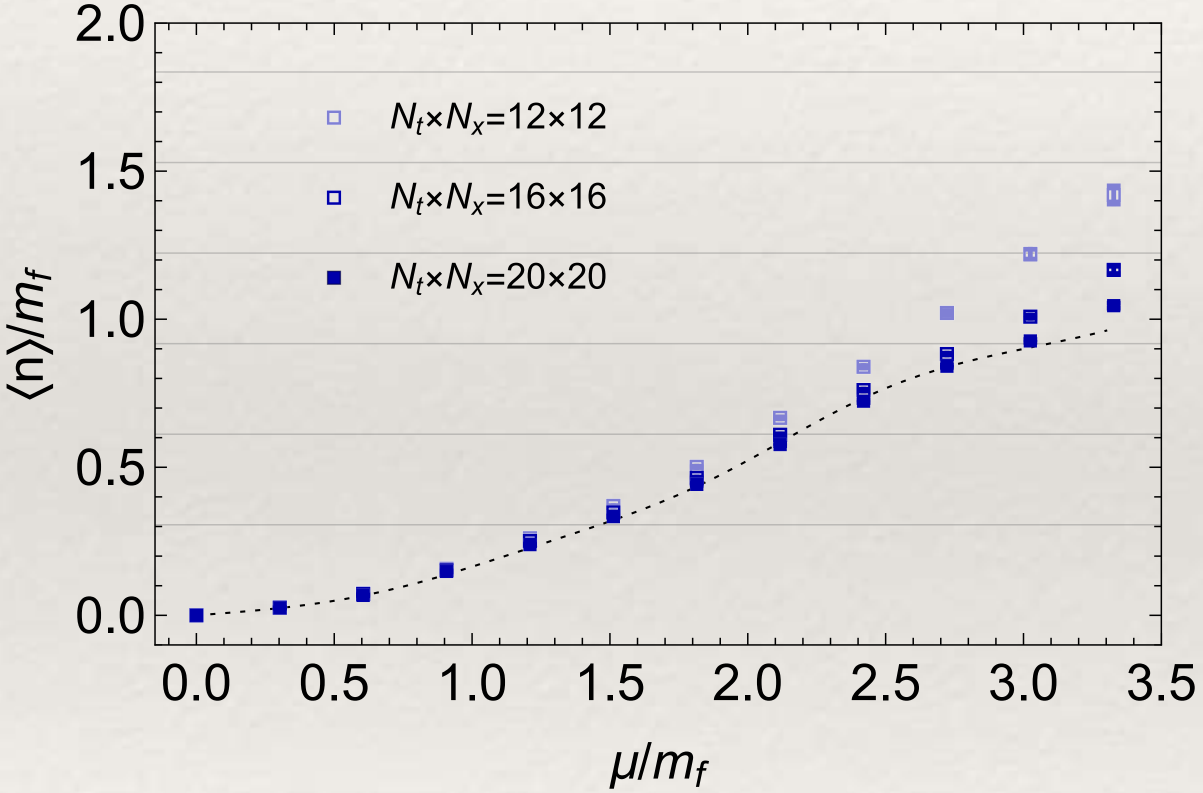


[Alexandru, GB, Bedaque, Ridgway, Warrington, Phys. Rev. D95, 014502]

