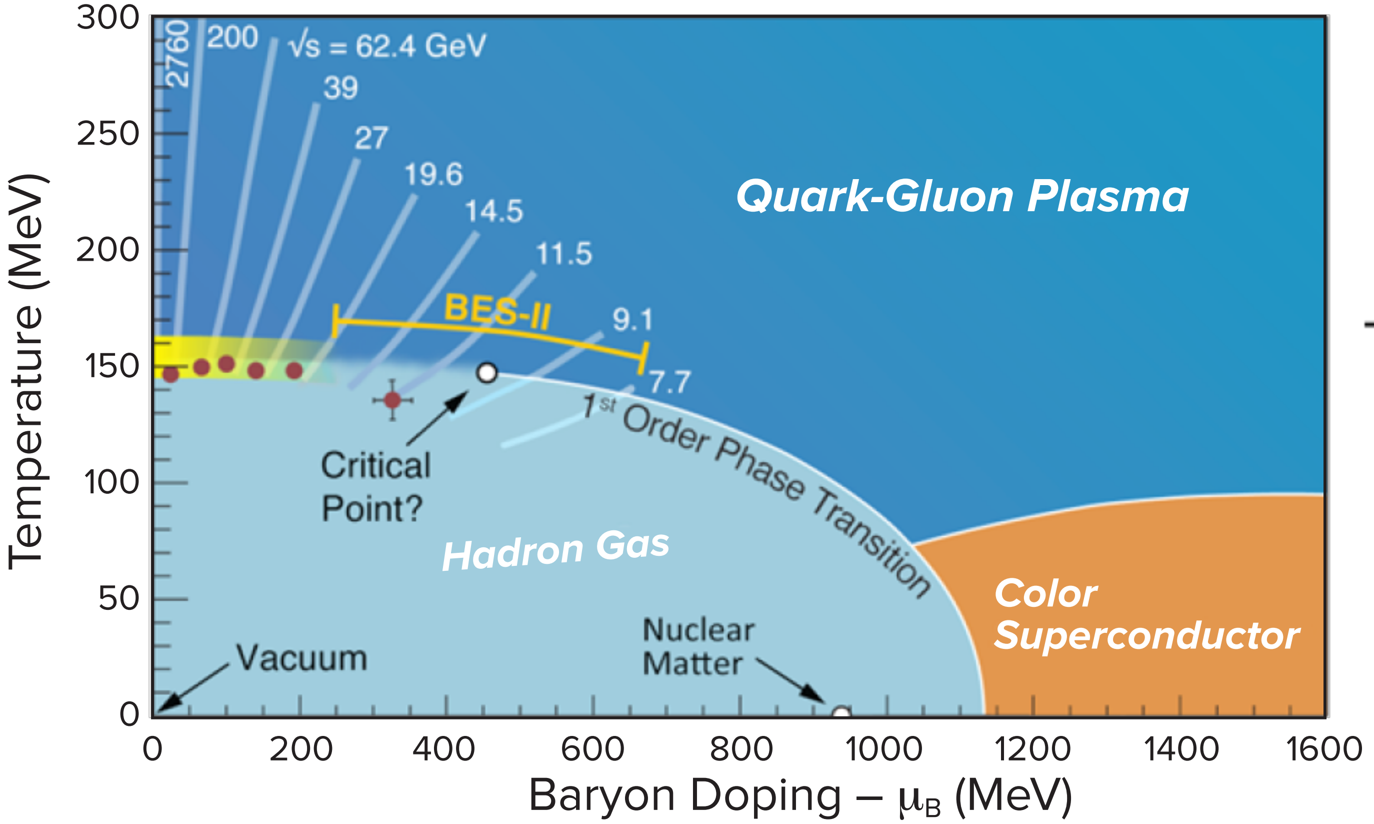
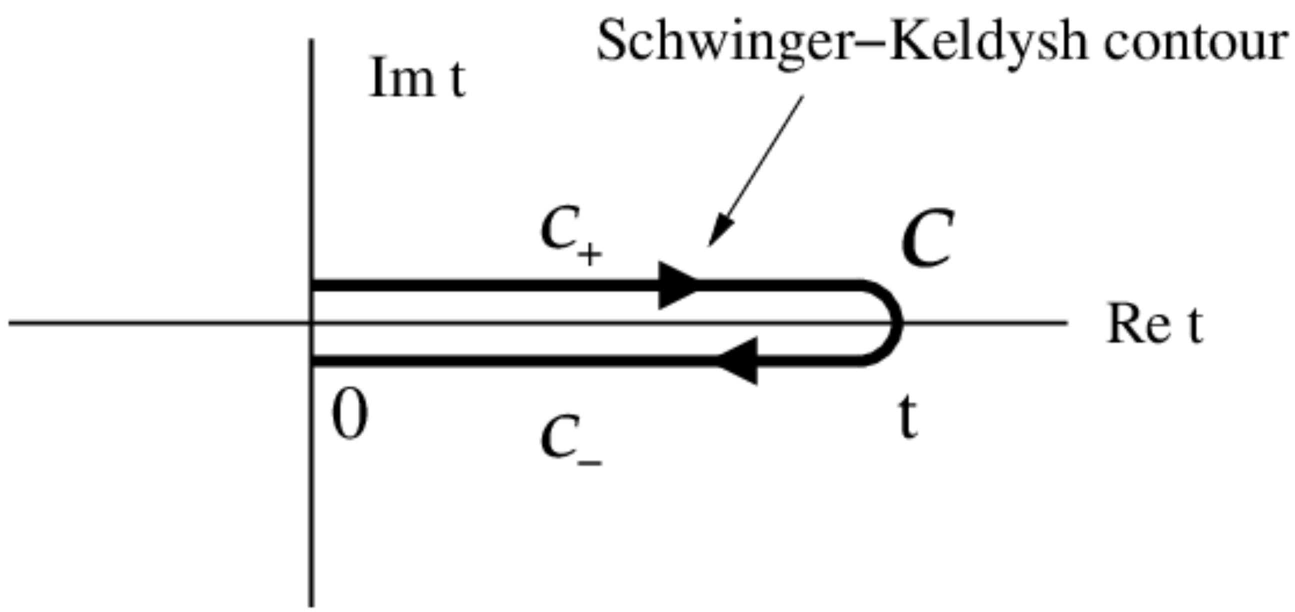
Decoding the Path Integral: Resurgent Asymptotics and Extreme QFT

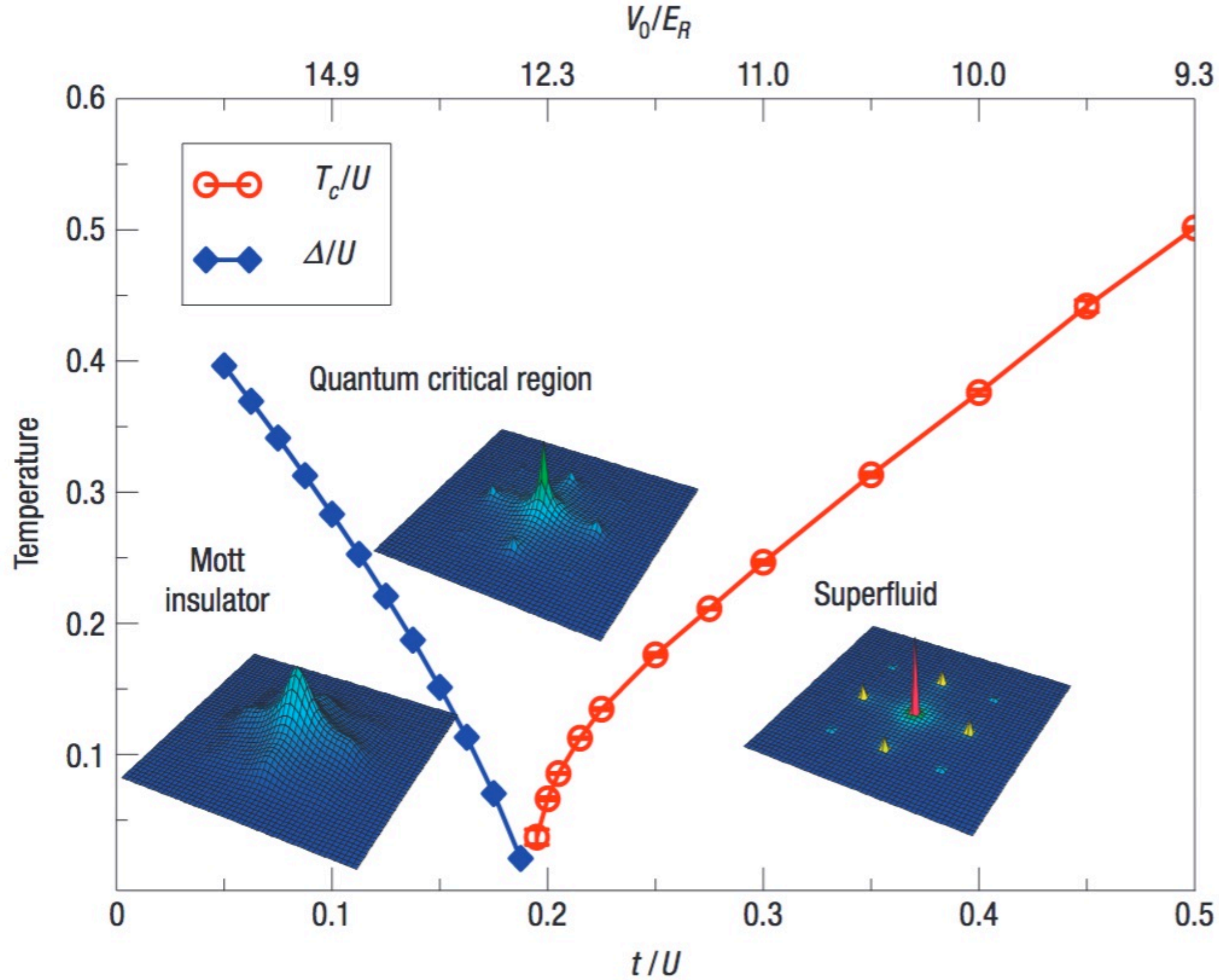
Gerald Dunne
University of Connecticut

Theoretical Physics Colloquium, Arizona State, October 28, 2020

Physical Motivation







Physical Motivation: Quantum Physics in Extreme Conditions

- QCD phase diagram
- non-equilibrium physics at strong-coupling
- quantum systems in extreme background fields
- back-reaction physics
- transition to hydrodynamics

extreme = strongly-coupled; high density; ultra-fast driving; ultra-cold; strong fields; strong curvature; heavy ion collisions; …

- perturbation theory
- non-perturbative semi-classical methods: “instantons"
- non-perturbative numerical methods: Monte Carlo
- asymptotics

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“resurgence”: new form of asymptotics that unifies these approaches

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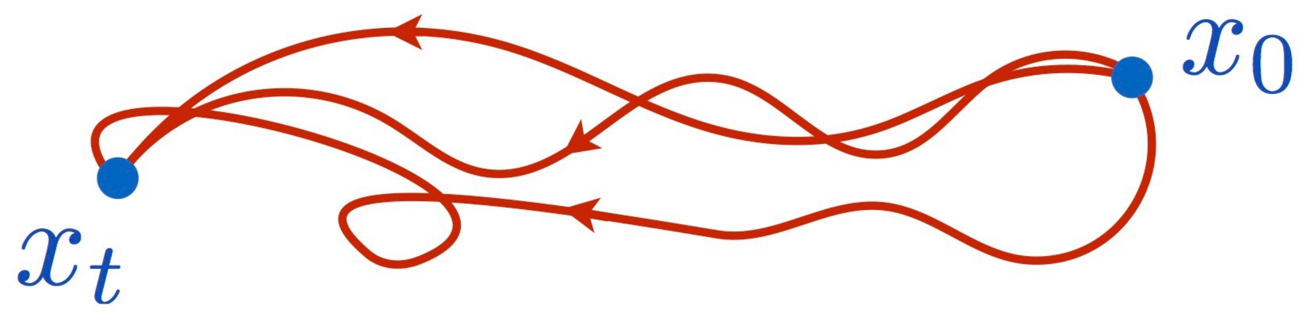
“resurgence”: new form of asymptotics that unifies these approaches

technical problem: what does a quantum path integral really mean?

