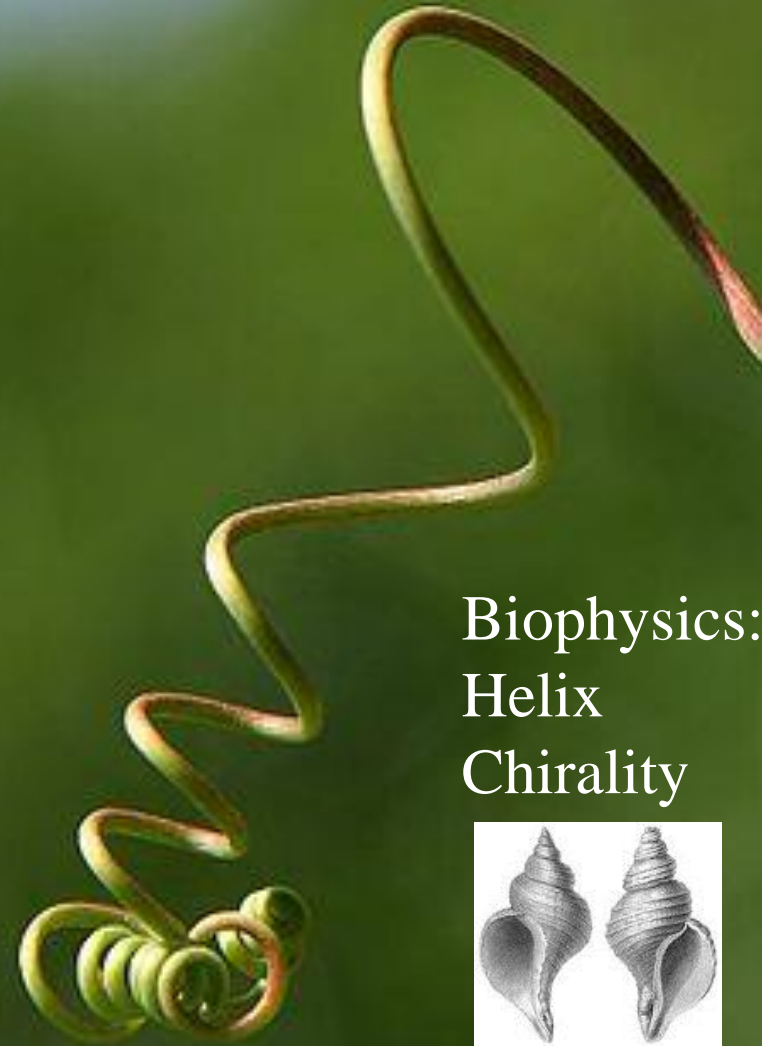# Helical magnetic fields in the early universe

- What is magnetic helicity
- The myth of catastrophic quenching
- Magn helicity in decaying turbulence
- The chiral magnetic effect
- Gravitational waves from the resulting turbulence

Axel Brandenburg (Nordita, Stockholm)

1

ελιξ

Curling tendril
→ Climbing plant

Biophysics:
Helix
Chirality

The degree of knottedness of tangled vortex lines

By H. K. MOFFATT

Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge

Let $\mathbf{u}(\mathbf{x})$ be the velocity field in a fluid of infinite extent due to a vorticity distribution $\boldsymbol{\omega}(\mathbf{x})$ which is zero except in two closed vortex filaments of strengths κ_1, κ_2. It is first shown that the integral

$$I = \int \mathbf{u} \cdot \boldsymbol{\omega} \, dV$$

is equal to $\alpha \kappa_1 \kappa_2$ where α is an integer representing the degree of linkage of the two filaments; $\alpha = 0$ if they are unlinked, ± 1 if they are singly linked. The invariance of I for a continuous localized vorticity distribution is then established for barotropic inviscid flow under conservative body forces. The result is interpreted in terms of the conservation of linkages of vortex lines which move with the fluid.
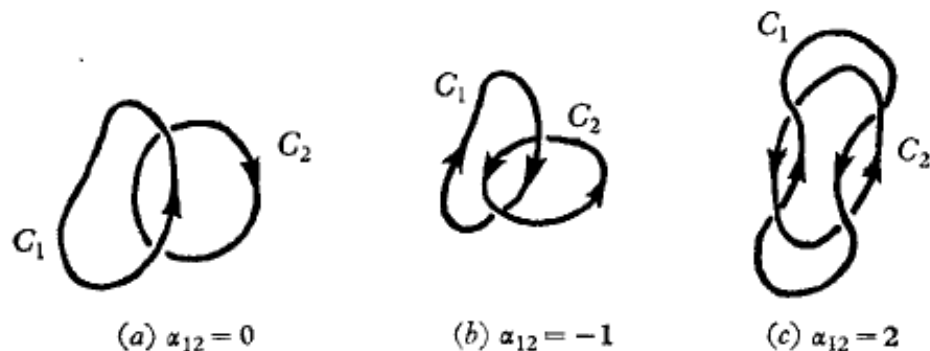
3

FIGURE 1. The degree of linkage of two closed filaments C_1, C_2. The choice of sign in (b), (c) is determined by the relative orientation of the two filaments.
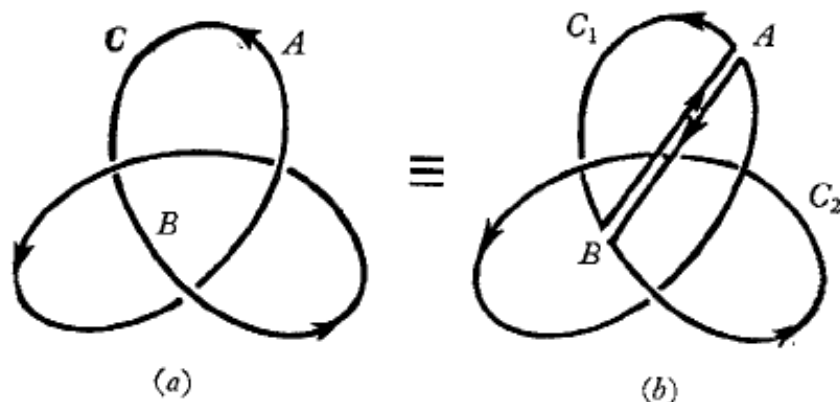

FIGURE 2. Decomposition of a knotted vortex line. To get from (a) to (b), two equal and opposite vorticity segments are inserted between the points A and B. C_1 and C_2 are evidently unknotted but linked.

† The term 'winding number' (anzahl der umschlingungen) and the expression given below for it, equation (11), can be traced to a paper by Gauss (1833) which was concerned with the magnetic field produced by two or more electric current circuits. It is the simplest (but by no means the only) topological invariant of two linked curves (see, for example, Crowell & Fox 1964, and the references given therein).

Moffatt coined the term in hydro/MHD

The quantity $\mathbf{u} . \boldsymbol{\omega}$ admits a simple, essentially kinematical, interpretation. The fluid particles in any small volume element dV undergo at any instant a superposition of three motions: the (uniform) velocity $\mathbf{u}_0$ of any representative point 0 of the element, an irrotational uniform strain $\nabla\phi$ relative to I, and a rigid body rotation $2\boldsymbol{\omega}_0$ where $\boldsymbol{\omega}_0$ is the vorticity at 0. The streamlines of the flow $\mathbf{u} - \nabla\phi$ passing near 0 are (locally) helices about the streamline through 0, and the contribution

$$\mathbf{u} . \boldsymbol{\omega} \, dV \approx \mathbf{u}_0 . \boldsymbol{\omega}_0 \, dV$$

to I from dV is positive or negative according as the screw of these helices is right-handed or left-handed. The term *helicity* is used in particle physics for the scalar product of the momentum and spin of a particle, and it would seem to be a natural candidate in the present context to describe the quantity $\mathbf{u} . \boldsymbol{\omega} \, dV$; the quantity $\mathbf{u} . \boldsymbol{\omega}$ may then be described as the *helicity per unit volume* of the flow. Equation (17) then expresses the result that the total helicity within any closed vortex surface (on which $\boldsymbol{\omega} . \mathbf{n} = 0$) is constant.

Magnetic helicity measures linkage of flux

$$H = \int_V \mathbf{A} \cdot \mathbf{B}\, dV$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\Phi_1 \qquad S_1 \qquad \Phi_2$$
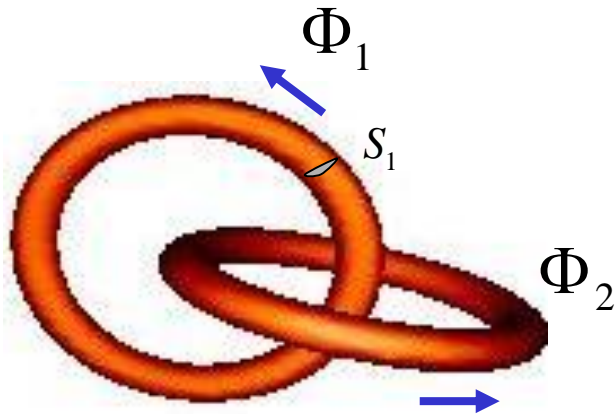
$$H = \pm 2\Phi_1\Phi_2$$

Therefore the unit is Maxwell squared

$$H_1 = \int_{L_1} \mathbf{A} \cdot d\ell \int_{S_1} \mathbf{B} \cdot d\mathbf{S}$$

$$= \int_{S_2} \nabla \times \mathbf{A} \cdot d\mathbf{S} = \Phi_2$$

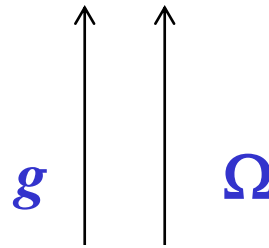$$= \Phi_1$$

What produces helicity?