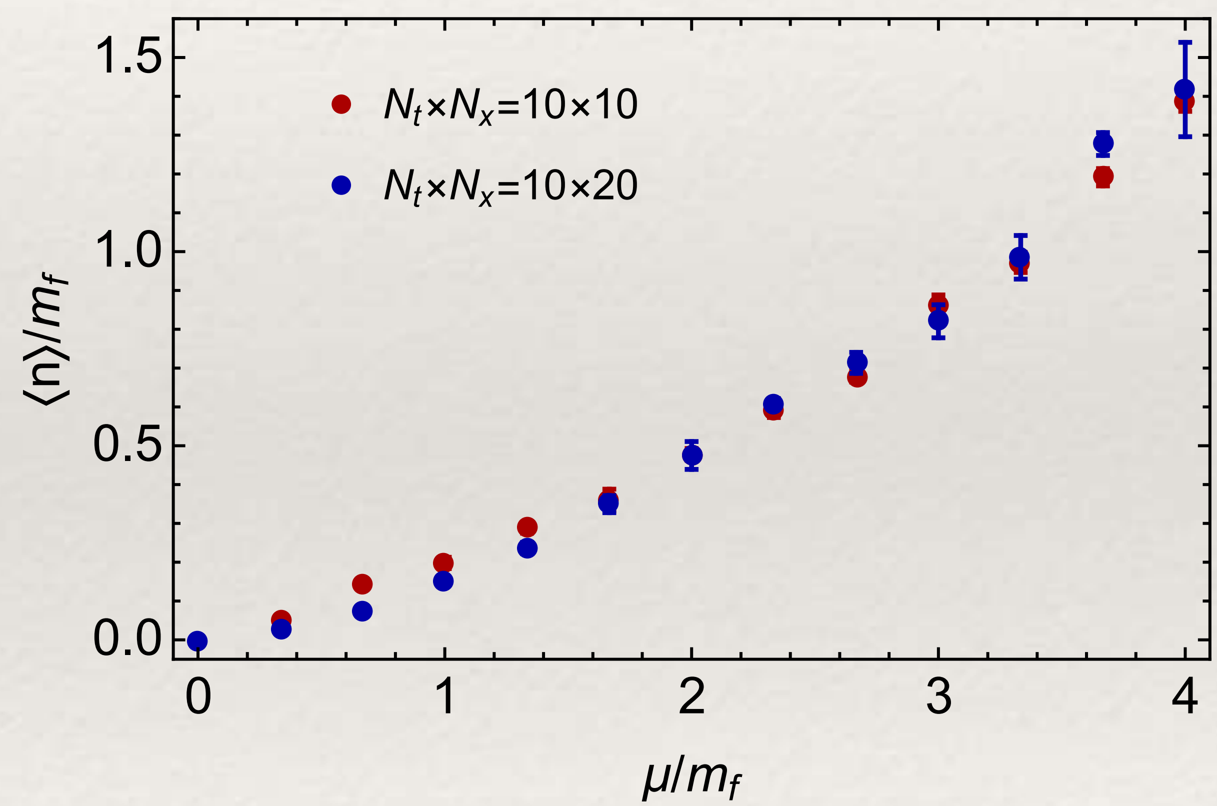
Many body physics - 2d Thirring model

Equation of state

continuum limit

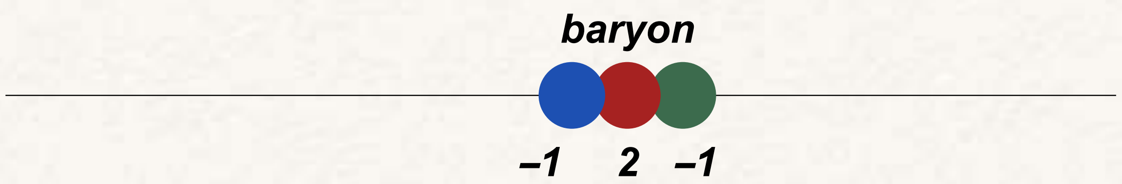


thermodynamic limit



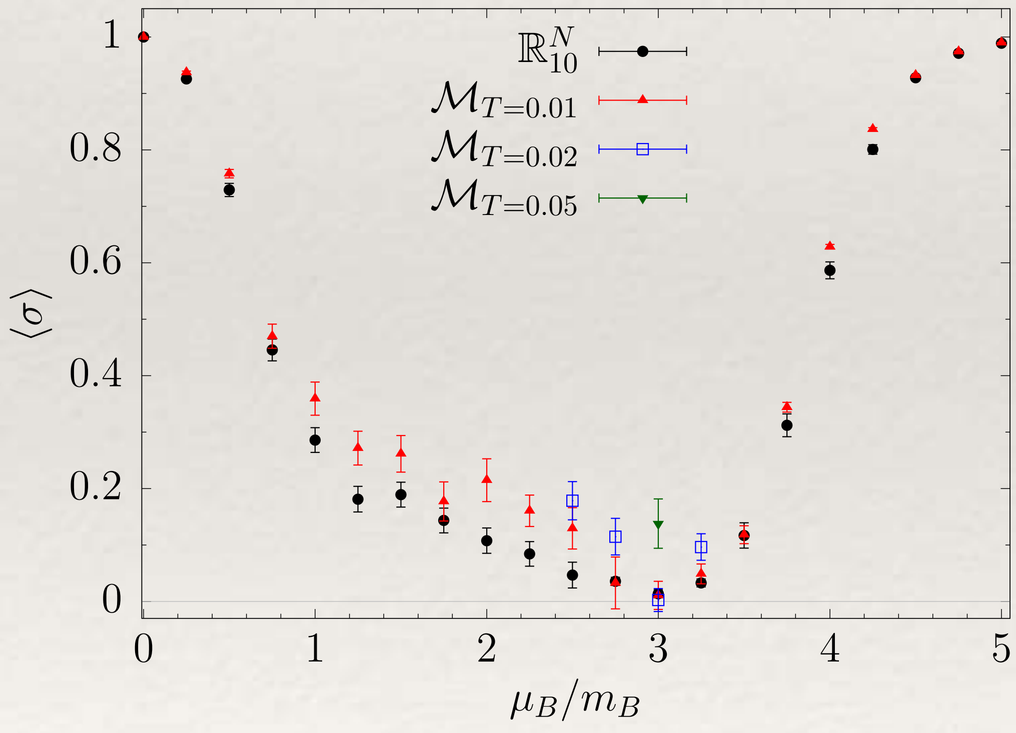
[Alexandru, GB, Bedaque, Ridgway, Warrington, Phys. Rev. D95, 014502]