The Feynman Path Integral

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



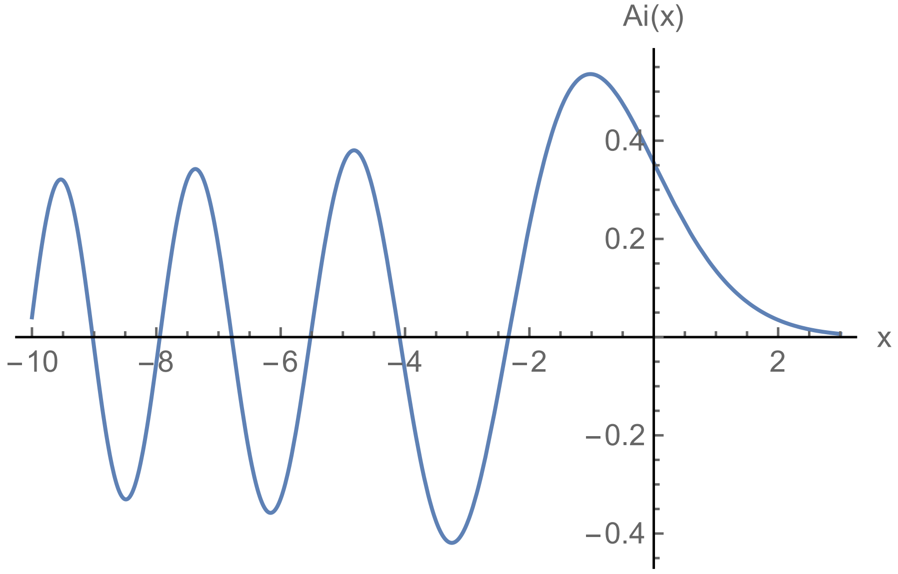
$$\text{QM:} \quad \int \mathcal{D}x(t)\ \exp\left[\frac{i}{\hbar} S[x(t)]\right]$$

$$\text{QFT:} \quad \int \mathcal{D}A(x^\mu)\ \exp\left[\frac{i}{g^2} S\left[A(x^\mu)\right]\right]$$

- stationary phase approximation: classical physics
- quantum perturbation theory: fluctuations about trivial saddle point
- other saddle points: non-perturbative physics
- <u>resurgence</u>: saddle points are related by analytic continuation, so perturbative and non-perturbative physics are *unified*

Stokes and the Airy Function: “Stokes Phenomenon”

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{i\left(\frac{1}{3}t^3 + x\,t\right)}$$



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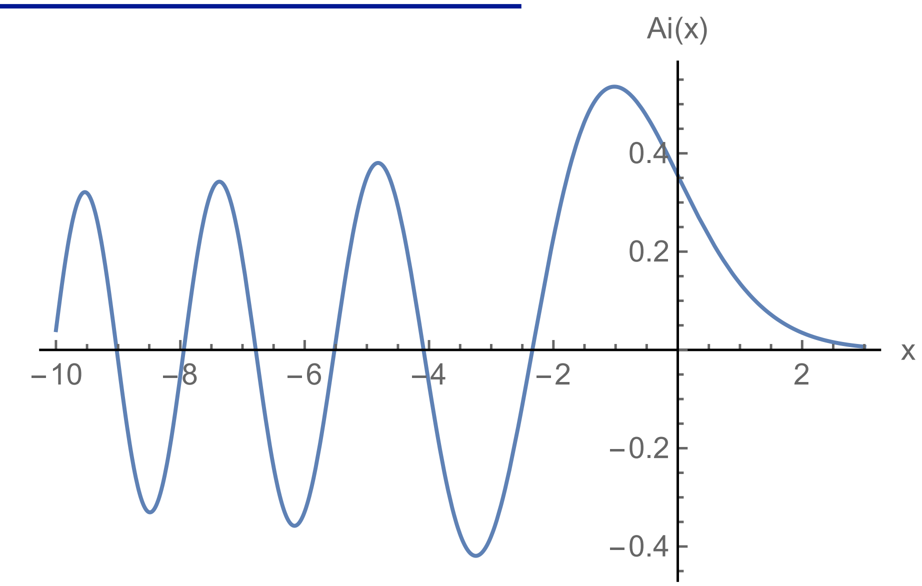
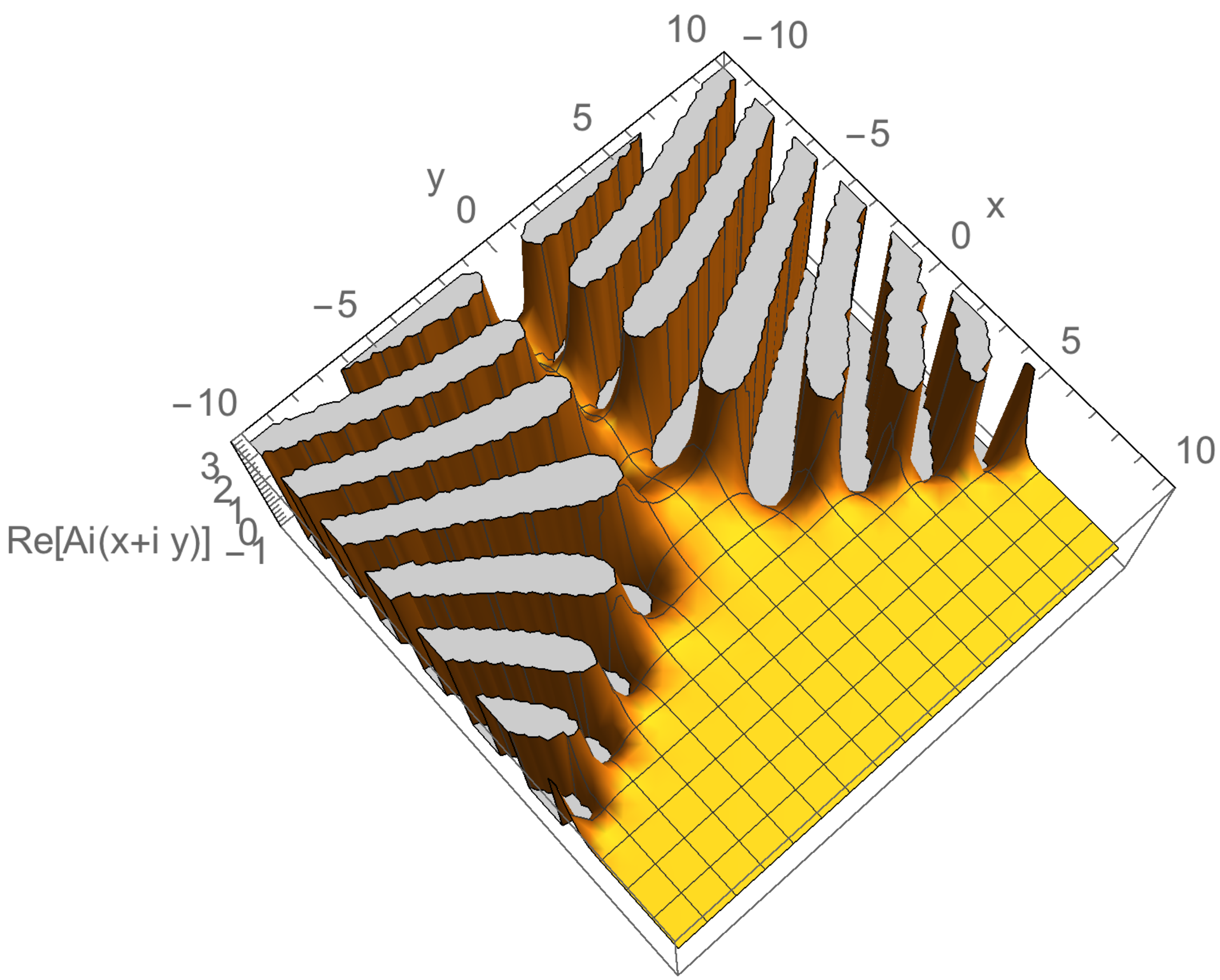


- Stokes transitions occur in the complex x plane



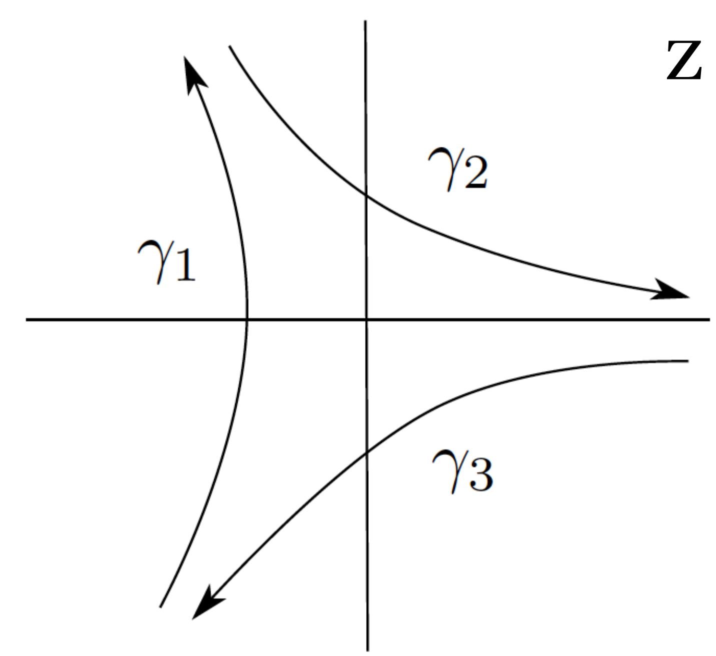
Airy Stokes sectors

anti-Stokes lines: $\arg(x) = \pm\frac{\pi}{3}, \pi$

Stokes lines: $\arg(x) = \pm\frac{2\pi}{3}, 0$

- non-perturbative connection formulae connect sectors

$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi\, i}\int_\gamma dz\, e^{-r^{3/2}\left(\frac{1}{3}z^3 - e^{i\theta}\, z\right)}$$



Analytic Continuation of Path Integrals

since we need complex analysis and contour deformation to make sense of oscillatory exponential integrals, it is natural to explore similar methods for (infinite dimensional) path integrals

$$\int \mathcal{D}x(t)\, \exp\left[\frac{i}{\hbar}\, S[x(t)]\right] \longleftrightarrow \int \mathcal{D}x(t)\, \exp\left[-\frac{1}{\hbar}\, S[x(t)]\right]$$

goal: a satisfactory formulation of the functional integral should be able to describe Stokes transitions

idea: seek a computationally viable constructive definition of the path integral using ideas from resurgent trans-series

Resurgent Trans-Series

resurgence: "new" idea in mathematics

Ecalle 1980s; Dingle 1960s; Stokes 1850

perturbative series $\longrightarrow$ "trans-series"

$$f(\hbar) = \sum_p c_{[p]}\, \hbar^p \quad \longrightarrow \quad f(\hbar) = \sum_k \sum_p \sum_l c_{[kpl]}\, e^{-\frac{k}{\hbar}}\, \hbar^p\, (\ln \hbar)^l$$

physics:
- unifies perturbative and non-perturbative physics

mathematics:
- trans-series is well-defined under analytic continuation
- expansions about different saddles are related
- exponentially improved asymptotics

Resurgent Functions

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

J. Ecalle, 1980

implication: fluctuations about different singularities are related

conjecture: this structure occurs for all "natural problems"