north

south

equator
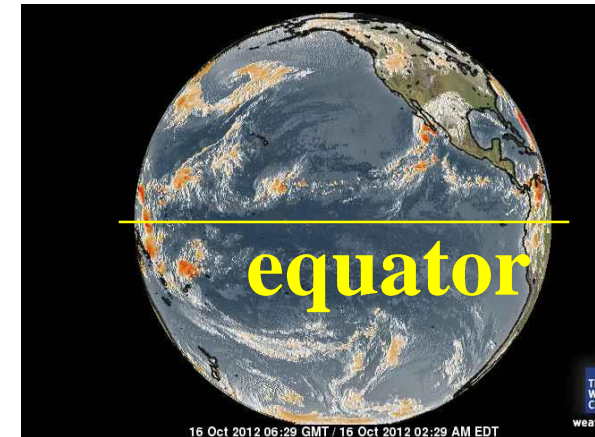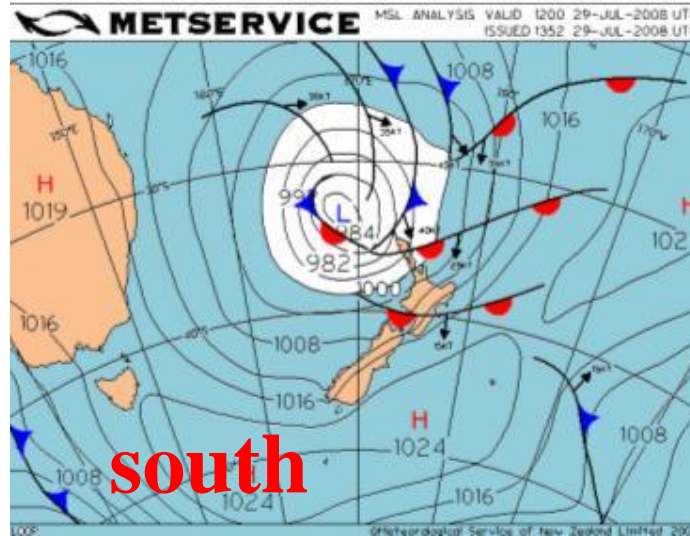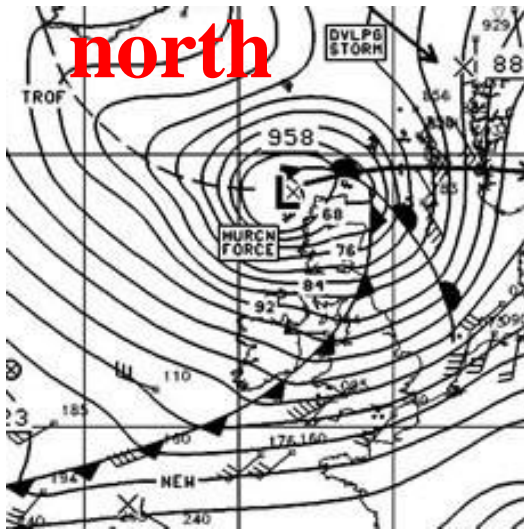
Cyclones:
Down: faster
Up: slower

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

g Ω

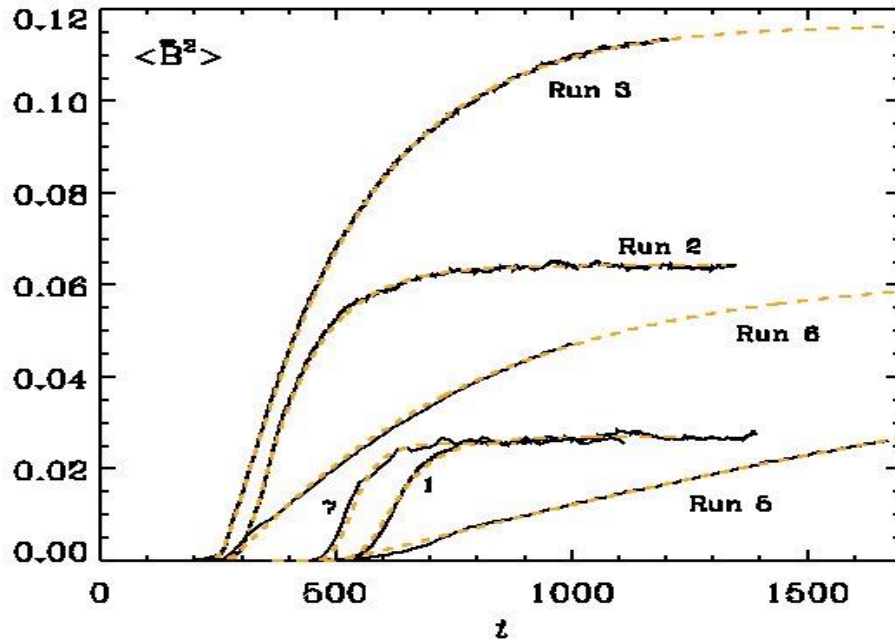$\langle \boldsymbol{\omega} \cdot \mathbf{u} \rangle < 0$

g Ω
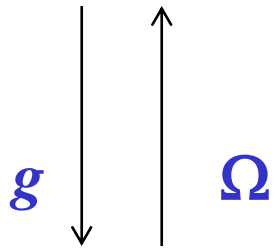
$\langle \boldsymbol{\omega} \cdot \mathbf{u} \rangle > 0$

No preferred helicity,
but + and – possible
if result of instability

Catastrophic quenching

$$\frac{d}{dt}\langle \mathbf{A}\cdot\mathbf{B}\rangle = -2\eta\langle \mathbf{J}\cdot\mathbf{B}\rangle$$

$$\langle \mathbf{J}\cdot\mathbf{B}\rangle = \langle \overline{\mathbf{J}}\cdot\overline{\mathbf{B}}\rangle + \langle \mathbf{j}\cdot\mathbf{b}\rangle$$

$$\langle \overline{\mathbf{J}}\cdot\overline{\mathbf{B}}\rangle \approx -k_1\langle \overline{\mathbf{B}}^2\rangle, \quad \langle \mathbf{j}\cdot\mathbf{b}\rangle \approx k_{\mathrm{f}}\langle \mathbf{b}^2\rangle$$

$$k_1^{-1}\frac{d}{dt}\langle \overline{\mathbf{B}}^2\rangle = -2\eta k_1\langle \overline{\mathbf{B}}^2\rangle + 2\eta k_{\mathrm{f}}\langle \mathbf{b}^2\rangle$$

molecular value!!

$$\langle \overline{\mathbf{B}}^2\rangle = \langle \mathbf{b}^2\rangle\frac{k_f}{k_1}\left[1 - e^{-2\eta k_1^2(t-t_{\mathrm{s}})}\right]$$

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Large-scale magnetic fields from hydromagnetic turbulence in the very early universe

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Early Universe

Energy momentum tensor

$$T^{\mu\nu} = (p+\rho)U^\mu U^\nu + pg^{\mu\nu}$$
$$+ \frac{1}{4\pi}\left(F^{\mu\sigma}F^\nu{}_\sigma - \frac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right),$$

$$\widetilde{\mathbf{S}} = R^4\mathbf{S}, \quad \widetilde{p} = R^4 p, \quad \widetilde{\rho} = R^4\rho, \quad \widetilde{\mathbf{B}} = R^2\mathbf{B},$$
$$\widetilde{\mathbf{J}} = R^3\mathbf{J}, \quad \text{and} \quad \widetilde{\mathbf{E}} = R^2\mathbf{E}.$$

$$\frac{\partial\widetilde{\mathbf{S}}}{\partial\widetilde{t}} = -(\boldsymbol{\nabla}\cdot\mathbf{v})\widetilde{\mathbf{S}} - (\mathbf{v}\cdot\boldsymbol{\nabla})\widetilde{\mathbf{S}} - \boldsymbol{\nabla}\widetilde{p} + \widetilde{\mathbf{J}}\times\widetilde{\mathbf{B}}.$$

Conformal time, rescaled equations

$$\widetilde{t} = \int dt/R. \qquad \mathbf{S} = (p+\rho)\gamma^2\mathbf{v}.$$

$$\frac{\partial\widetilde{\mathbf{B}}}{\partial\widetilde{t}} = -\boldsymbol{\nabla}\times\widetilde{\mathbf{E}}, \quad \boldsymbol{\nabla}\cdot\widetilde{\mathbf{B}} = 0,$$

the MHD equations in an expanding universe with zero curvature are the same as the relativistic MHD equations in a nonexpanding universe, provided the dynamical quantities are replaced by the scaled ''tilde'' variables, and provided conformal time $\widetilde{t}$ is used. The effect of this is, as usual, that
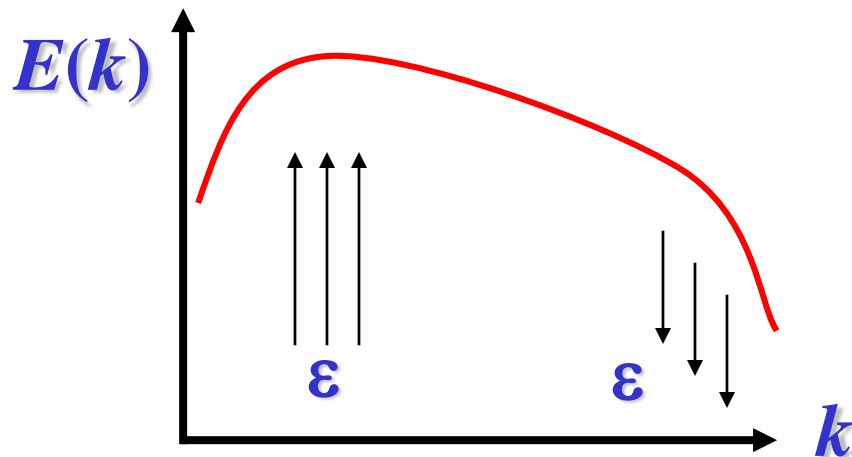
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Kolmogorov turbulence
→ forward cascade

nonlinearity

$$\left(\cos kx\right)^2 = \tfrac{1}{2}\cos 2kx + \tfrac{1}{2}$$

$$k \rightarrow 2k$$

constant flux ε [cm²/s³]

$$\int E(k)dk = \tfrac{1}{2}\left\langle \mathbf{u}^2 \right\rangle \qquad \mathbf{[cm^3/s^2]}$$
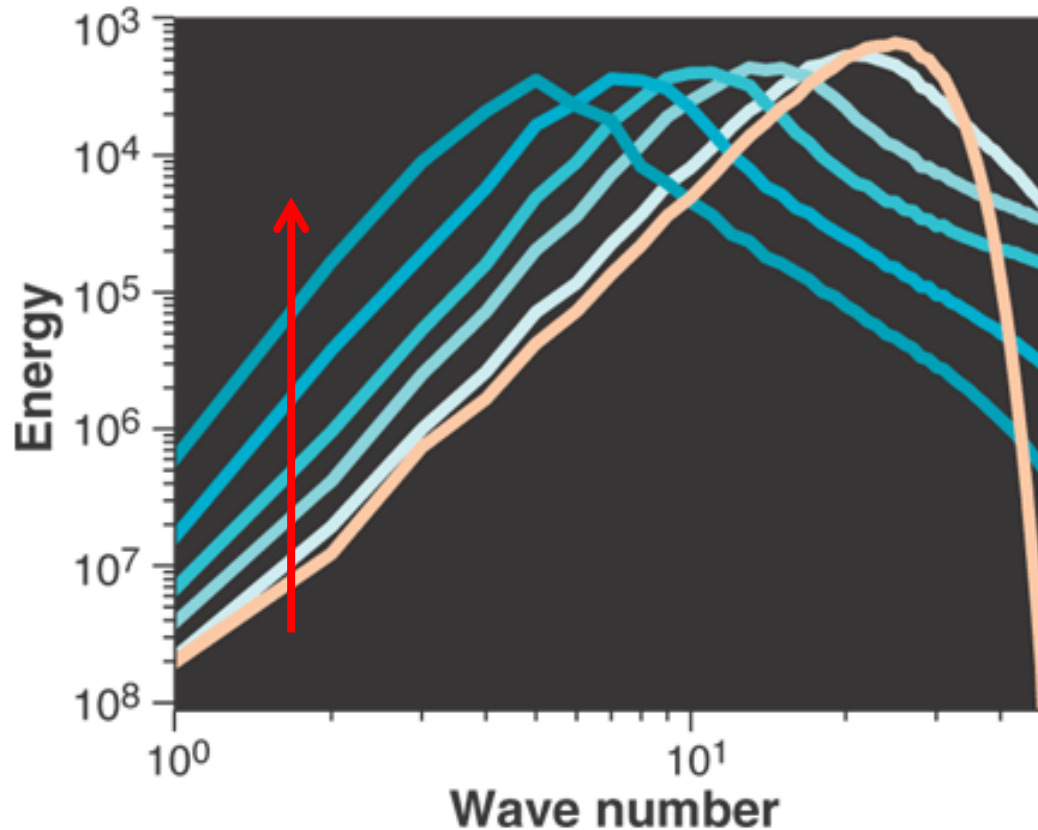
$$E(k) = C_K \varepsilon^a k^b$$