Gauge theories - 2d QED

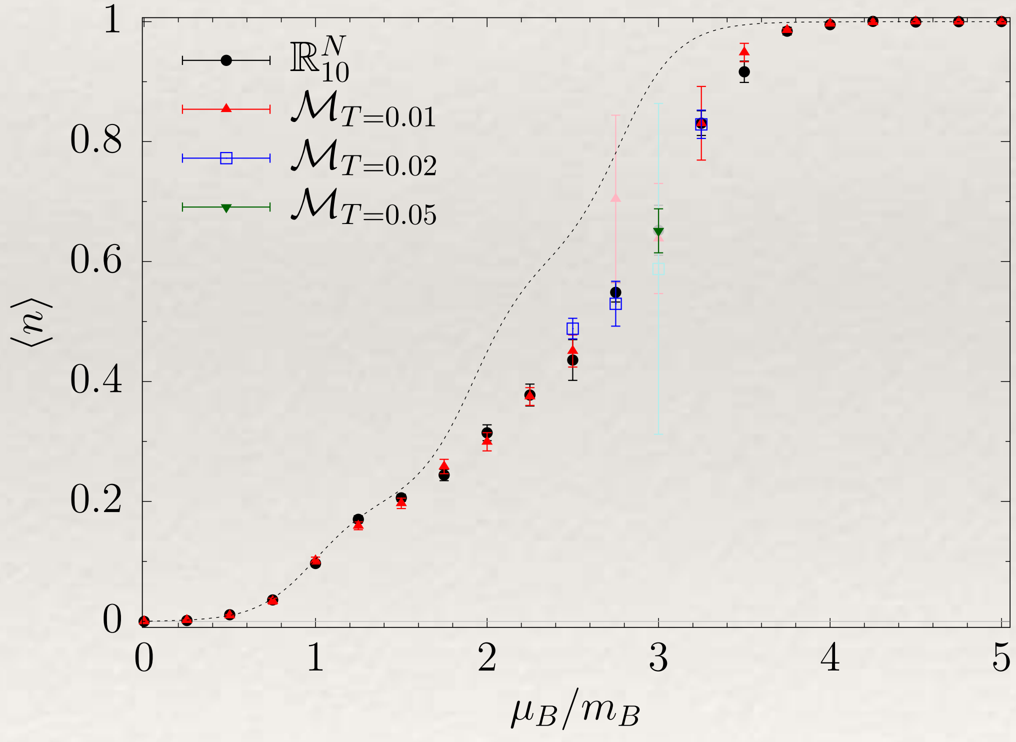


QED with 3 ``quarks" with charges q=2,-1,-1

$$S = \sum_{a=1}^{3} \int d^2x \left[F^2 + \bar{\psi}^a \left(\gamma^\mu(\partial_\mu - g q_a A_\mu) + m - \mu\gamma^0 \right) \psi^a \right]$$