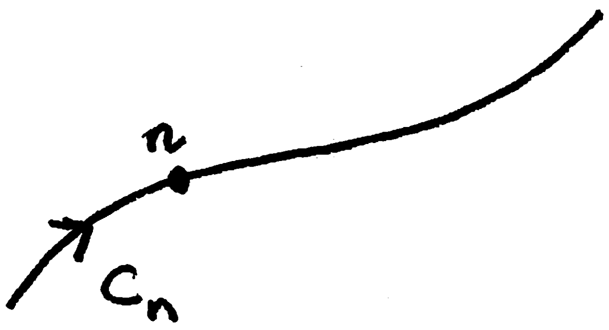
Resurgence in Exponential Integrals

steepest descent integral through saddle point "n":

$$I^{(n)}(\hbar) = \int_{C_n} dx\, e^{\frac{i}{\hbar}\, f(x)} = \frac{1}{\sqrt{1/\hbar}}\, e^{\frac{i}{\hbar}\, f_n}\, T^{(n)}(\hbar)$$

all fluctuations beyond the Gaussian approximation:

$$T^{(n)}(\hbar) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, \hbar^r$$



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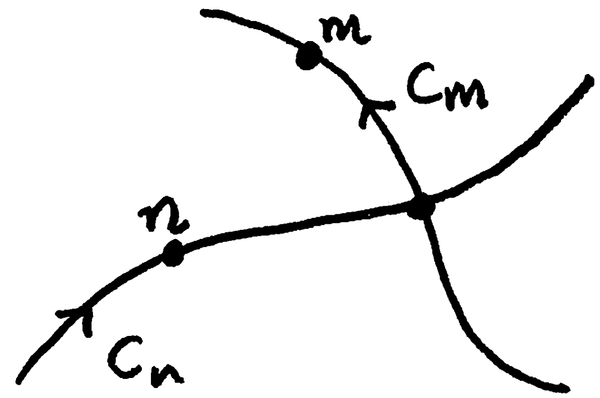

$$T^{(n)}(\hbar) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, \hbar^r$$

straightforward complex analysis implies:

universal large orders of fluctuation coefficients: $(F_{nm} \equiv f_m - f_n)$

$$T_r^{(n)} \sim \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(\pm 1)}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)}\, T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)}\, T_2^{(m)} + \dots \right]$$

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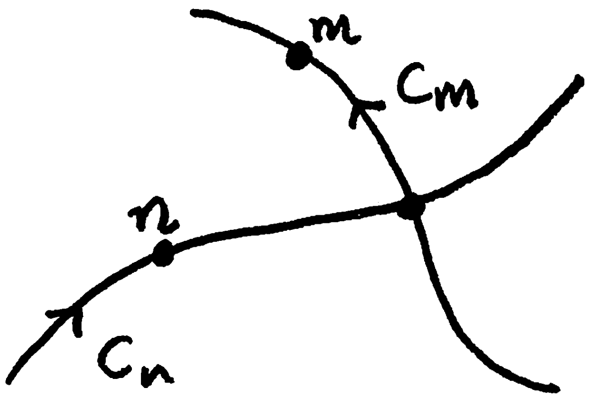

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fluctuations about different saddles are quantitatively related

Resurgence in Exponential Integrals

canonical example: Airy function: 2 saddle points

$$T_r^{\pm} = (\pm 1)^r \frac{\Gamma\left(r+\frac{1}{6}\right)\Gamma\left(r+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^r r!} = \left\{1, \pm\frac{5}{48}, \frac{385}{4608}, \pm\frac{85085}{663552}, \dots\right\}$$

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large orders of fluctuation coefficients:

$$T_r^{+} \sim \frac{(r-1)!}{(2\pi)\left(\frac{4}{3}\right)^r}\left(1 - \left(\frac{4}{3}\right)\frac{5}{48}\frac{1}{(r-1)} + \left(\frac{4}{3}\right)^2\frac{385}{4608}\frac{1}{(r-1)(r-2)} - \dots\right)$$

generic "large-order/low-order" resurgence

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generic “large-order/low-order” resurgence

amazing fact: this resurgent large-order/low-order behavior has been found in matrix models, QM, QFT, string theory, …

the only natural way to explain this is via analytic continuation of functional integrals

Perturbation Theory

perturbation theory works, but it is generically divergent

perturbation theory encodes non-perturbative information

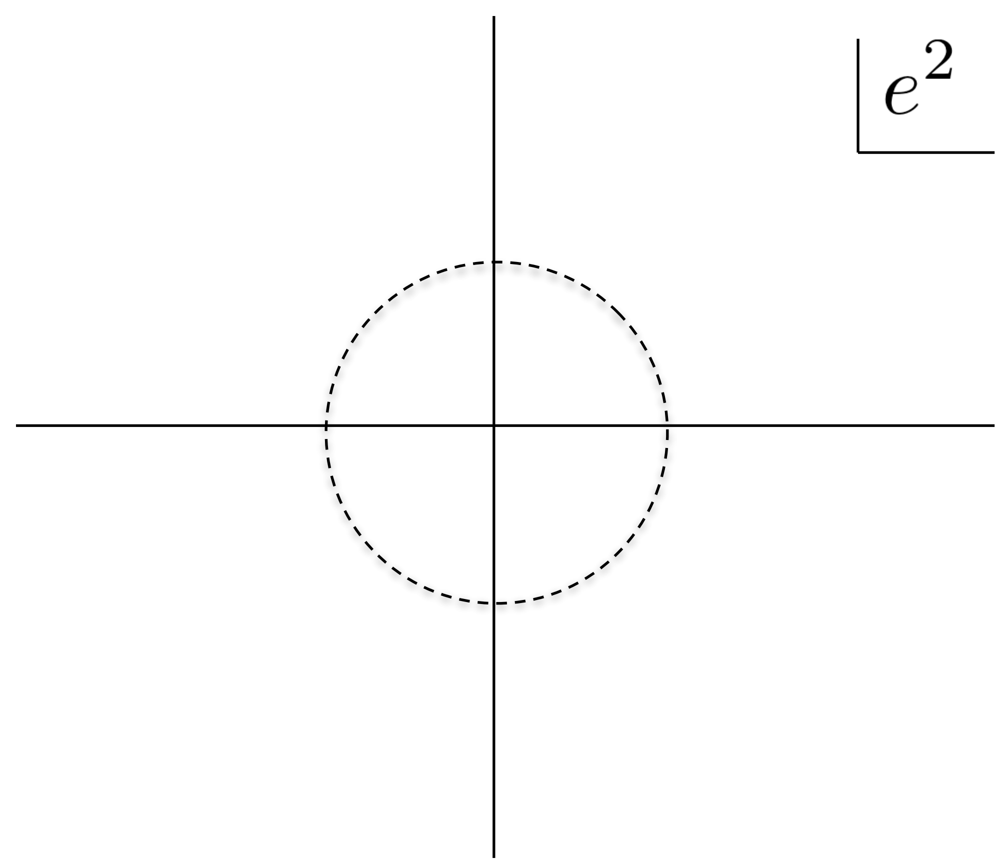
Divergence of Perturbation Theory in Quantum Electrodynamics

F. J. Dyson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

(Received November 5, 1951)

An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

$$F = a_0 + a_1 e^2 + a_2 e^4 + a_3 e^6 + \ldots$$



PHYSICAL REVIEW VOLUME 85, NUMBER 4 FEBRUARY 15, 1952

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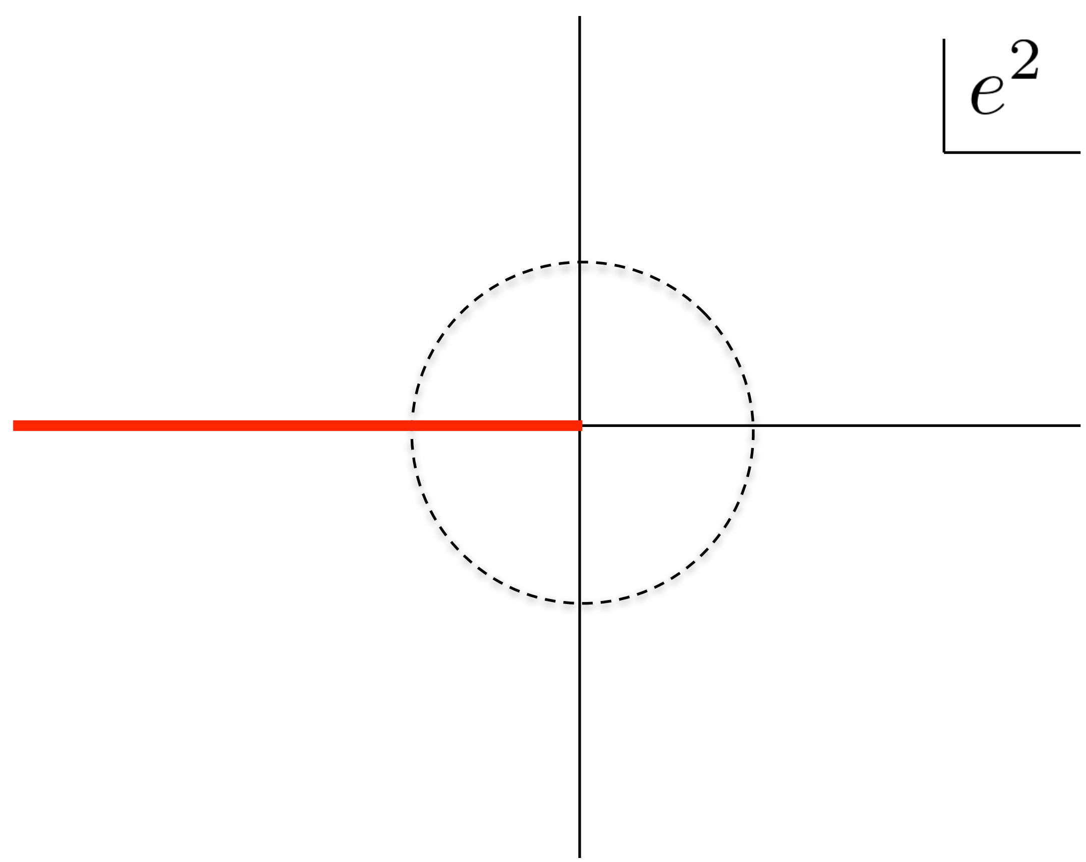


$e^2 < 0$

unstable

Borel Summation: the Physics of Divergent Series

Borel transform of a divergent series with $c_n \sim n!$

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \quad \rightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\, t^n$$

Borel sum of the divergent series:

$$\mathcal{S}[f](g) = \frac{1}{g} \int_0^{\infty} dt\, e^{-t/g}\, \mathcal{B}[f](t)$$

- the singularities of **B[f](t)** provide a physical encoding of the global asymptotic behavior of **f(g)**, which is also much more mathematically efficient than the asymptotic series
- singularities of Borel transform ⟷ non-perturbative physics
- singularities on positive Borel t axis: exponentially small imaginary part

QM Perturbation Theory: Zeeman & Stark Effects

Zeeman : divergent, alternating, asymptotic series

$$a_n \sim (-1)^n (2n)!$$

Borel singularities on the negative Borel axis.