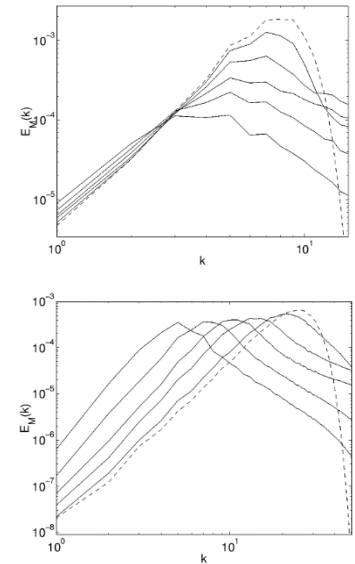cm : $3 = 2a - 1$

s : $2 = 3a$

a=2/3, b= −5/3

Magnetic helicity: inverse cascade

Initial slope
$E \sim k^4$

helical vs nonhelical

**Christensson et al.
(2001, PRE 64, 056405)**

Helical decay law: Biskamp & Müller (1999)

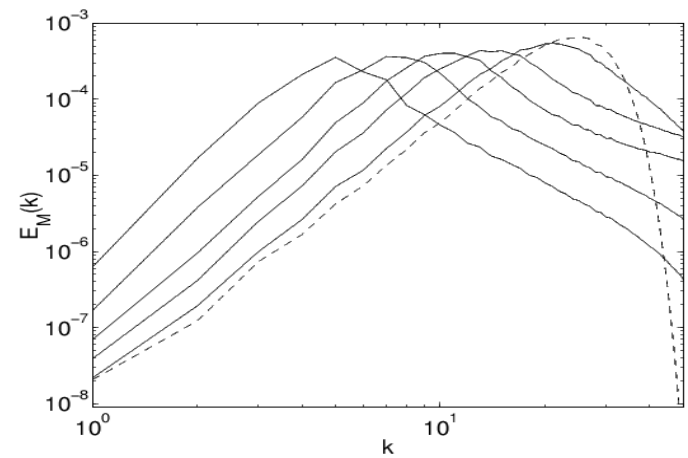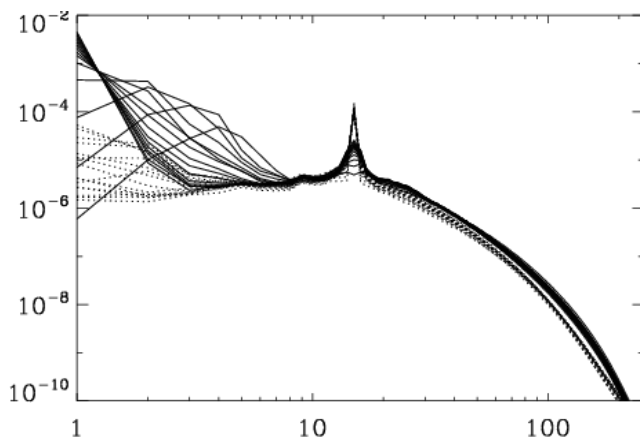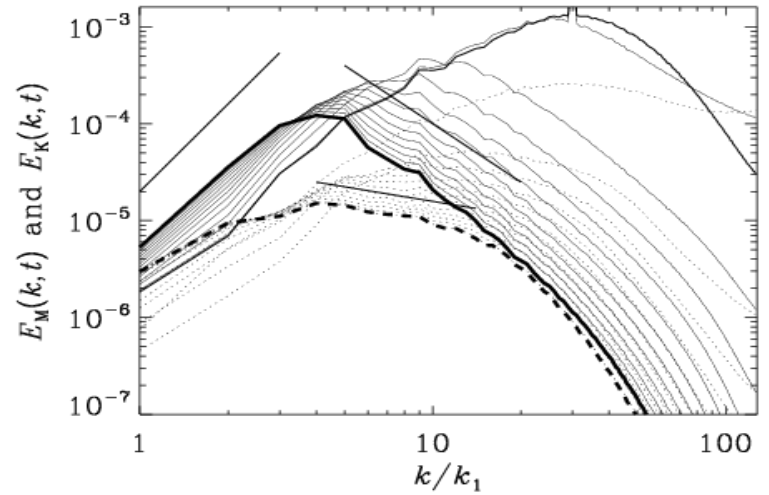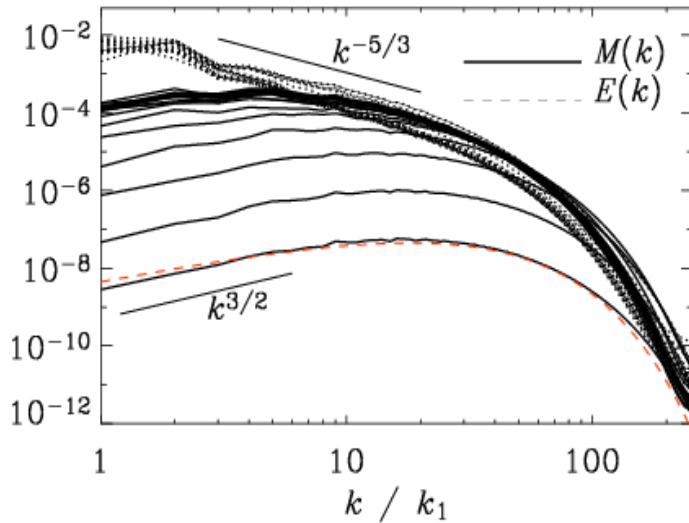$$H = EL = \text{const}$$

$$\varepsilon = U^3 / L = E^{3/2} / L$$

$$\varepsilon = -\mathrm{d}E / \mathrm{d}t$$

$$\varepsilon = -\mathrm{d}E / \mathrm{d}t = E^{3/2} / L = E^{5/2} / H$$

$$E \propto t^{-2/3}$$

Helical – nonhelical

Self-similar turbulent decay

$$E_{\mathrm{M}}(k\xi_{\mathrm{M}}(t), t) \approx \xi_{\mathrm{M}}^{-\beta_{\mathrm{M}}} \phi(k\xi_{\mathrm{M}}).$$

$$\int \dot{E}_i(k, t)\, \mathrm{d}k = \mathcal{E}_i \qquad \xi_i(t) = \int_0^\infty k^{-1} E_i(k, t)\, \mathrm{d}k / \mathcal{E}_i(t)$$

instantaneous
scaling exponents

$$p_i(t) = d \ln \mathcal{E}_i / d \ln t, \quad q_i(t) = d \ln \xi_i / d \ln t,$$

$$1 + \beta_{\mathrm{M}} = p_{\mathrm{M}} / q_{\mathrm{M}}$$

β	p	q	inv.	dim.
4	$10/7 \approx 1.43$	$2/7 \approx 0.286$	$\mathcal{L}$	$[x]^7[t]^{-2}$
3	$8/6 \approx 1.33$	$2/6 \approx 0.333$		
2	$6/5 = 1.20$	$2/5 = 0.400$		
1	$4/4 = 1.00$	$2/4 = 0.500$	$\langle \boldsymbol{A}_{2\mathrm{D}}^2 \rangle$	$[x]^4[t]^{-2}$
0	$2/3 \approx 0.67$	$2/3 \approx 0.667$	$\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$	$[x]^3[t]^{-2}$
-1	$0/2 = 0.00$	$2/1 = 1.000$		

Collapsed spectra and *pq* diagrams
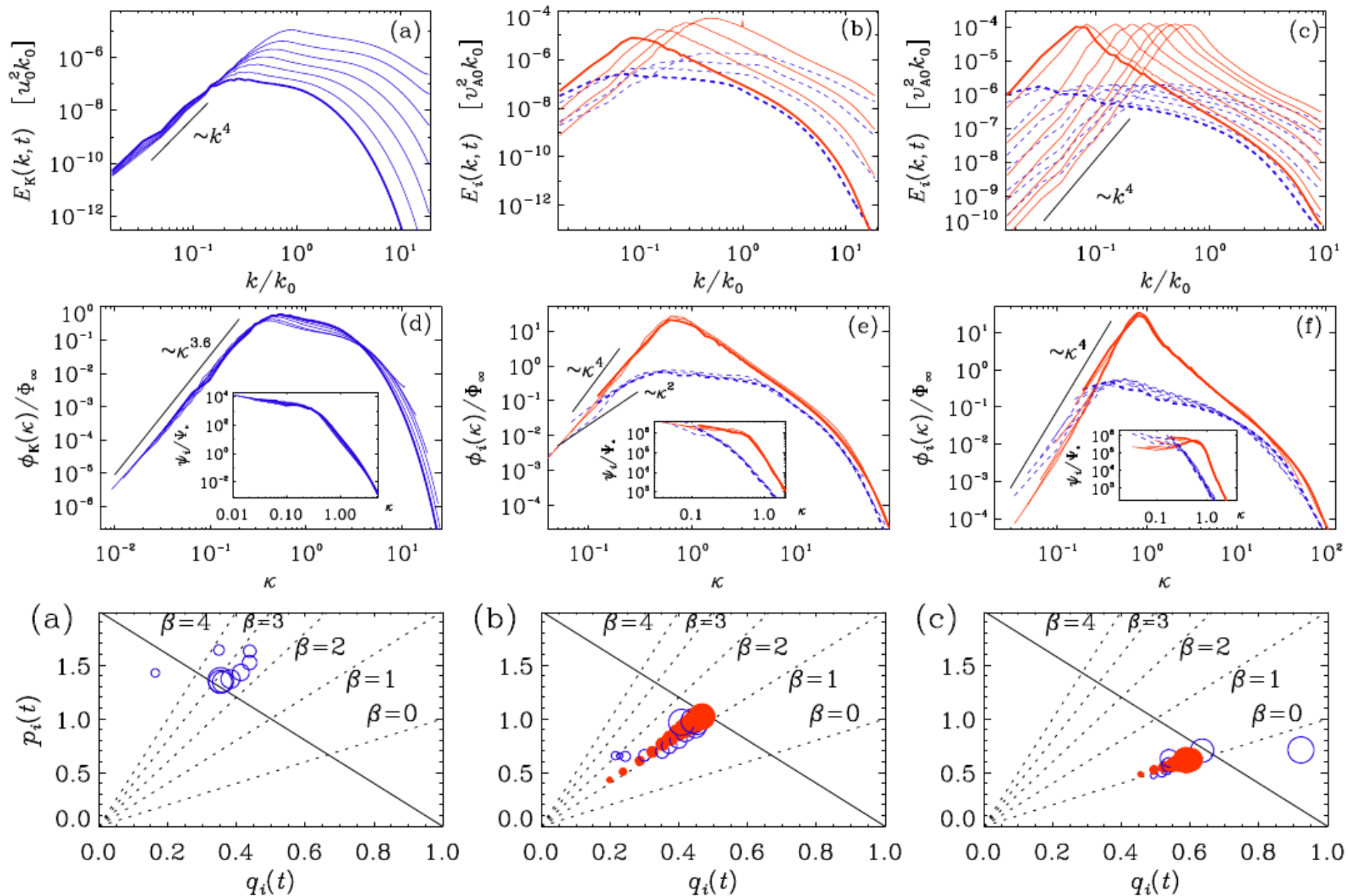
Chiral magnetic effect: caused by chirality of fermions