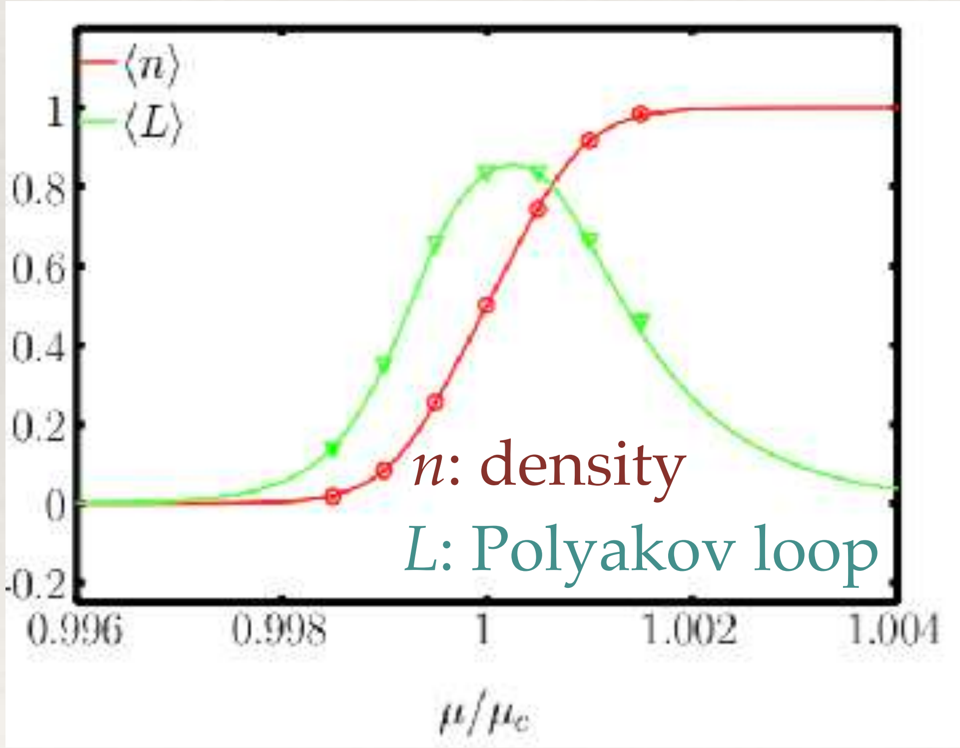
sign problem



equation of state

Gauge theories - heavy dense QCD

- In the limit $m_q \to \infty$ ➡ effective theory of Polyakov loops
- Still has a sign problem for $\mu \neq 0$ but easier to simulate
- Exploratory study on a few-site lattice with
  $\mathscr{M} \sim \sum$ ``Lefschetz thimbles" (fixed points of flow+fluctuations)