physics: Magnetic field causes (real) energy level shifts

Stark : divergent, non-alternating, asymptotic series

$$a_n \sim (2n)!$$

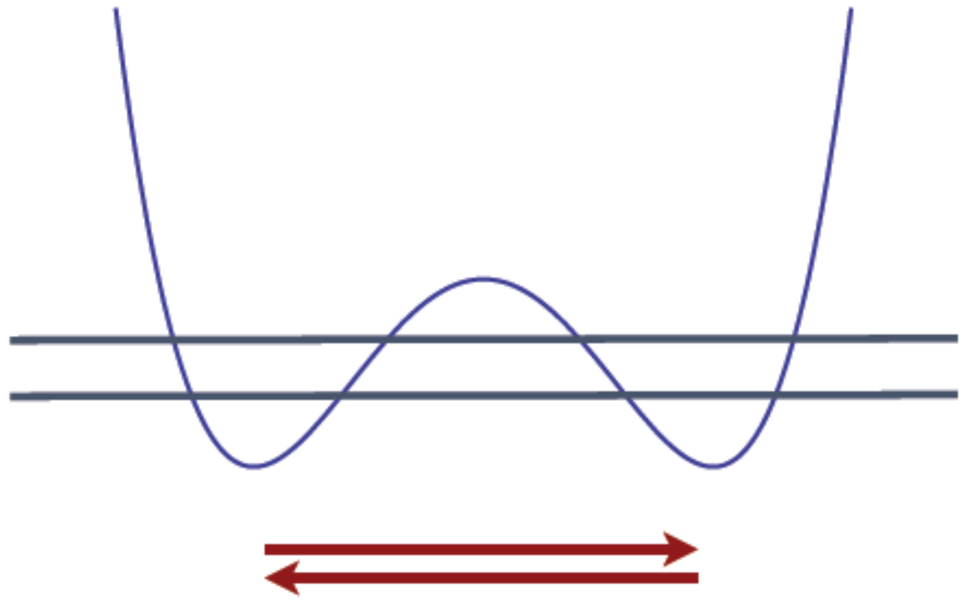
Borel singularities on the positive Borel axis.

physics: Electric field causes (real) energy level shifts

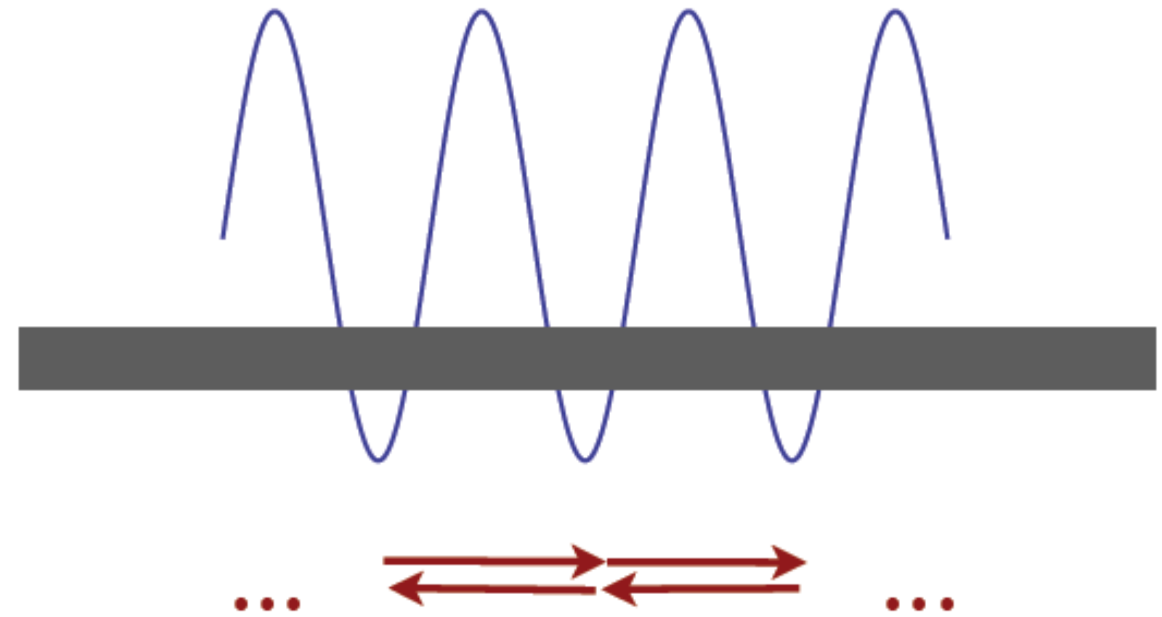
and ionization (imaginary, exponentially small) but not so fast …

the story becomes even more interesting …

Instantons and Non-Perturbative Physics



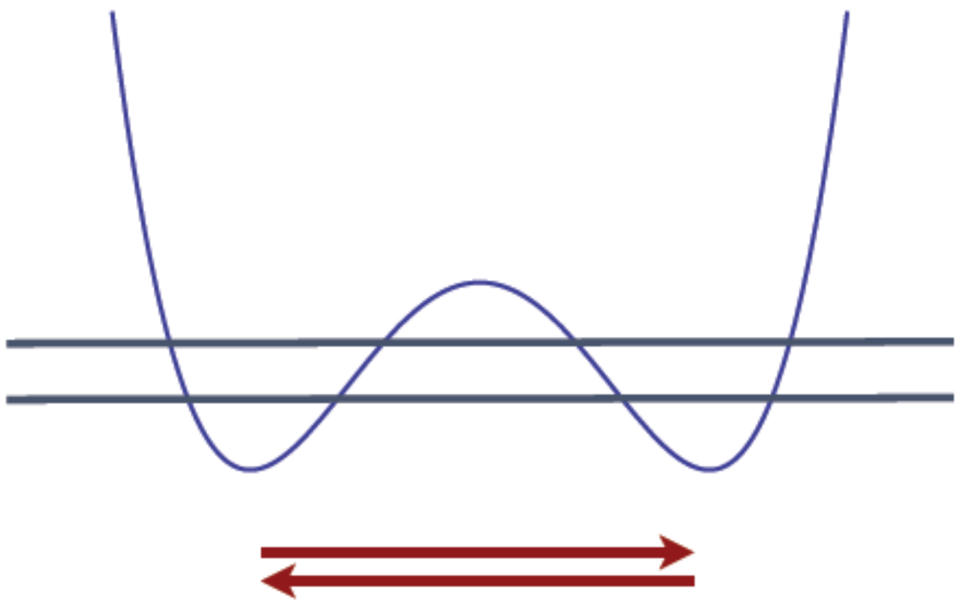
(phase transitions)



(band structure)

- exponentially small non-perturbative splitting due to tunneling
- Yang-Mills theory and QCD have aspects of both systems

Instantons and Non-Perturbative Physics



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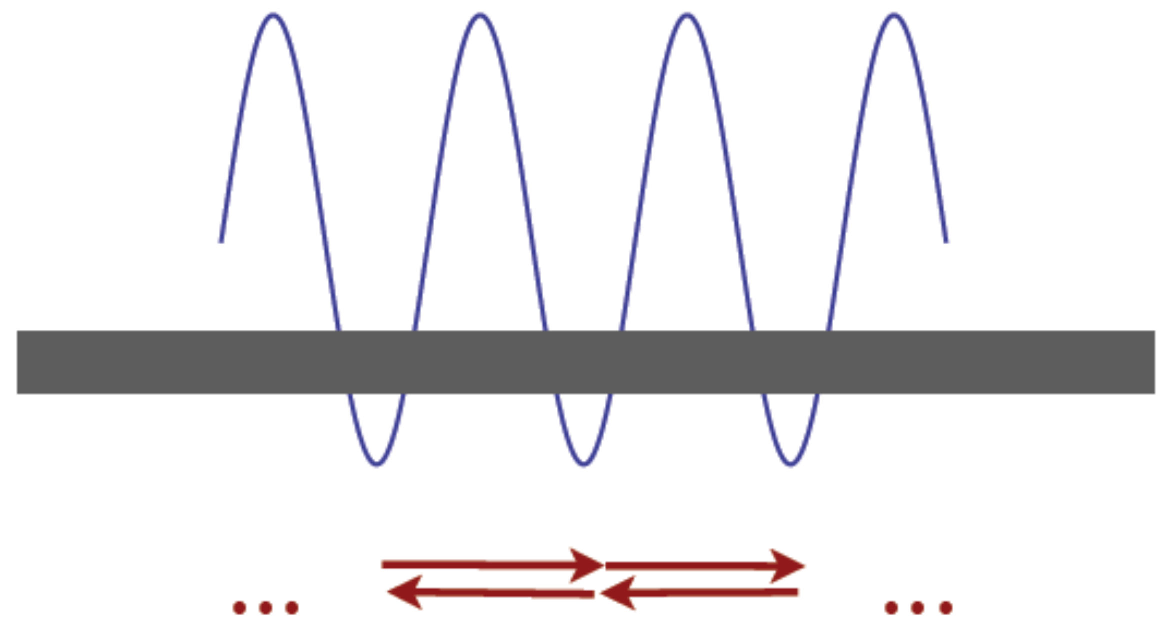

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surprise: perturbation theory is non-alternating divergent

but these systems are stable ???

A Brilliant Resolution: "BZJ Cancelation Mechanism"

E. B. Bogomolny, 1980; J. Zinn-Justin et al, 1980

perturbation theory + Borel: $\longrightarrow$ $+i \, \exp\left[-\frac{2\, S_I}{\hbar}\right]$

non-perturbative instanton & anti-instanton interaction: $\longrightarrow$ $-i \, \exp\left[-\frac{2\, S_I}{\hbar}\right]$

unphysical imaginary parts exactly cancel

separately, each of these perturbative and non-perturbative computations is inconsistent; but combined as a trans-series they are consistent

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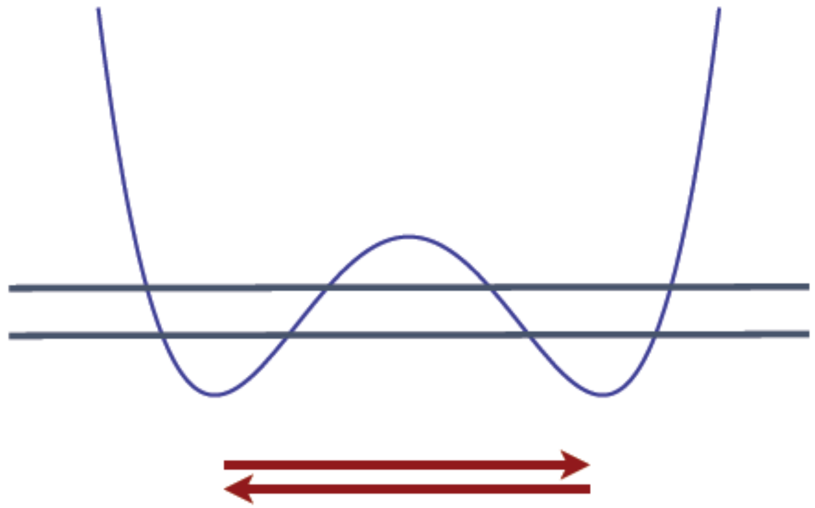
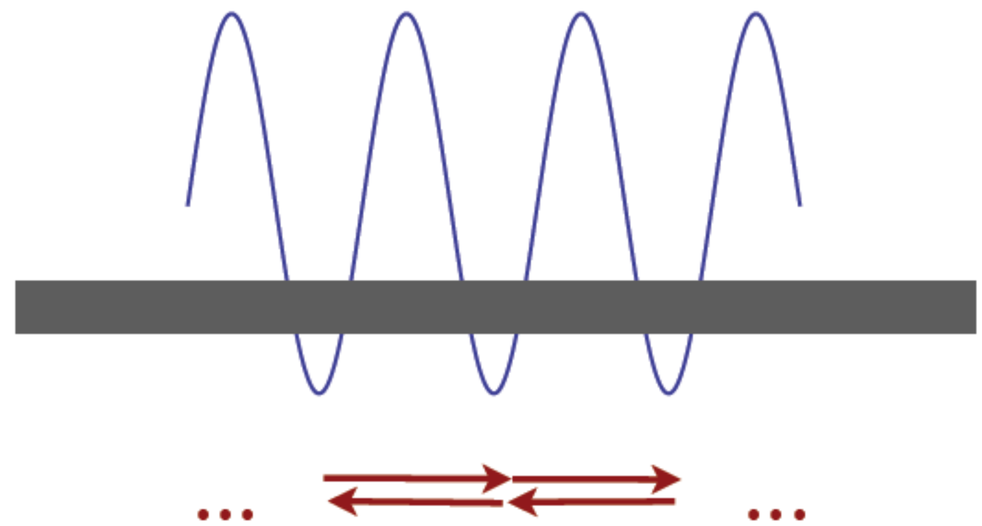
tip-of-the-iceberg: perturbative/non-perturbative relations