e^-

p

e^-

n

$\bar{\nu}_e$
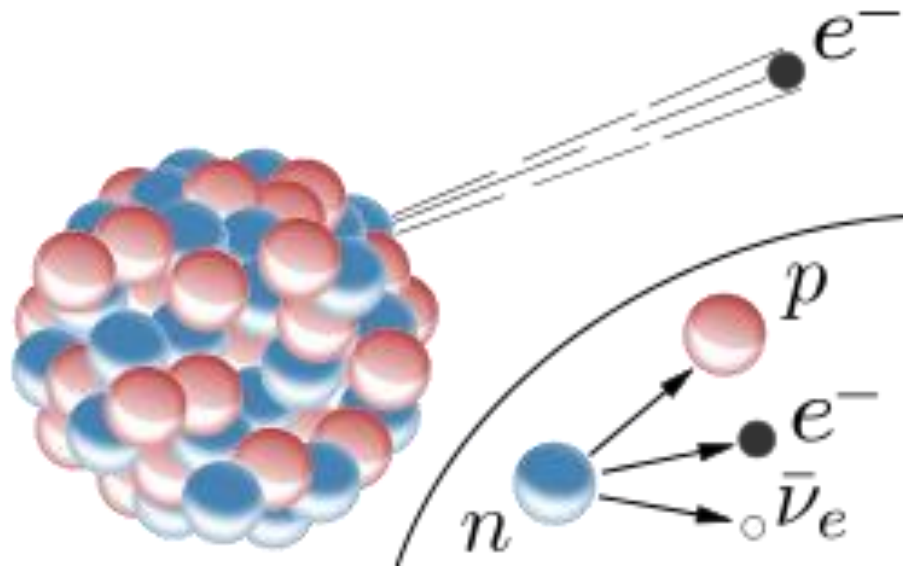
electrons left-handed
positrons: right-handed

Electrons have handedness

Right-handed:

p

S

Left-handed:

p

S
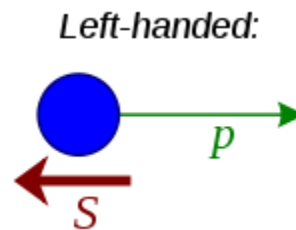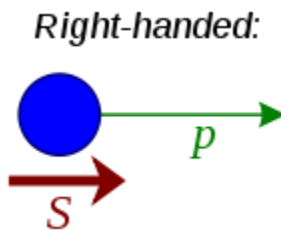
But also spontaneous
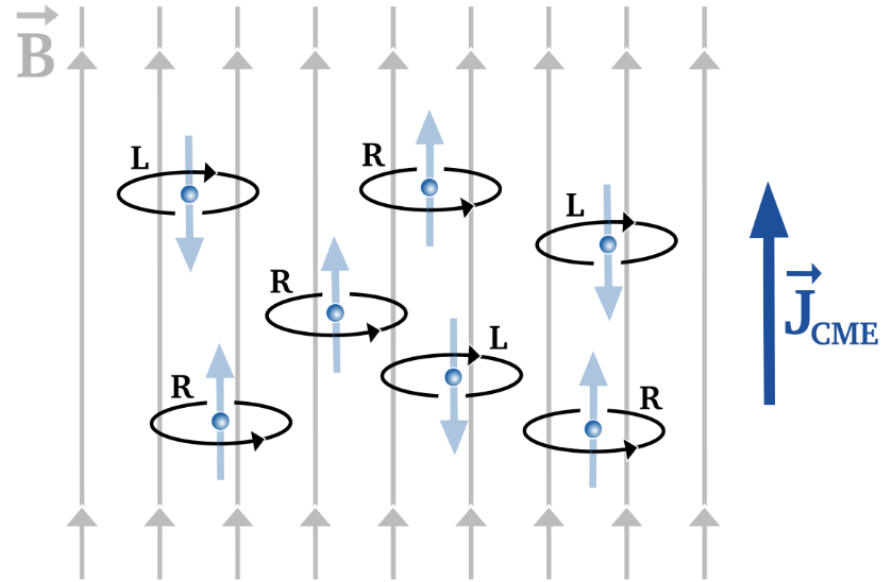spin flip unless

$$k_B T >> m_e c^2$$

$6 \times 10^9 \text{K}$

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Chiral magnetic effect

In the presence of *B*-field chiral electrons produce a current $J=...+\mu B$ with

$$\mu = 24\,\alpha_{em}(n_L - n_R)(\hbar c/k_B T)^2$$

α_{em} is the fine structure constant → quantum effect

μ is a pseudoscalar

Leads to a dynamo effect

$$\frac{\partial \boldsymbol{A}}{\partial t} = \eta(\mu\boldsymbol{B} - \boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{U} \times \boldsymbol{B}$$

$$\sigma = \left|\mu k\right| - \eta k^2$$

Total chirality conserved

Uncurled induction equation & chemical potential

$$\frac{\partial \boldsymbol{A}}{\partial t} = \eta(\mu \boldsymbol{B} - \boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{U} \times \boldsymbol{B},$$

$$\frac{D\mu}{Dt} = -\lambda\,\eta(\mu \boldsymbol{B} - \boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \boldsymbol{B} + D\nabla^2\mu - \Gamma_{\mathrm{f}}\mu,$$

(spin flipping)

Coupling between electromagnetic field and chem potential

$$\lambda \equiv 3\hbar c\,(8\alpha_{\mathrm{em}}/k_{\mathrm{B}}T)^2$$

Conservation equation

$$\tfrac{1}{2}\lambda\,\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle + \langle \mu \rangle = \mathrm{const} \equiv \mu_0 \quad (\text{for } \Gamma_{\mathrm{f}} \ll \eta\mu_0^2)$$

Therefore

$$\langle \boldsymbol{B}^2 \rangle\, \xi_{\mathrm{M}} \lesssim \mu_0/\lambda.$$
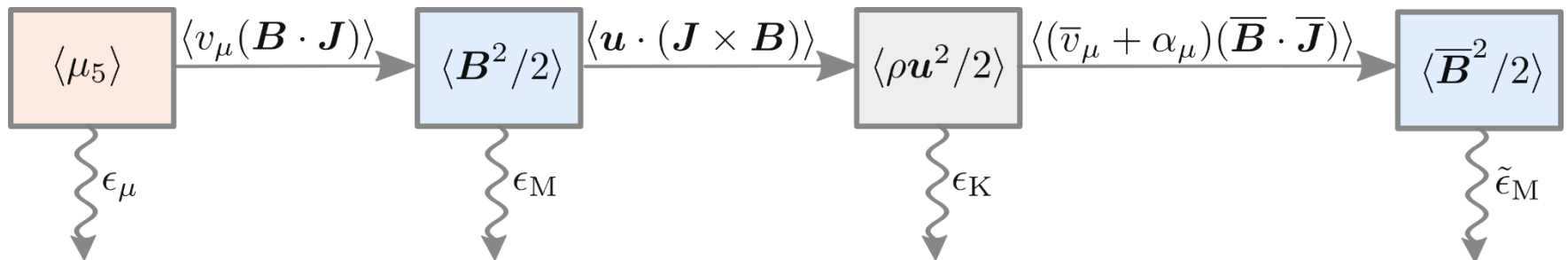
Total system of equations

Momentum & continuity equations

$$\rho \frac{DU}{Dt} = (\nabla \times B) \times B - \nabla p + \nabla \cdot (2\rho\nu S), \qquad (4)$$

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot U, \qquad (5)$$

Together with:

$$\frac{\partial A}{\partial t} = \eta(\mu B - \nabla \times B) + U \times B,$$

$$\frac{D\mu}{Dt} = -\lambda\,\eta(\mu B - \nabla \times B) \cdot B + D\nabla^2\mu - \Gamma_{\rm f}\mu,$$