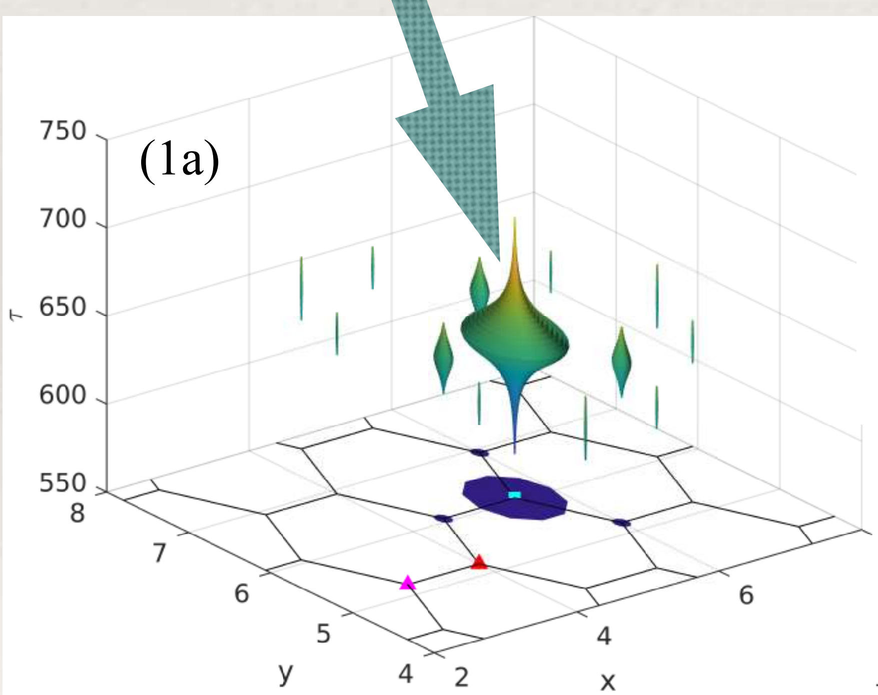


[Zambello, Di Renzo, Phys. Rev. D95, 014502]

Many body physics - Hubbard model

2d Hubbard model away from half filling on a Honeycomb lattice

fixed point of flow = saddle point of $S[\phi]$



conventional MC

average sign

	$\langle \cos \mathrm{Im} S \rangle$	$\langle \cos \arg J \rangle$	$\langle \Sigma_G \rangle$
BSS-QMC	0.2363±0.0032		0.2363±0.0032
HMC,α=1.0	0.9627±0.0038	0.427±0.014	0.351±0.015
HMC,α=0.8	0.797±0.022	0.915±0.008	0.644±0.028

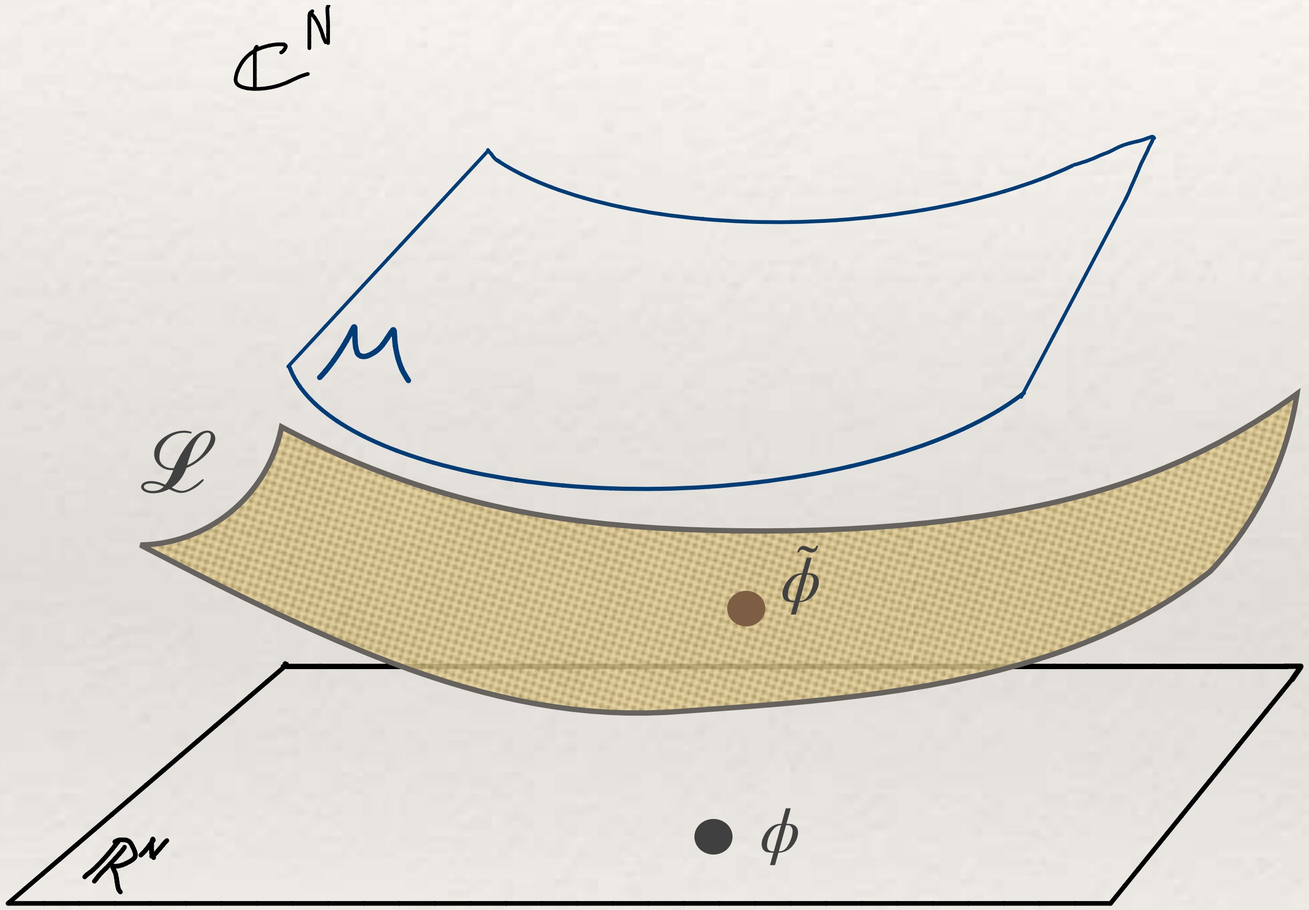
deformation ~ ∑ "thimbles"