"Resurgence": cancelations occur to all orders; the trans-series expression for the energy is real & well-defined

Resurgence in Quantum Mechanical Instanton Models





trans-series for energy, including non-perturbative splitting:

$$E_{\pm}(\hbar, N) = E_{\text{pert}}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right] \mathcal{P}_{\text{inst}}(\hbar, N) + \ldots$$

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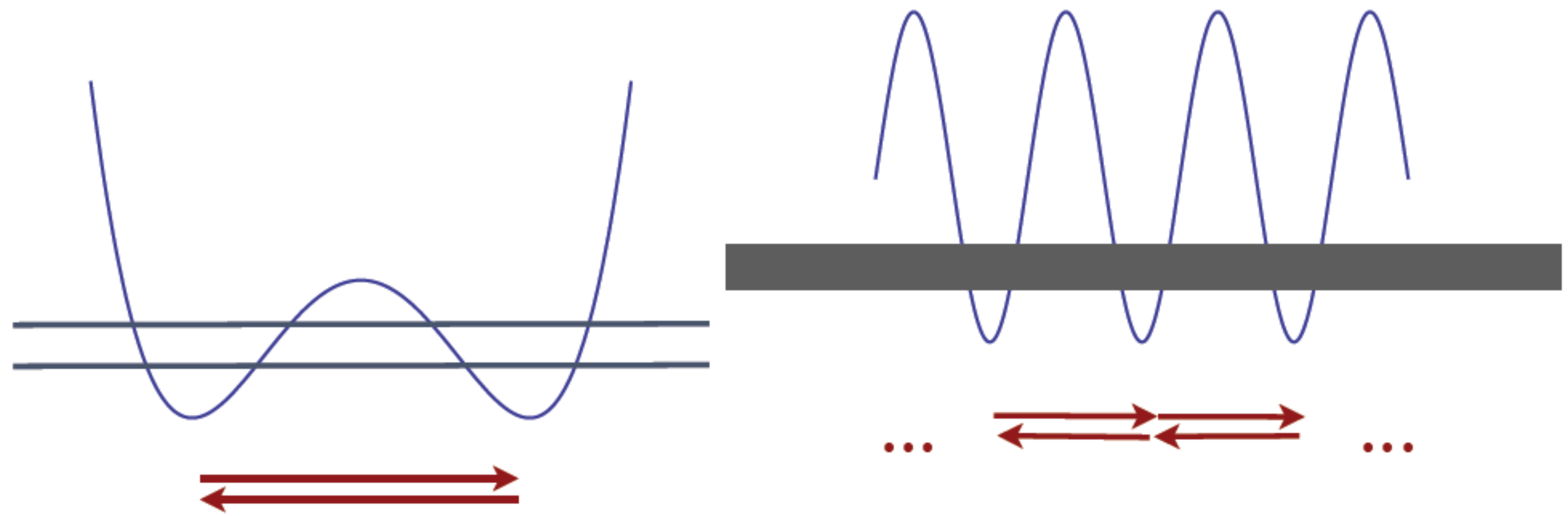

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fluctuations about first non-trivial saddle:

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

perturbation theory encodes everything … to all orders

Resurgent Functions

resurgent relations seen in QM path integrals with an infinite number of saddles

Parametric Resurgence and Phase Transitions

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right)$$

- in practice, we are interested in many parameters

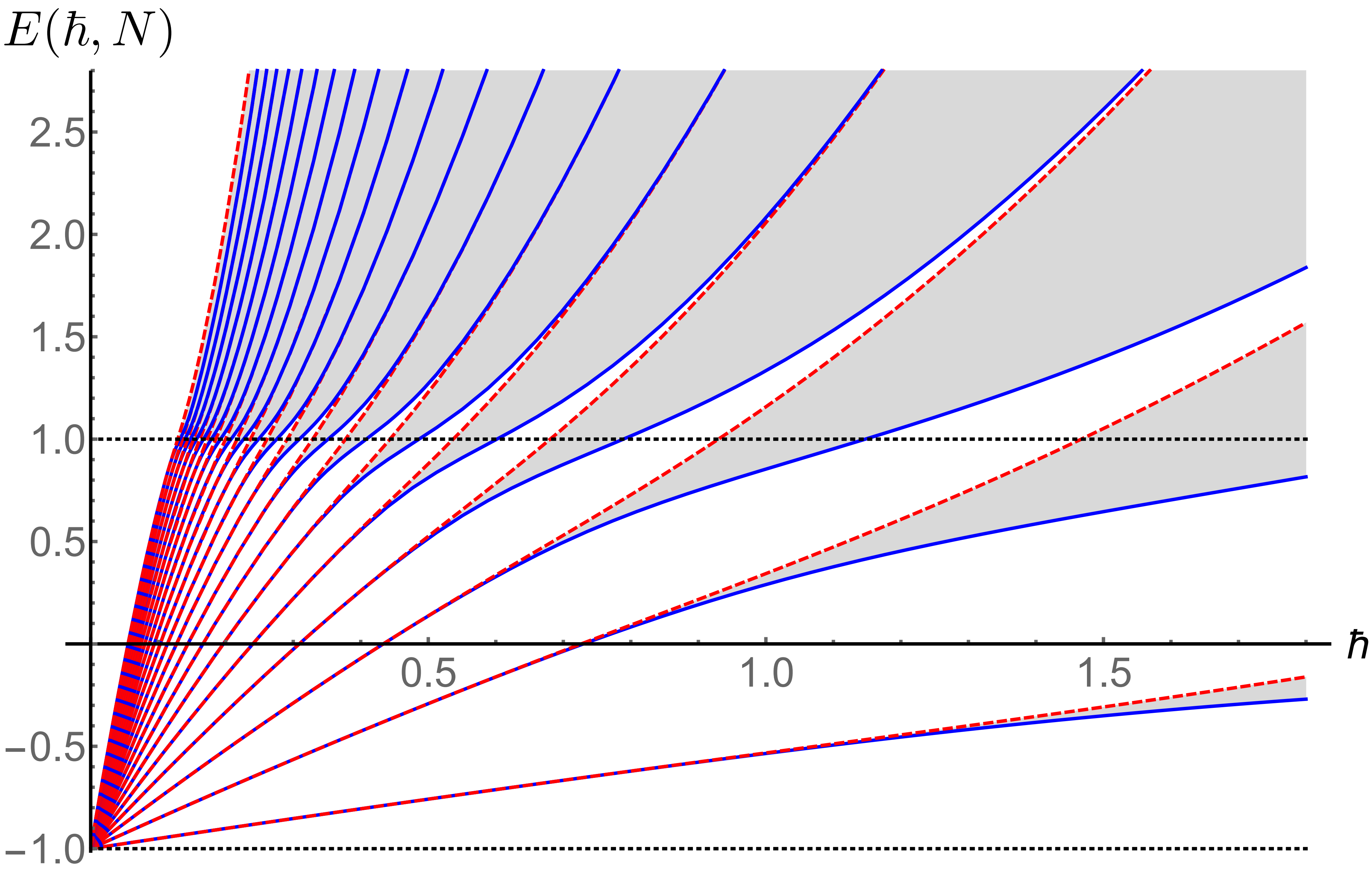
$$Z(\hbar) \;\to\; Z(\hbar, \text{masses}, \text{couplings}, \mu, T, B, \ldots)$$

- e.g., for a phase transition: large N ``thermodynamic limit''

$$Z(\hbar) \;\to\; Z(\hbar, N), \text{ and } N \to \infty$$

- multiple parameters: different limits are possible
- "uniform" 't Hooft limit: $N \to \infty \;,\; \hbar \to 0 \;:\; \hbar\, N = \text{fixed}$
- trans-series transmutes into different form in the large N limit
- hallmark of a phase transition

Phase Transition in the Periodic Potential Spectrum



- N= band/gap label; $\hbar$ =coupling
- phase transition: narrow bands vs. narrow gaps: $\hbar N = \frac{8}{\pi}$
- real instantons vs. complex instantons
- phase transition = "instanton condensation"
- universal phase transition

Neuberger, 1981;
Basar, GD, 1501.05671,
GD, Unsal, 1603.04924

Resurgence in QFT: Euler-Heisenberg Effective Action

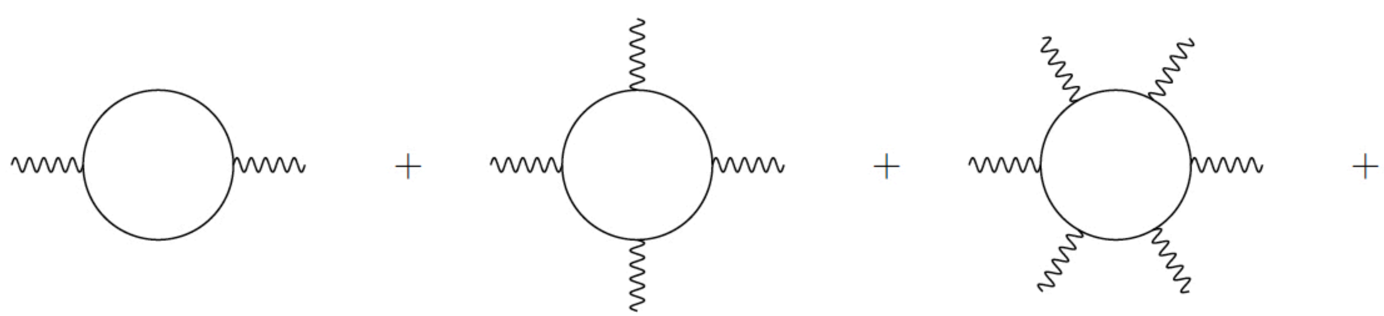
Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \mathrm{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \mathrm{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$



- paradigm of an effective field theory
- integral representation = Borel sum
- analogue of Stark effect ionization and Dyson’s argument

Stokes Phase Transition in QFT

- Schwinger effect with monochromatic E field: $E(t) = \mathcal{E}\, \cos(\omega\, t)$
- Keldysh adiabaticity parameter: $\gamma \equiv \frac{m\, c\, \omega}{e\, \mathcal{E}}$
- WKB: $\Gamma_{\rm QED} \sim \exp\left[-\pi \frac{m^2\, c^3}{e\, \hbar\, \mathcal{E}}\, g(\gamma)\right]$

(Keldysh, 1964; Brezin/Itzykson, 1980; Popov, 1981)

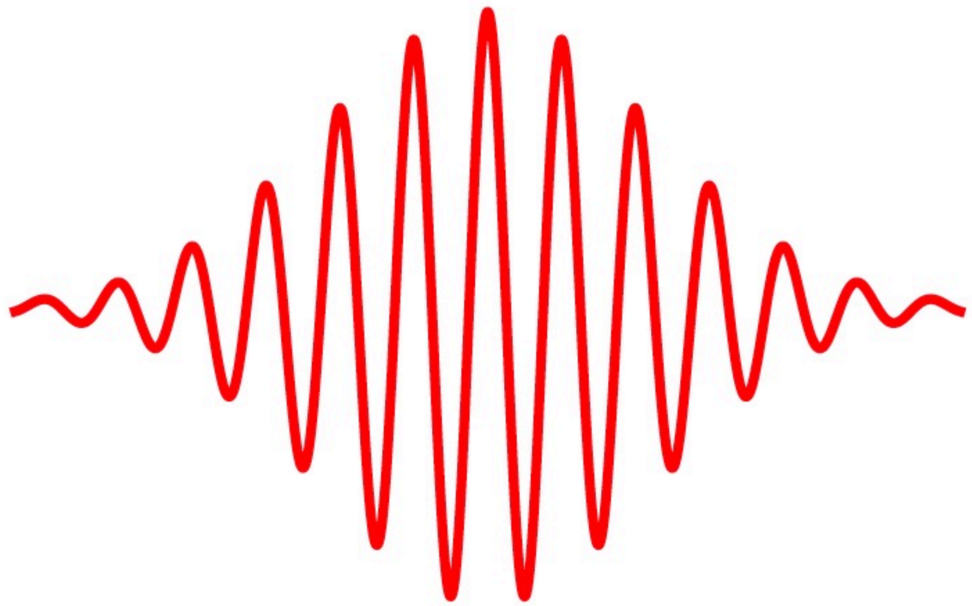
$$\Gamma_{\rm QED} \sim \begin{cases} \exp\left[-\pi \frac{m^2\, c^3}{e\, \hbar\, \mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(tunneling)} \\ \left(\frac{e\, \mathcal{E}}{m\, c\, \omega}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(multiphoton)} \end{cases}$$