Characteristic velocities

Two velocities from chiral magnetic effect

$$v_\lambda = \mu / (\overline{\rho}\lambda)^{1/2}, \qquad v_\mu = \mu\eta,$$

Different orderings

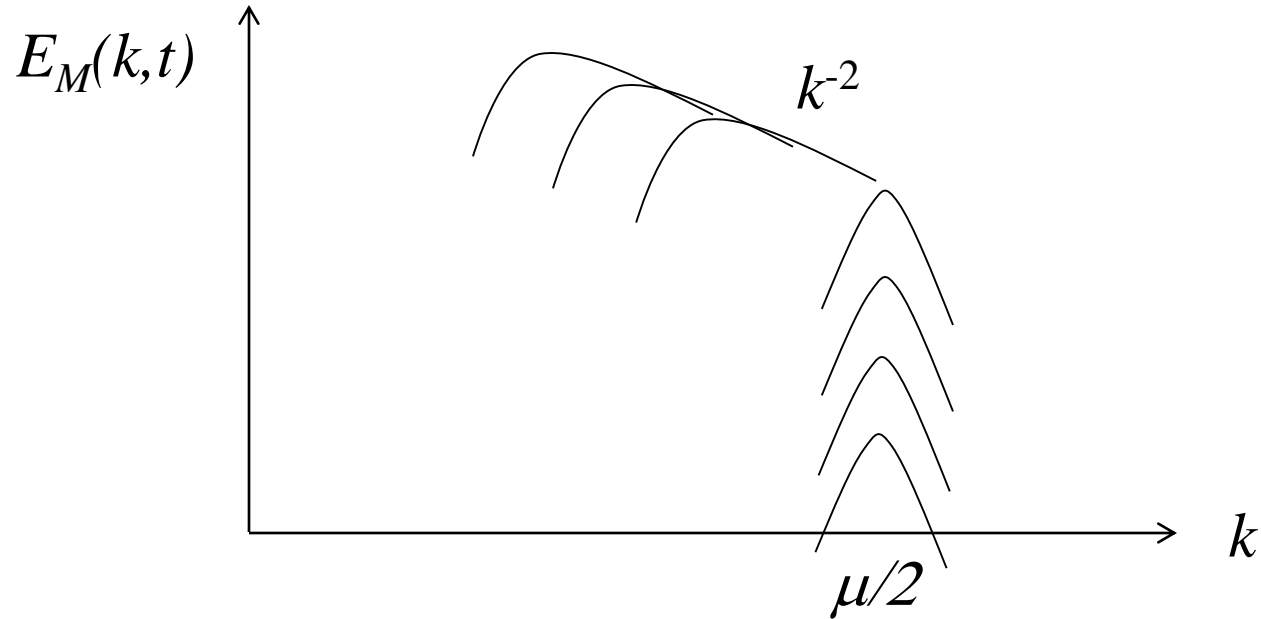$$c_s > v_\lambda > v_\mu > \eta k_1 \quad \text{(regime I)},$$

$$c_s > v_\mu > v_\lambda > \eta k_1 \quad \text{(regime II)}.$$

Dimensional arguments

- $[E(k,t)] = [\rho]\ \mathrm{cm}^3\ \mathrm{s}^{-2} = \mathrm{g}\ \mathrm{s}^{-2}$
- $E(k,t) = C_\mu\ \rho\ \mu^a \eta^b$
- cm: $3 = -a + 2b$
- s: $-2 = -b$
- $\rightarrow b = 2,\ a = 2b - 3 = 4 - 3 = 1$
- $E(k,t) = C_\mu\ \rho\mu\eta^2$
- $\rightarrow$ to determine C_μ from simulations

The spectrum from chiral effect
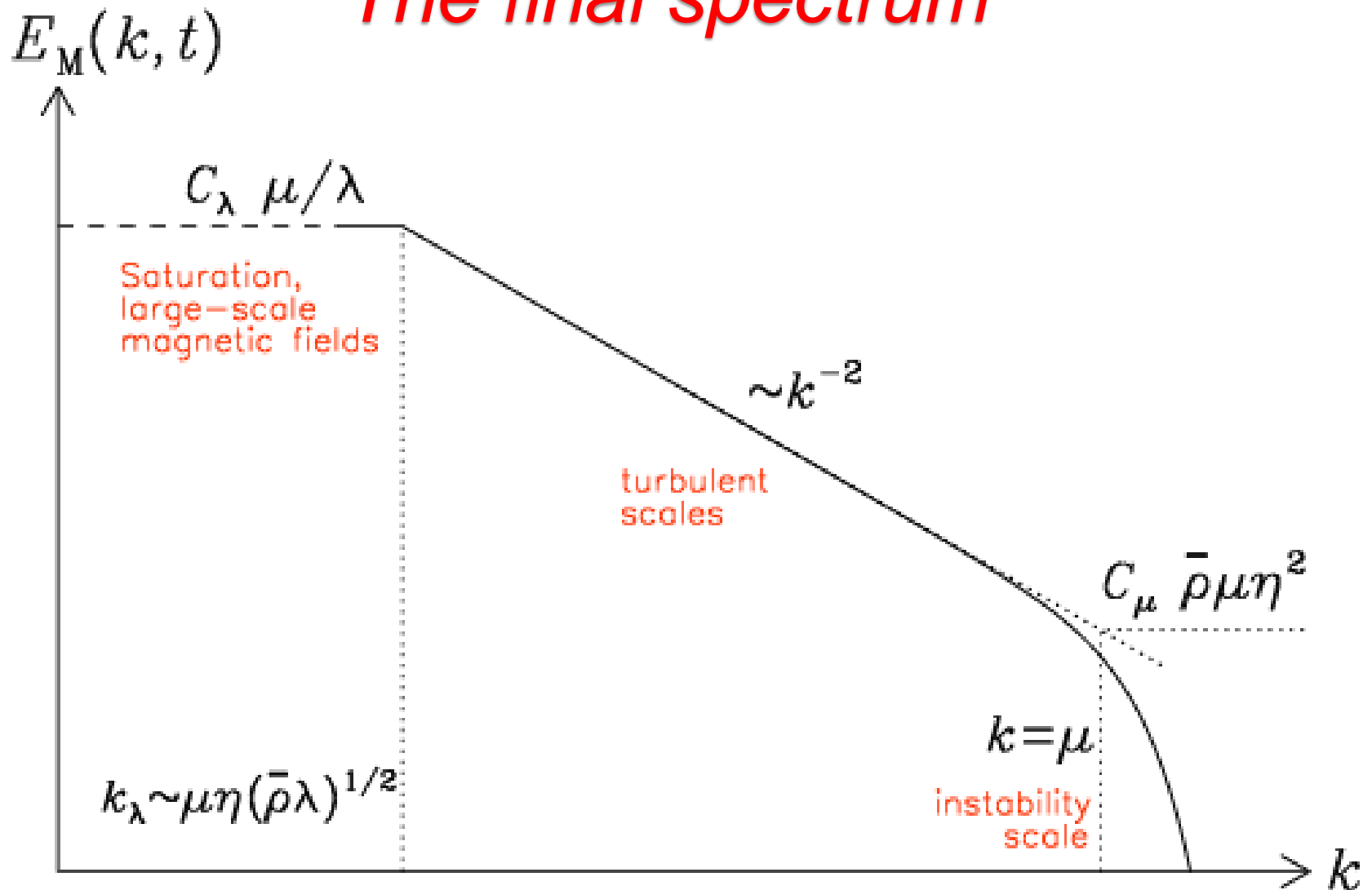
Spectrum build-up from high wavenumbers

Governed solely by chiral chemical potential

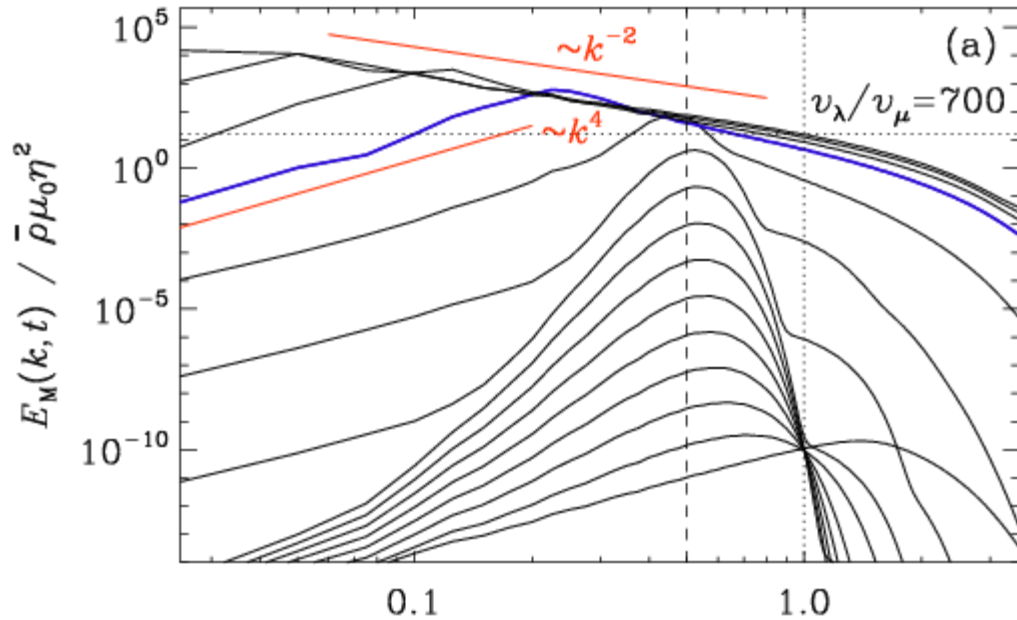$$E_{\mathrm{M}}(k, t) = C_\mu \, \bar{\rho}\mu^3\eta^2 k^{-2},$$

Eventual saturation

- $[E(k,t)\,] = g\ s^{-2}$
- $E(k,t) = C_\lambda\, \rho^a \mu^b \lambda^c$
- g: $1 = a - c$
- cm: $0 = -b - c$
- s: $-2 = 2c$
- → $c = -1,\ b = -c = 1,\ a = 1 + c = 0$
- $E(k,t) = C_\lambda\, \mu/\lambda$
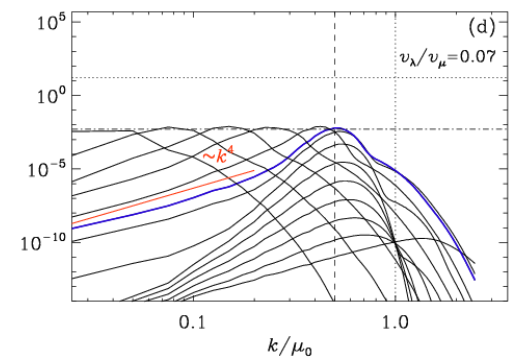- → to determine C_λ from simulations