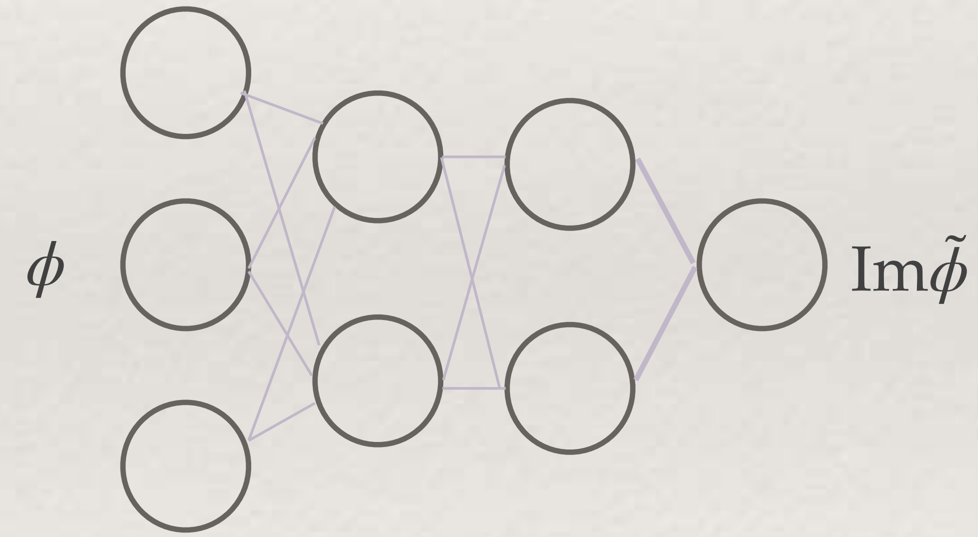
[Ulybyshev, Winterowd, Zafeiropoulos PRD 101 (1), 014508]

Other deformations: "Learnifolds"