- phase transition: tunneling vs. multi-photon “ionization”
- phase transition: real vs. complex instantons (GD, Dumlu,1004.2509, 1102.2899)
- the same transition as in the Mathieu equation
- applications to worldline representation of QFT (Raju’s colloquium)

Resurgence in QFT: Ultra-Fast Dynamics

time evolution of quantum systems with ultra-fast perturbations

$|0\rangle_{\rm in}$



$|0\rangle_{\rm out}$

$t = -\infty$ intermediate time $t = +\infty$

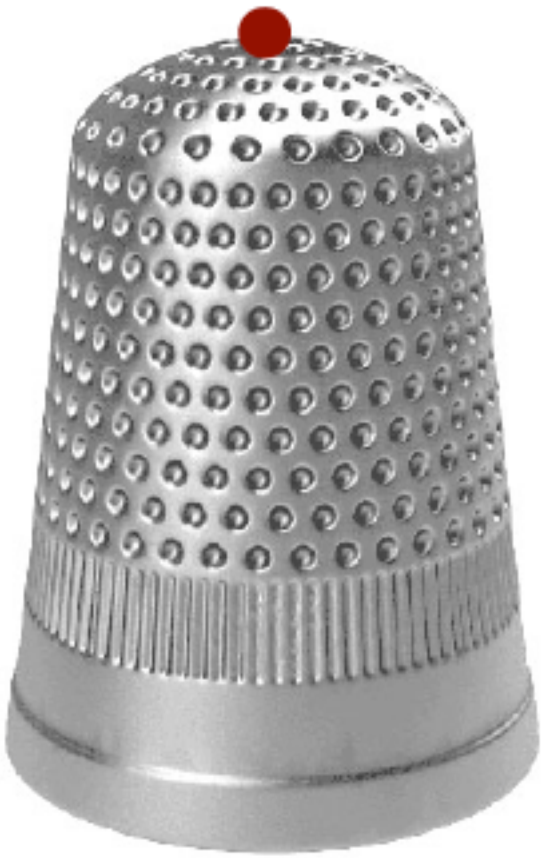
- the adiabatic/gradient expansion is divergent
- resurgence: expansion can be (Borel) resummed to a <u>universal form</u>
- novel quantum interference effects: complex saddles
- applications in QFT, and in AMO and CM physics

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \stackrel{?}{=} \sum_{\text{thimble}} \mathcal{N}_{\text{th}} \, e^{i\,\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \, \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

Lefschetz thimble = "functional steepest descents contour"

on a thimble, the path integral becomes
well-defined and computable



complexified gradient flow:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

CRISTOFORETTI *et al.* (2013)

Fujii et al (2013)

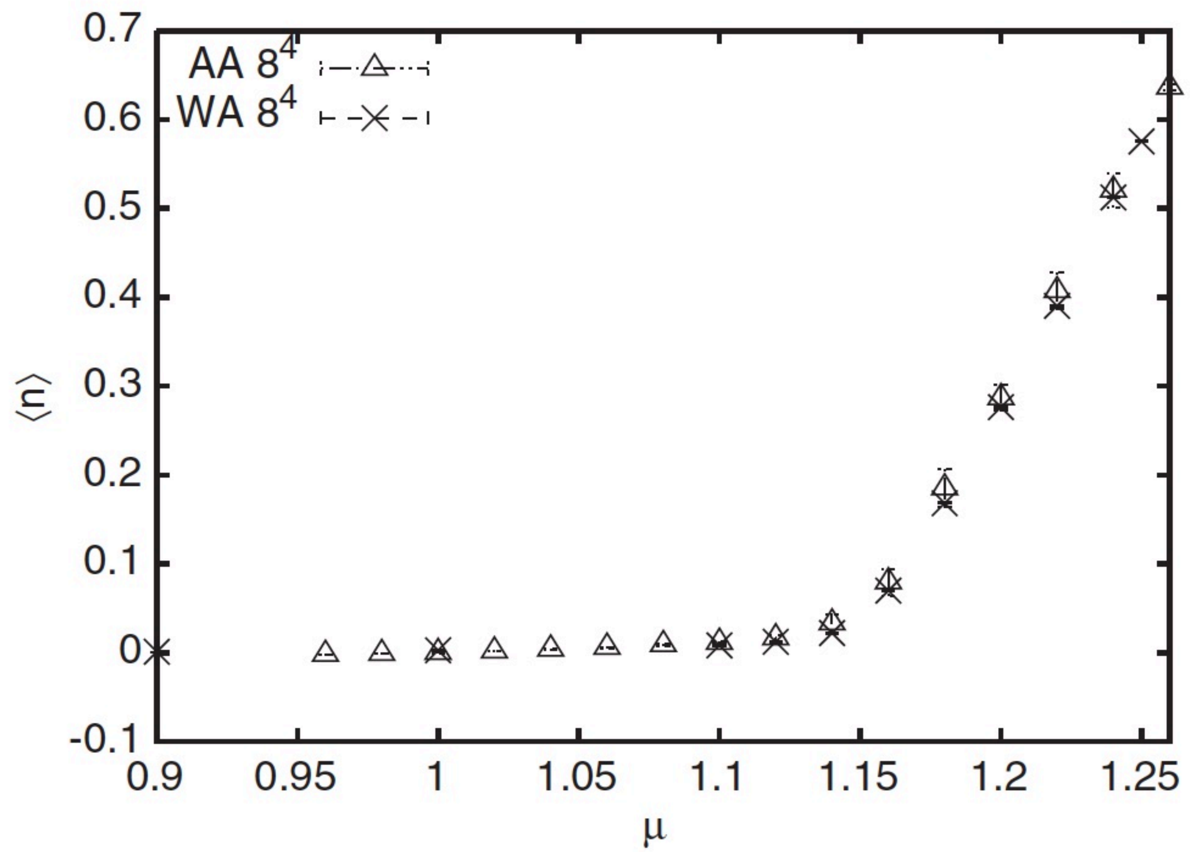
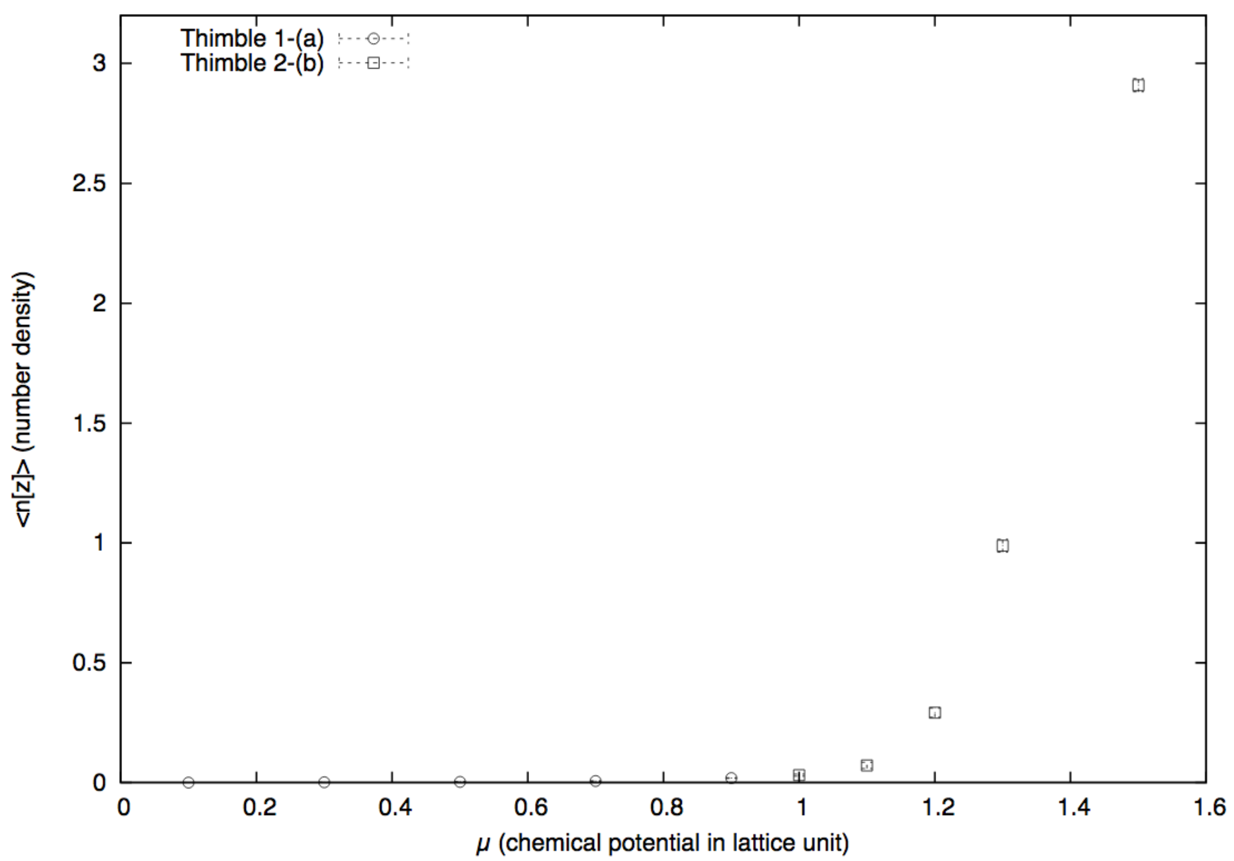


FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA)

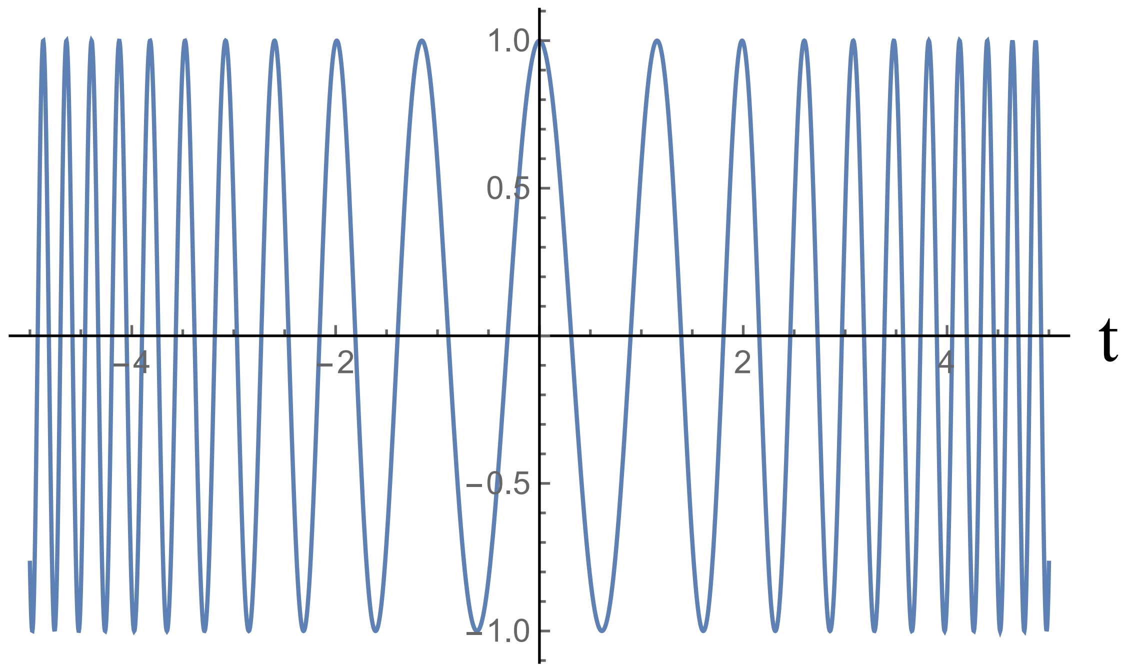


- 4d relativistic Bose gas: complex scalar field theory
- Monte Carlo on thimble softens the sign problem
- results comparable to "worm algorithm"