The final spectrum

$E_{\mathrm{M}}(k,t)$

$C_\lambda \; \mu/\lambda$

Saturation,
large−scale
magnetic fields

$\sim k^{-2}$

turbulent
scales

$C_\mu \; \bar{\rho}\mu\eta^2$

$k=\mu$

instability
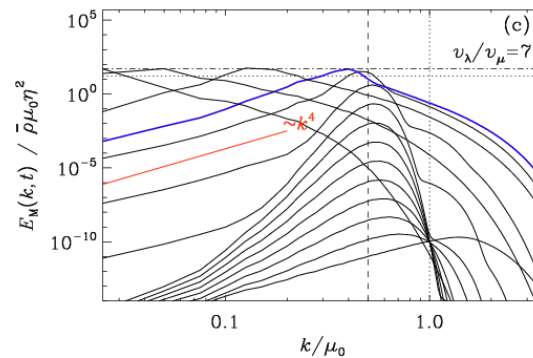scale

$k_\lambda \sim \mu\eta(\bar{\rho}\lambda)^{1/2}$

k

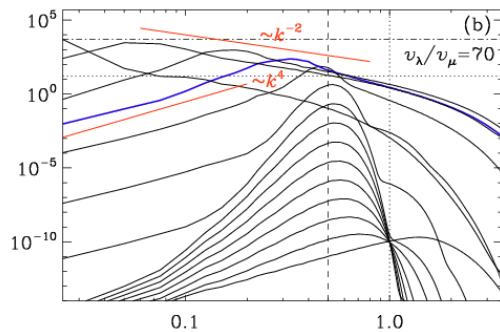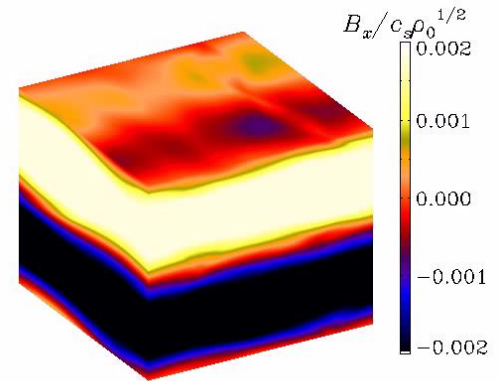# *Inverse cascade!*

Growth at one wavenumber
Then: saturation caused by
initial chemical potential

Early universe: use conservation law

Conservation equation

$$\tfrac{1}{2}\lambda \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle + \langle \mu \rangle = \mathrm{const} \equiv \mu_0 \quad (\text{for } \Gamma_{\mathrm{f}} \ll \eta \mu_0^2)$$

$$(n_{\mathrm{L}} - n_{\mathrm{R}}) + \frac{4\alpha_{\mathrm{em}}}{\hbar c} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = \mathrm{const.}$$

Maximally helical:

$$\langle \boldsymbol{B}^2 \rangle \, \xi_{\mathrm{M}} \lesssim \mu_0/\lambda$$

$$\langle \boldsymbol{B}^2 \rangle \, \xi_{\mathrm{M}} = \epsilon \, (k_{\mathrm{B}} T_0)^3 (\hbar c)^{-2},$$

Inserting actual numbers

Magnetic helicity

$=(0.2e\text{-}18G)^2 \text{ Mpc G}$

$$\langle \boldsymbol{B}^2 \rangle \xi_{\mathrm{M}} = \frac{\hbar c}{4\alpha_{\mathrm{em}}} \frac{g_0}{g_*} n_{\gamma 0} N_{\mathrm{f}} = 5 \times 10^{-38} \frac{N_{\mathrm{f}}}{10} g_{100}^{-1} \, \mathrm{G}^2 \, \mathrm{Mpc}. \quad (17)$$

Here, $g_0 = 3.36$ and $n_{\gamma 0} = 2\zeta(3)/\pi^2 (k_B T_0/\hbar c)^3 = 411 \, \mathrm{cm}^{-3}$

Inverse length scale

$$|\mu| \ll 4\alpha_{\mathrm{em}} \frac{k_{\mathrm{B}} T}{\hbar c} \approx 1.5 \times 10^{14} \, T_{100} \, \mathrm{cm}^{-1}.$$

# Evidence for Strong Extragalactic Magnetic Fields from Fermi Observations of TeV Blazars

Andrii Neronov* and Ievgen Vovk

Magnetic fields in galaxies are produced via the amplification of seed magnetic fields of unknown nature. The seed fields, which might exist in their initial form in the intergalactic medium, were never detected. We report a lower bound $B \geq 3 \times 10^{-16}$ gauss on the strength of intergalactic magnetic fields, which stems from the nonobservation of GeV gamma-ray emission from electromagnetic cascade initiated by tera–electron volt gamma rays in intergalactic medium. The bound improves as $\lambda_B^{-1/2}$ if magnetic field correlation length, λ_B, is much smaller than a megaparsec. This lower bound constrains models for the origin of cosmic magnetic fields.

Fig. 2. Light, medium, and dark gray: known observational bounds on the strength and correlation length of EGMF, summarized in (*25*). The bound from Big Bang nucleosynthesis (BBN) is from (*2*). The black hatched region shows the lower bound on the EGMF derived from observations of 1ES 0347-121 (cross-hatching) and 1ES 0229+200 (single diagonal hatching) in this paper. Orange hatched regions show the allowed ranges of B and λ_B for magnetic fields generated at the epoch of inflation (horizontal hatching), the electroweak phase transition (dense vertical hatching), QCD phase transition (medium vertical hatching), and epoch of recombination (light vertical hatching) (*25*). White ellipses show the range of measured magnetic field strengths and correlation lengths in galaxies and galaxy clusters.
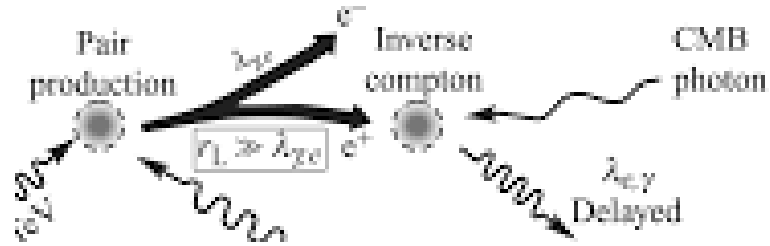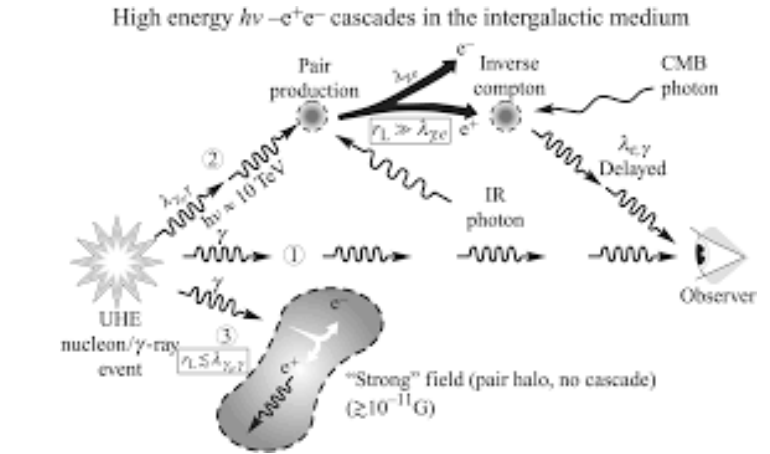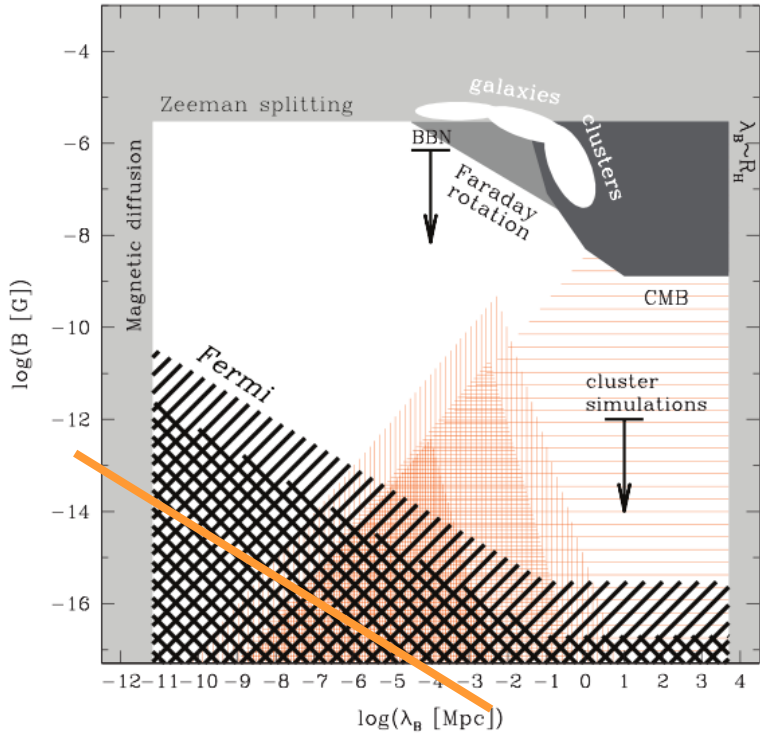
- Chiral magnetic effect alone may be too weak to explain B-field
- But the magnetic stress could still explain gravitational waves

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Cosmological GWs

Amplification of gravitational waves in an isotropic universe

L. P. Grishchuk

State Astronomical Institute
(Submitted April 1, 1974)
Zh. Eksp. Teor. Fiz. **67**, 825–838 (September 1974)

It is shown that weak gravitational waves in a nonstationary isotropic universe can be amplified to a greater degree than indicated by the adiabatic law. It is a necessary (but not a sufficient) condition for the amplification that there should exist such a stage in the evolution of the universe when the characteristic time for change in the background metric is less than the period of the wave. In an expanding universe the wave is the more amplified the more strongly does the rate of evolution of the universe differ from the one which is dictated by matter with the equation of state $p = \epsilon/3$, and the earlier the wave had been "started." The superadiabatic amplification of gravitational waves denotes the possibility of creation of gravitons. An exceptional position is occupied by the "hot" isotropic universe with $p = \epsilon/3$, in which the superadiabatic amplification of gravitational waves and the production of gravitons is impossible. An estimate is made of the converse reaction of the gravitons on the background metric. Apparently, the production of gravitons forbids at least those of the iostropic singularities near which $p > \epsilon/3$.