Machine learning, training set: points on $\mathscr{M}$

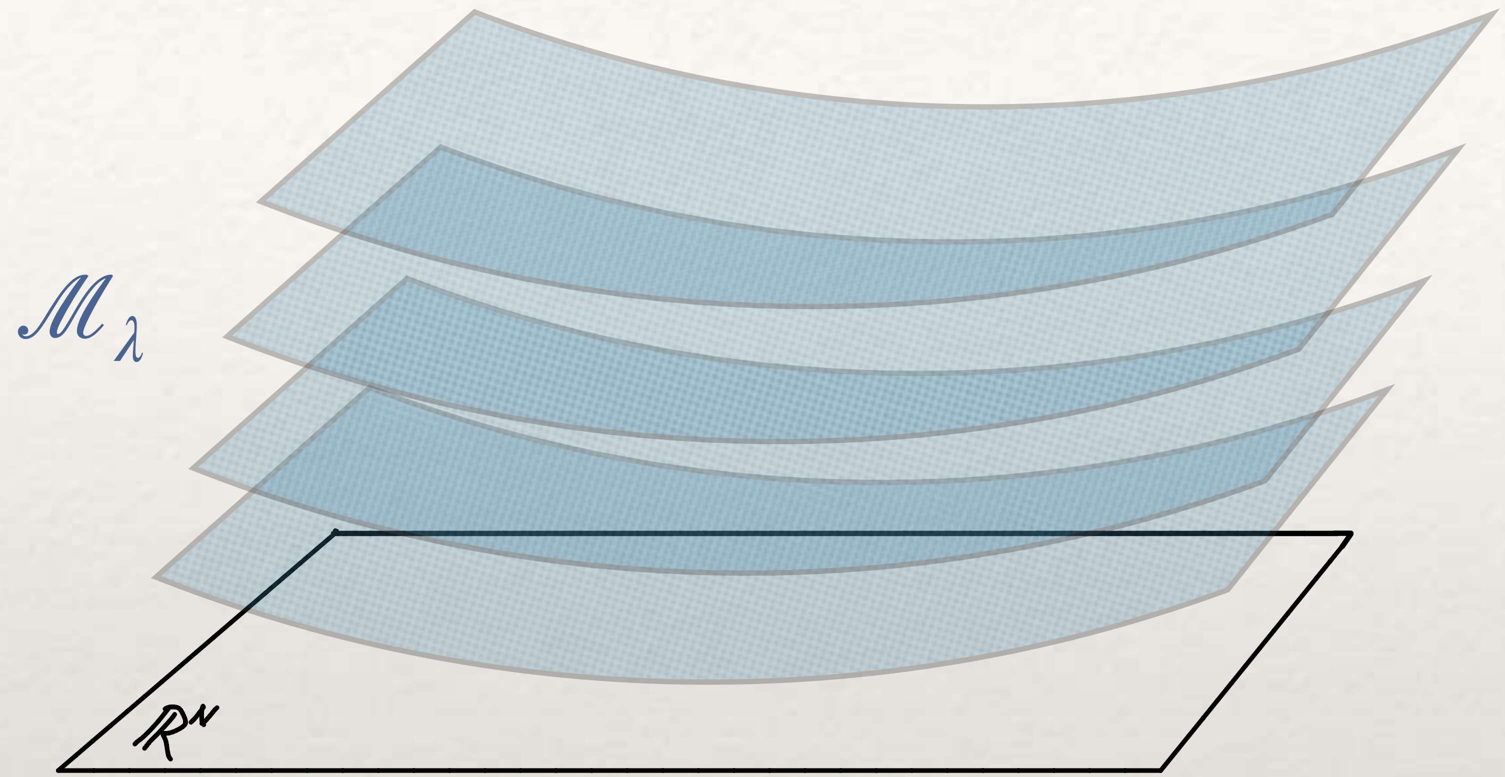
output: $\mathscr{L} \approx \mathscr{M}$





[Alexandru, Bedaque, Lamm, Lawrence *Phys.Rev.D* 96 (2017) 9, 094505]