Generalized Thimble Method

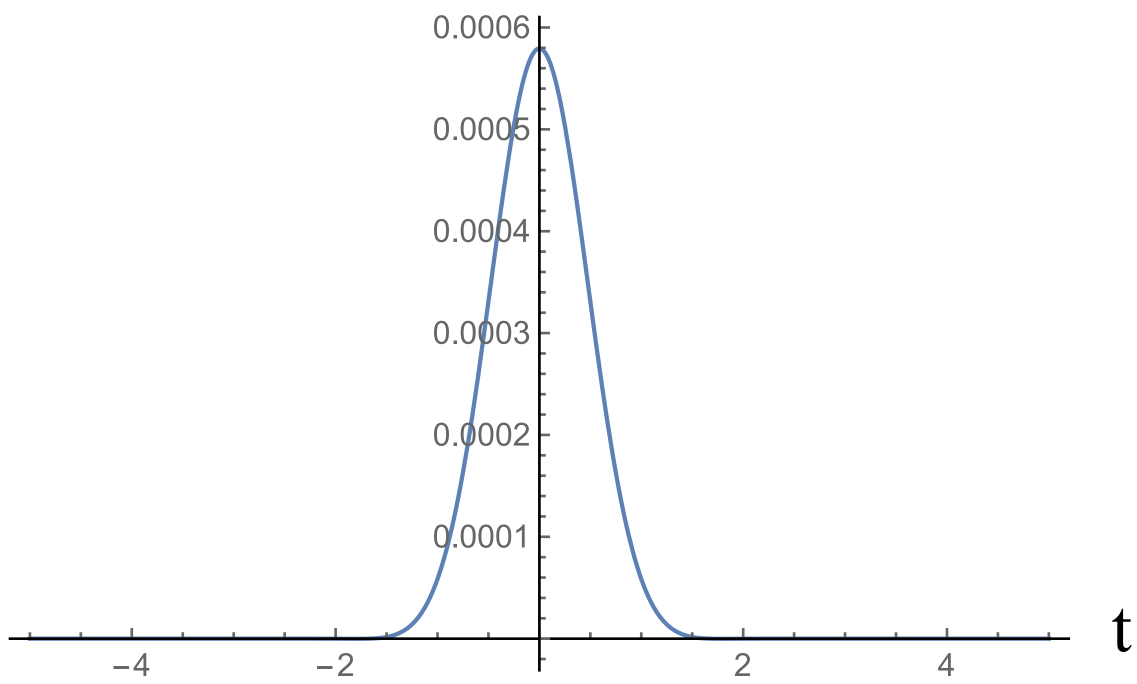
Alexandru, Basar, Bedaque et al 2016

idea: flow to an approximate Lefschetz thimble



$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\, e^{i\left(\frac{1}{3}t^3 + x\,t\right)}$$

exact steepest descents contour



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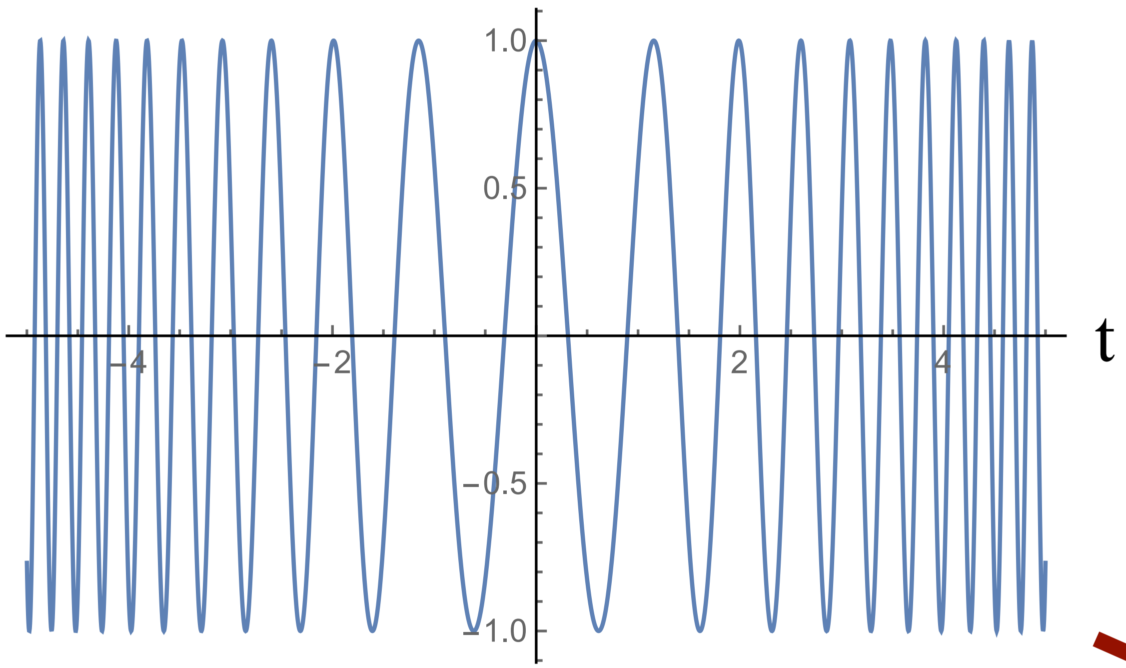

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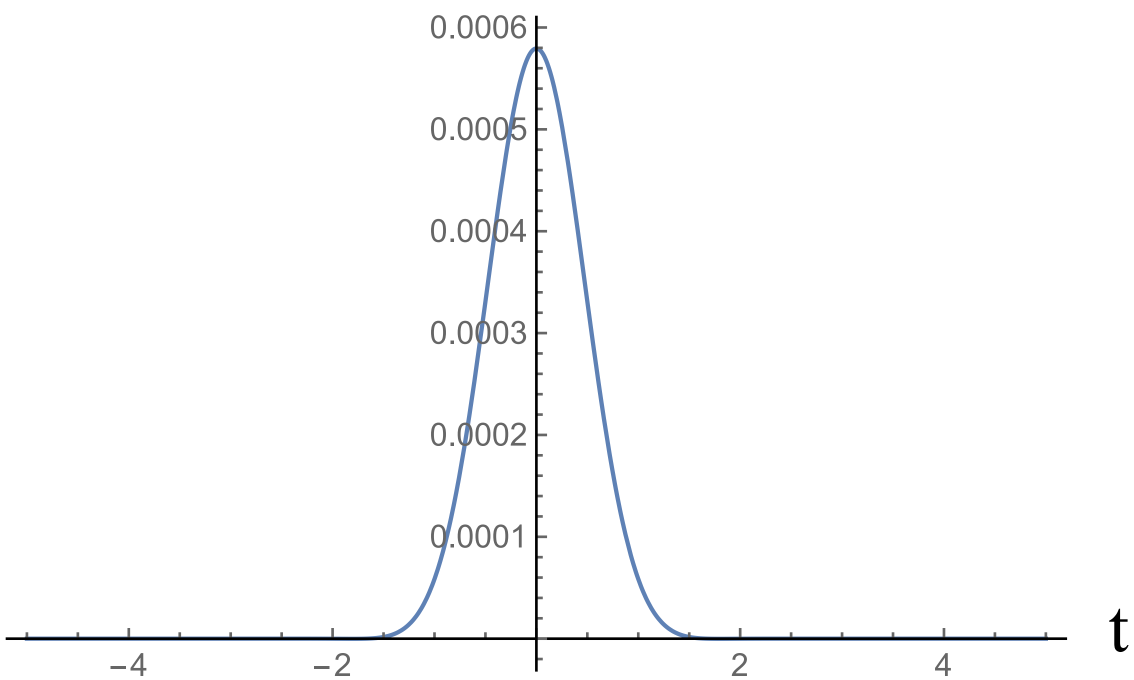
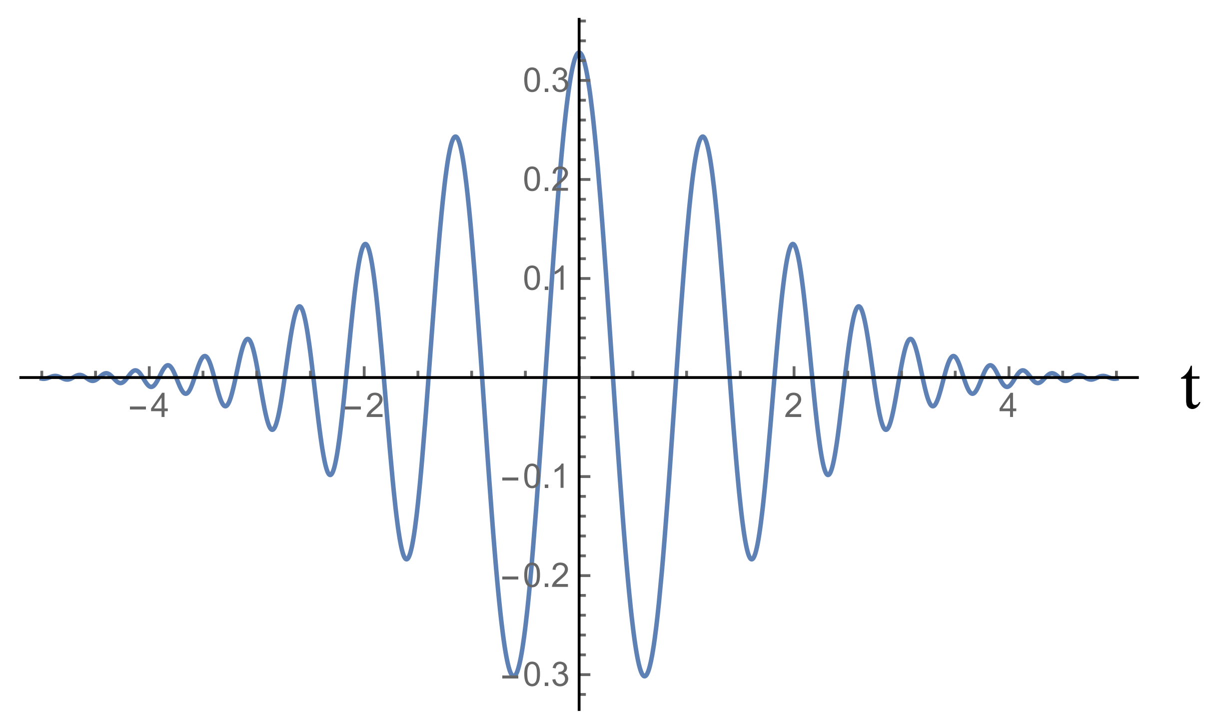
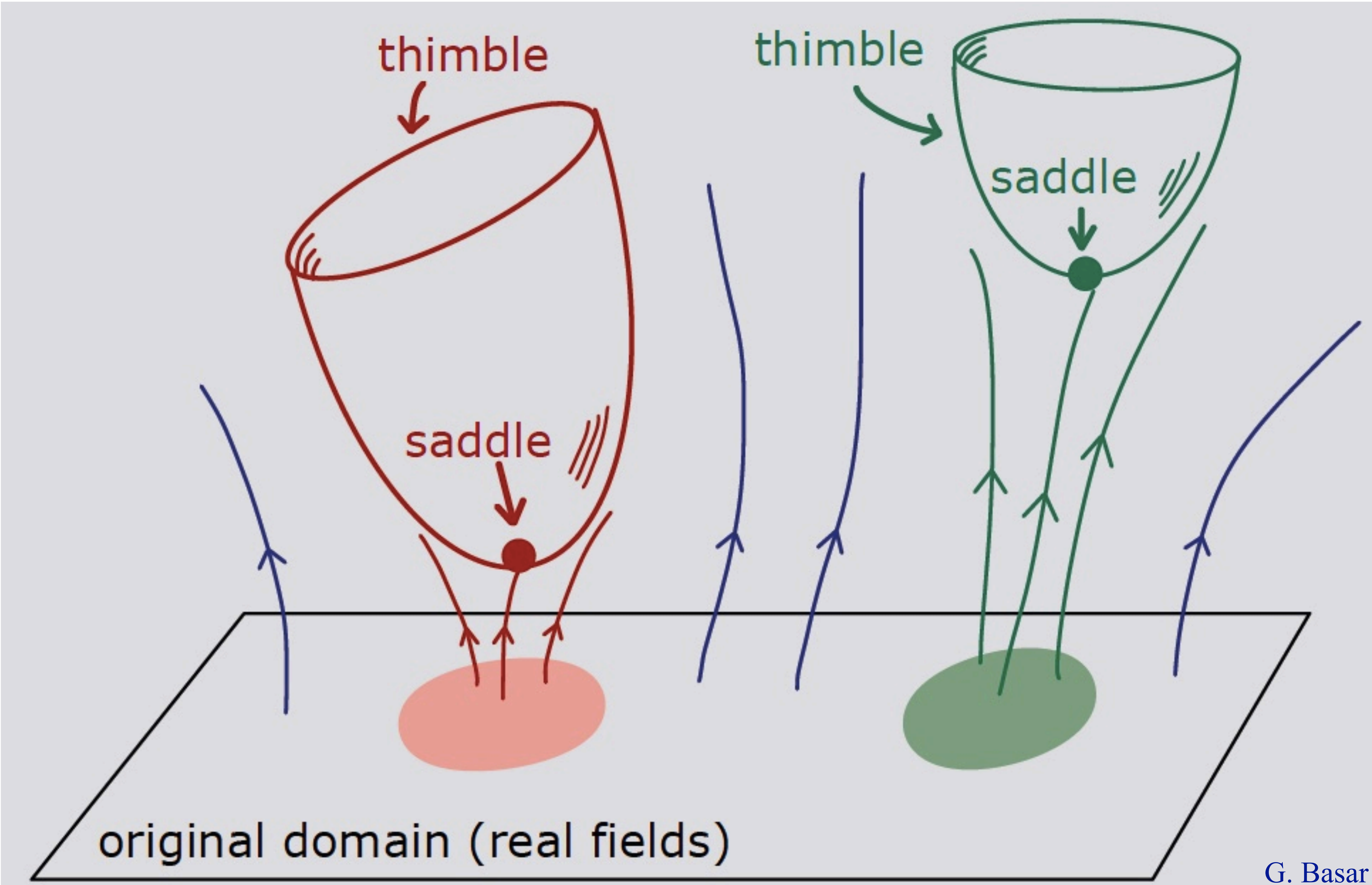