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Correspondence of spectra

$$\left(\partial_{\bar{t}}^2 - c^2 \nabla^2\right) h_{ij}(\boldsymbol{x}, \bar{t}) = 6\, T_{ij}^{\mathrm{TT}}(\boldsymbol{x}, \bar{t})/\bar{t}$$
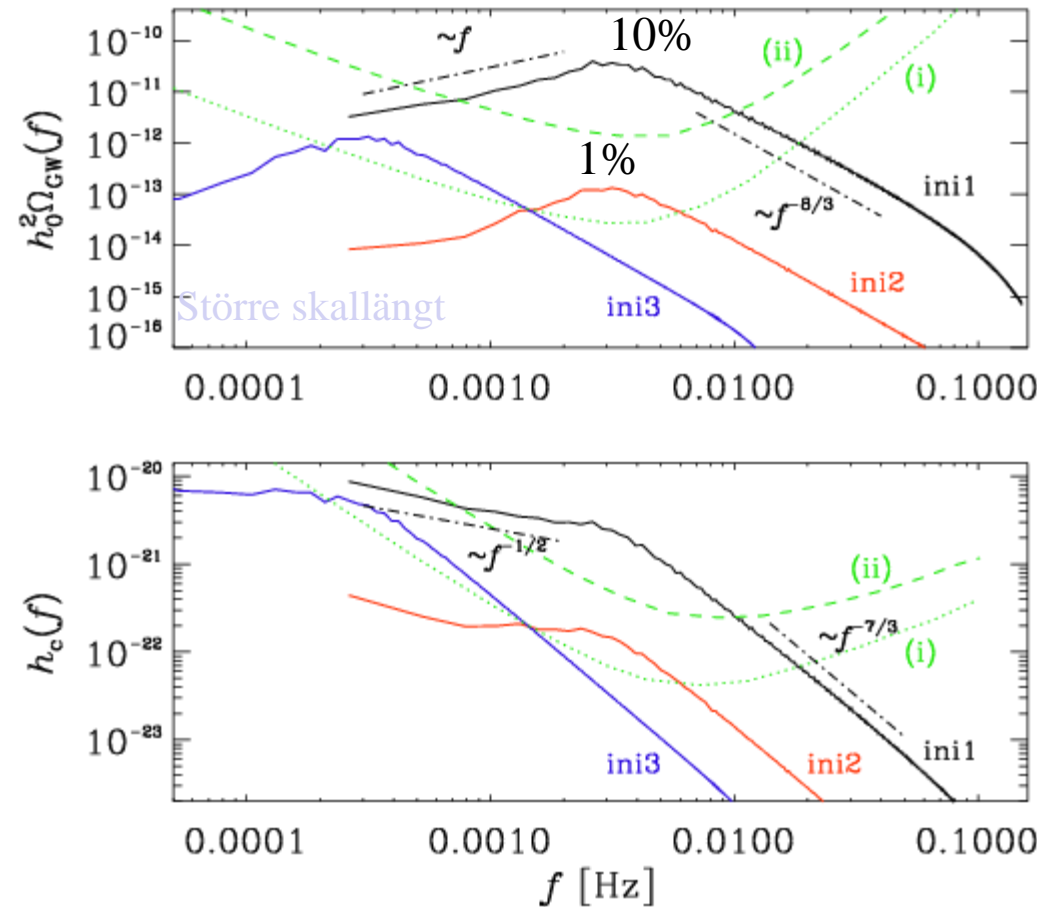
$$T_{ij} = (p + \rho)\, \gamma^2 u_i u_j - B_i B_j + (p + \boldsymbol{B}^2/2)\delta_{ij}$$

- If spectral slope of B is -5/3, then

- Spectral slope of B^2 is -5/3-2 = -11/3
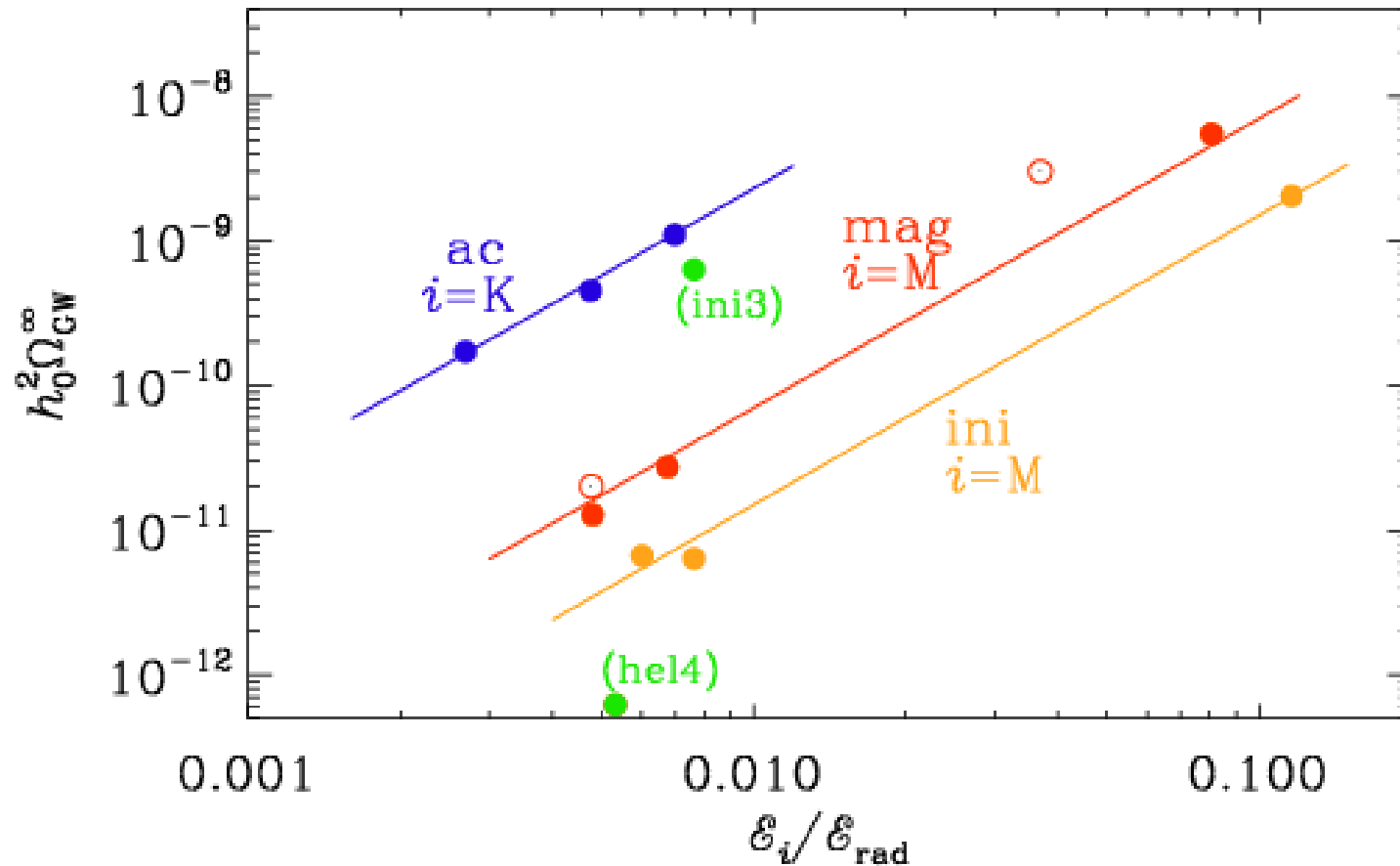
- But for slope 4, we don't get 4-2 = 2, but 0.

30
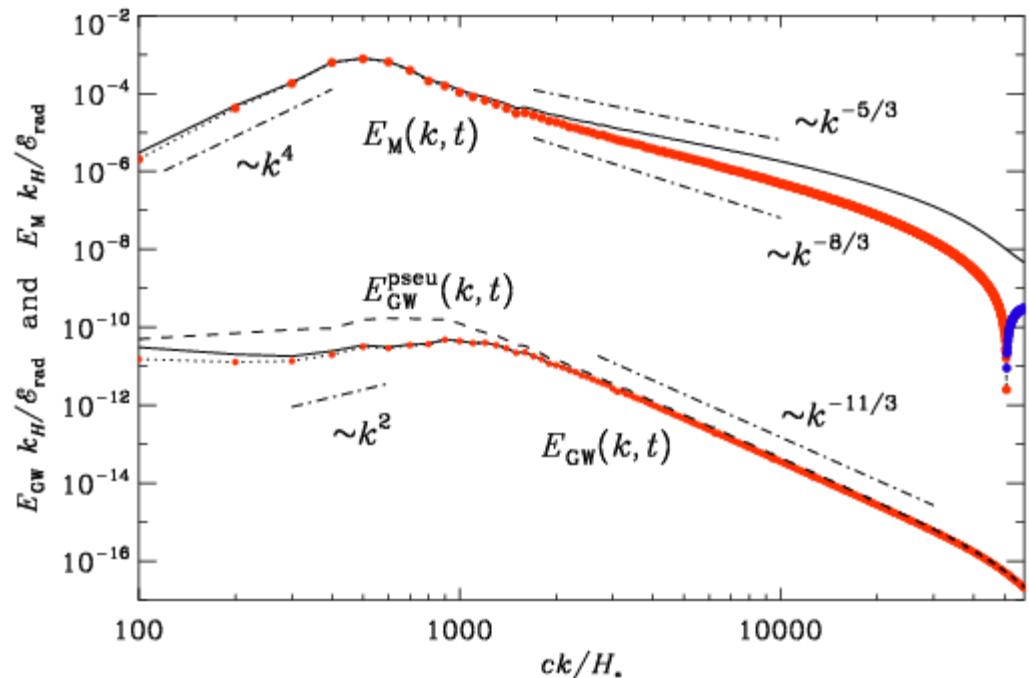
- Frequency ~3 mHz
  - Slope corresponds to turbulence spectrum
  - magnetic energy 10% or 1% or radiation

- Observable w/ LISA
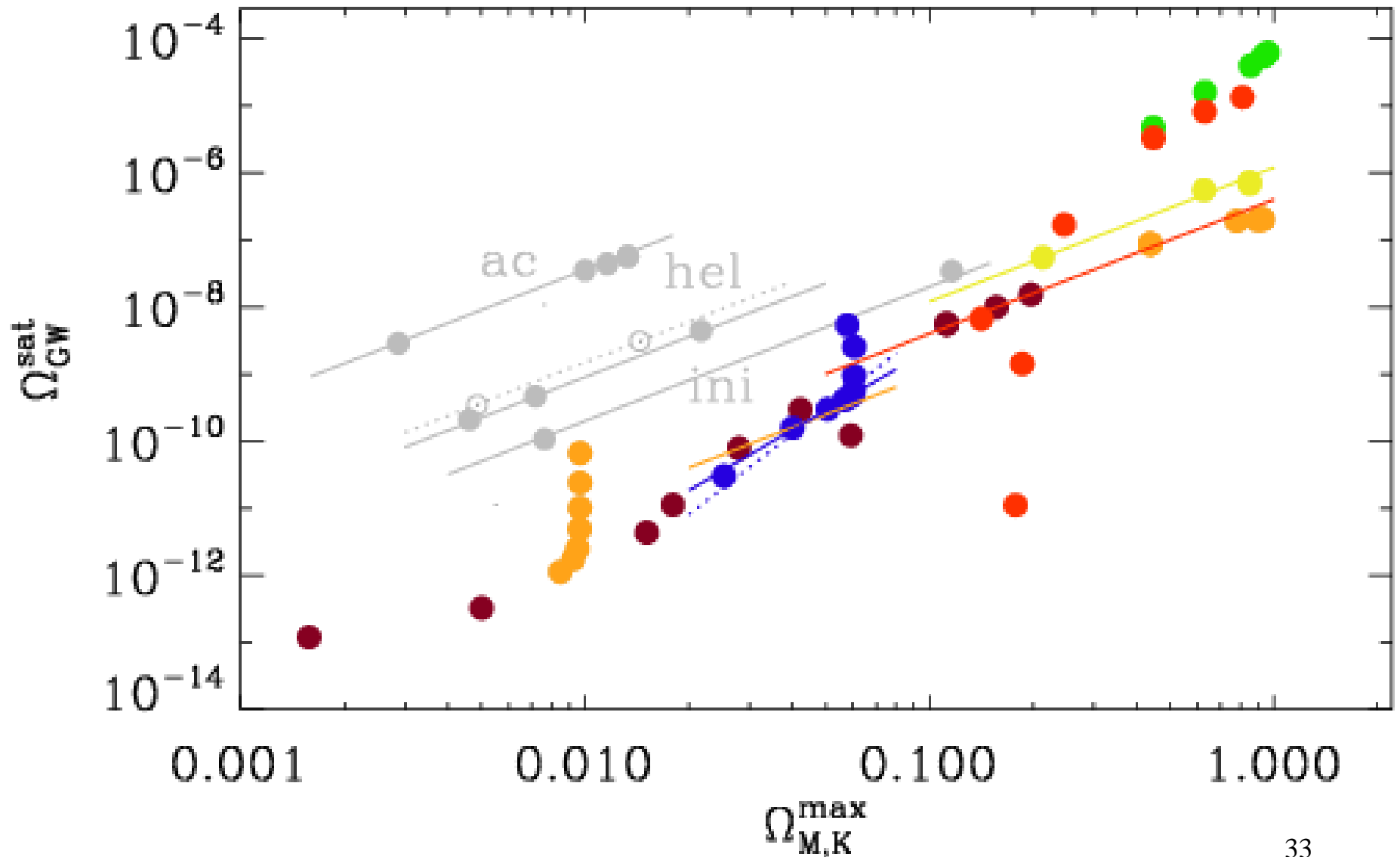 - arm length 2-5 milion km
  - Duration 2 or 5 yr