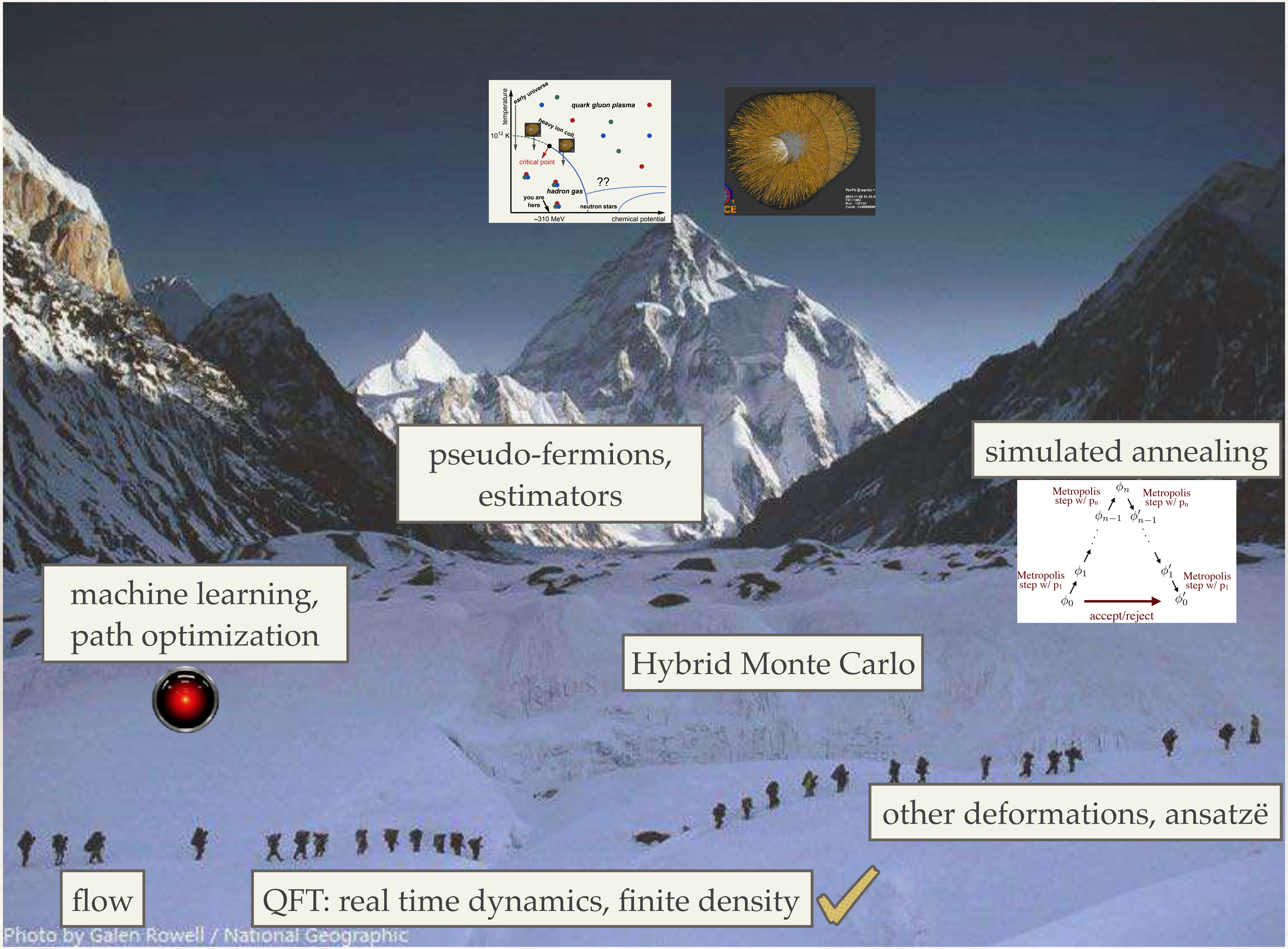
Sign optimized manifolds



within a family of manifolds $\mathscr{M}_\lambda$ minimize the sign problem

maximize the average phase: $\langle e^{-iS_I} \rangle_\lambda = \dfrac{\int_{\mathscr{M}_\lambda} d[\phi] e^{-S}}{\int_{\mathscr{M}_\lambda} d[\phi] e^{-S_R}}$

[Mori et al. '17-'19, Alexandru et al. '18, Bursa et al. '18, Kashiwa et al. '19, Detmold et al. '20]


temperature
early universe
quark gluon plasma
10^{12} K
heavy ion coll.
critical point
??
hadron gas
you are here
neutron stars
~310 MeV
chemical potential
pseudo-fermions,
estimators
simulated annealing
Metropolis step w/ p_n
ϕ_n
Metropolis step w/ p_n
ϕ_{n-1}
ϕ'_{n-1}
Metropolis step w/ p_1
ϕ_1
ϕ'_1
Metropolis step w/ p_1
ϕ_0
ϕ'_0
accept/reject
machine learning,
path optimization
Hybrid Monte Carlo
other deformations, ansatzë
flow
QFT: real time dynamics, finite density
Photo by Galen Rowell / National Geographic