Quadratic scaling

- Acoustic driving → strongest GW field
- Quadratic scaling

GWs from chiral magnetic effect

Circular Polarization of Gravitational Waves from Early-Universe Helical Turbulence

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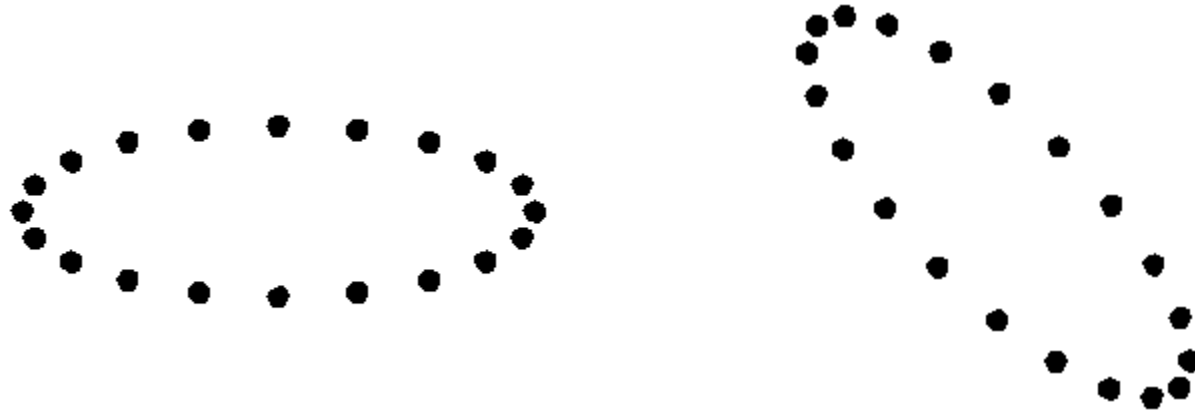[7] Physics and Astronomy Department, Stony Brook University, Stony Brook, New York 11794, USA
[8] Université de Paris, CNRS, Astrophysique et Cosmologie, Paris, F-75013, France
(Dated: November 12, 2020)
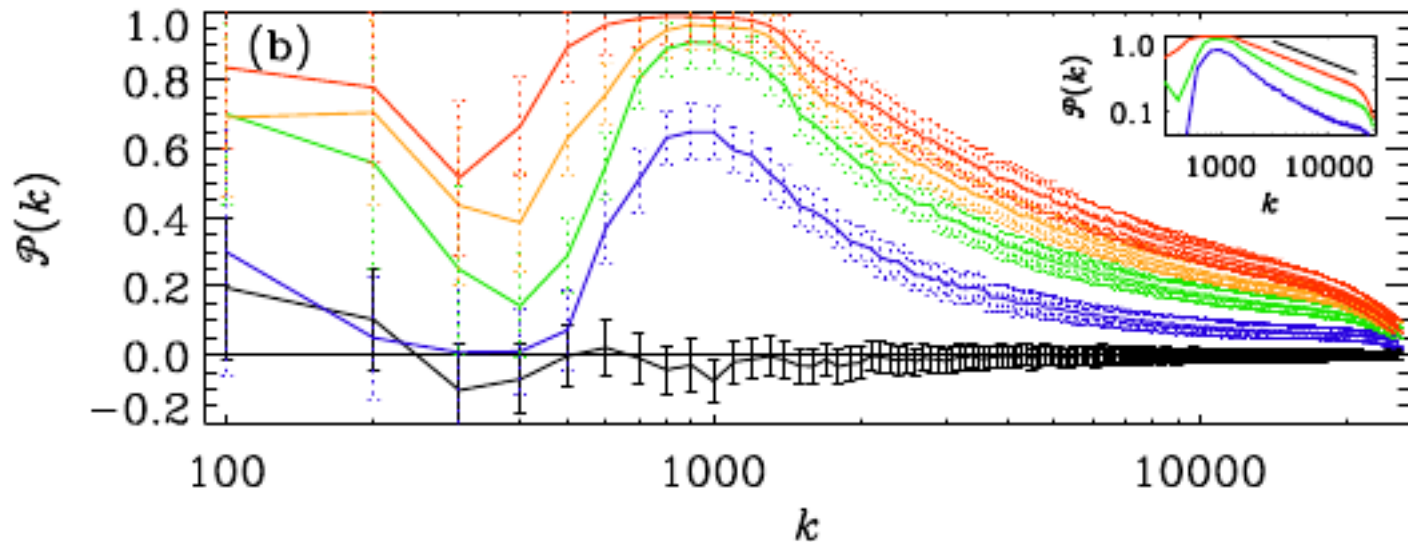
- Magnetic helicity causes circular polarized GWs
- Can reach 100%, and inverse cascade apparent
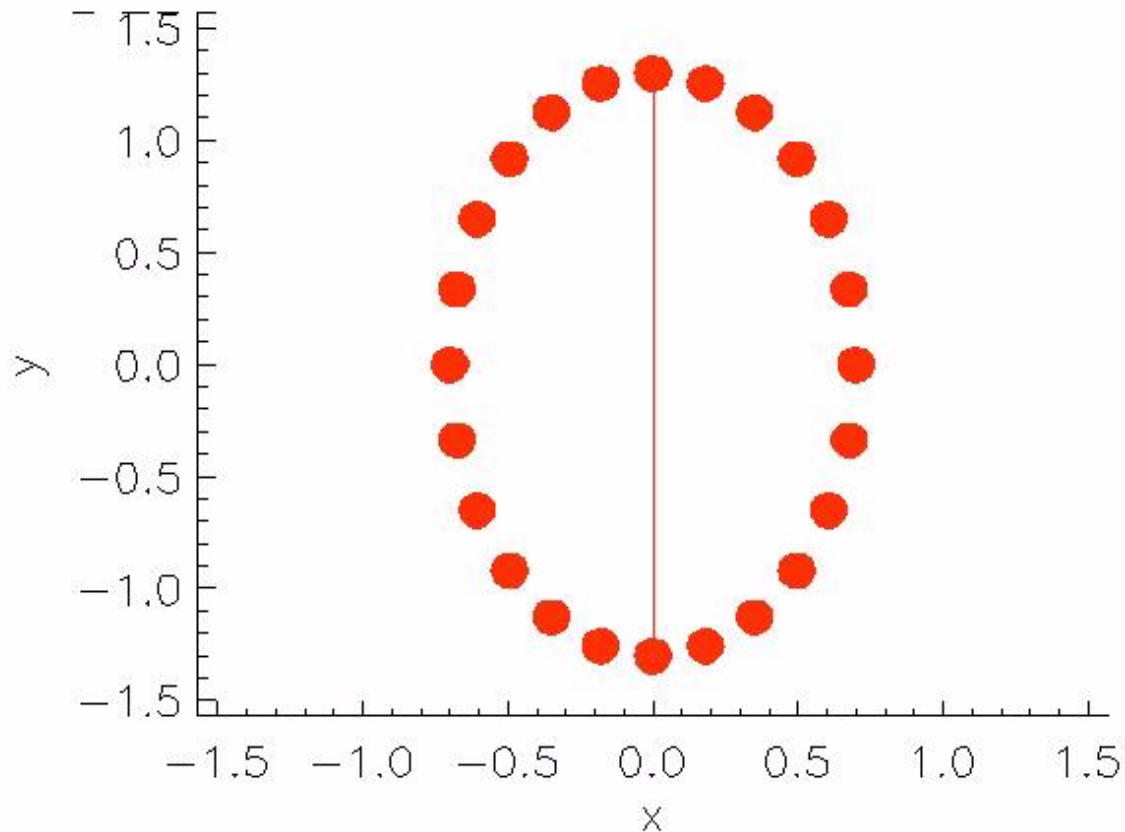
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Correspond to + and x modes

$$\begin{pmatrix} \langle h_+(\boldsymbol{n})h_+^*(\boldsymbol{n}') \rangle & \langle h_+(\boldsymbol{n})h_\times^*(\boldsymbol{n}') \rangle \\ \langle h_+^*(\boldsymbol{n})h_\times(\boldsymbol{n}') \rangle & \langle h_\times(\boldsymbol{n})h_\times^*(\boldsymbol{n}') \rangle \end{pmatrix}$$

$$\frac{\delta_{drc}(\boldsymbol{n}-\boldsymbol{n}')}{4\pi} \begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix}$$

Seto
(2006)

Circular polarization in space & time

- Both plus and cross polarization together
- Combine the two as a function of space & time
- Get circular polarization

Conclusions

- Magnetic helicity nearly perfectly conserved
  - Catastrophic quenching in periodic boxes
- Inverse cascade in decaying turbulence
  - Important in early Universe
- Can be initiated by chiral magnetic effect (CME)
  - But may not yet explain *lower* observational limits
- CME also drives gravitational waves
  - Currently somewhat too weak

Inverse cascading