approximate steepest descents contour



Generalized Thimble Method



recall that thimble structure can change as parameters change

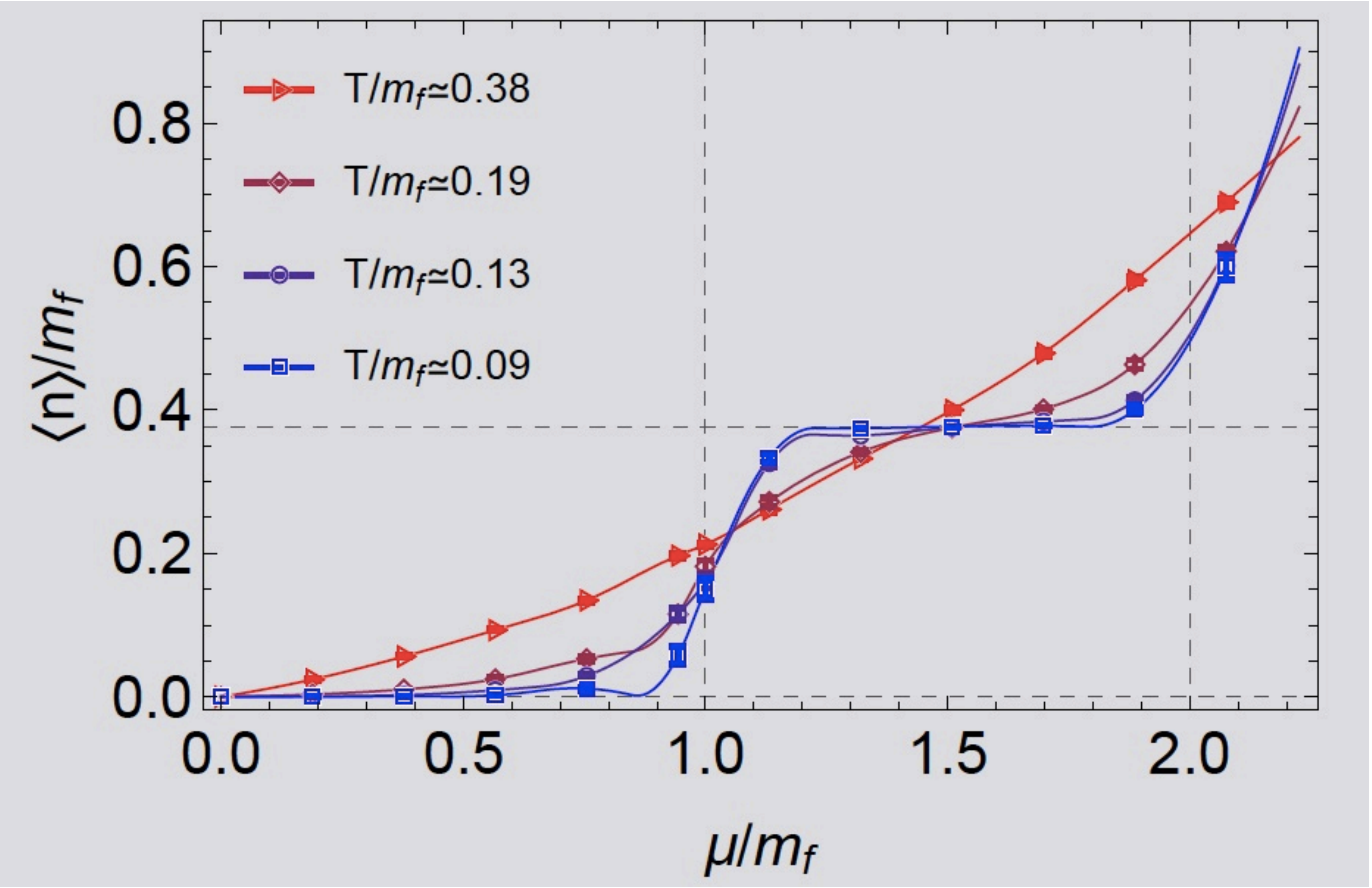
Phase Transitions in QFT: 2d Thirring Model

$$\mathcal{L} = \bar{\psi}^a \left(\gamma_\nu \partial_\nu + m + \mu\,\gamma_0\right) \psi^a + \frac{g^2}{2N_f} \left(\bar{\psi}^a \gamma_\nu \psi^a\right) \left(\bar{\psi}^b \gamma_\nu \psi^b\right)$$

- chain of interacting fermions: asymptotically free
- prototype for dense quark matter
- sign problem at nonzero density

Monte Carlo thimble computation

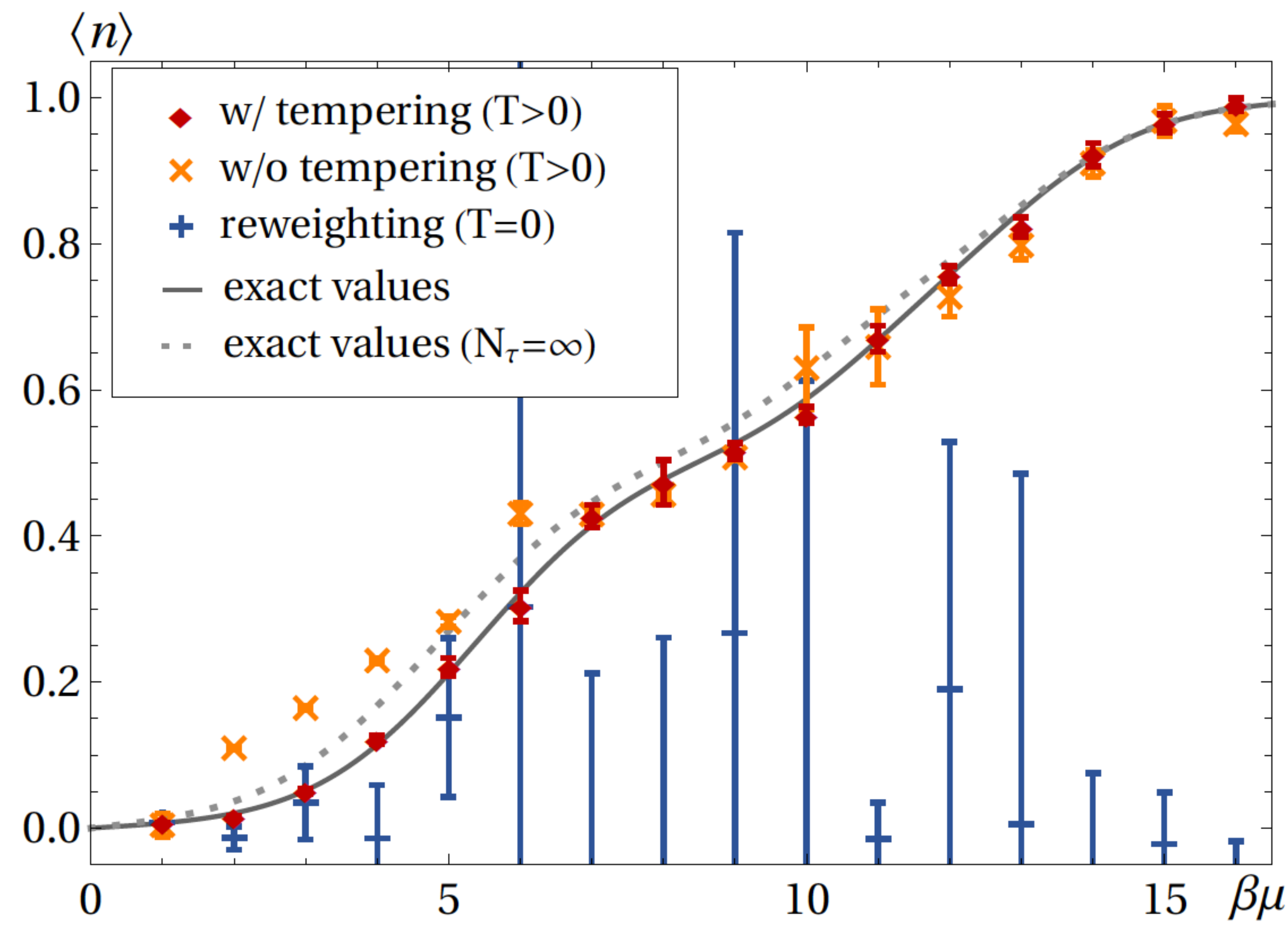
(Alexandru et al, 2016)



Tempered Lefschetz Thimble Method

(Fukuma et al, 2017, 2019,…)

- probe all relevant thimbles ???
- sign problem vs. ergodicity
- coupling → dynamical variable
- parallelized tempering
- e.g. 2d Hubbard model
- probes multiple thimbles



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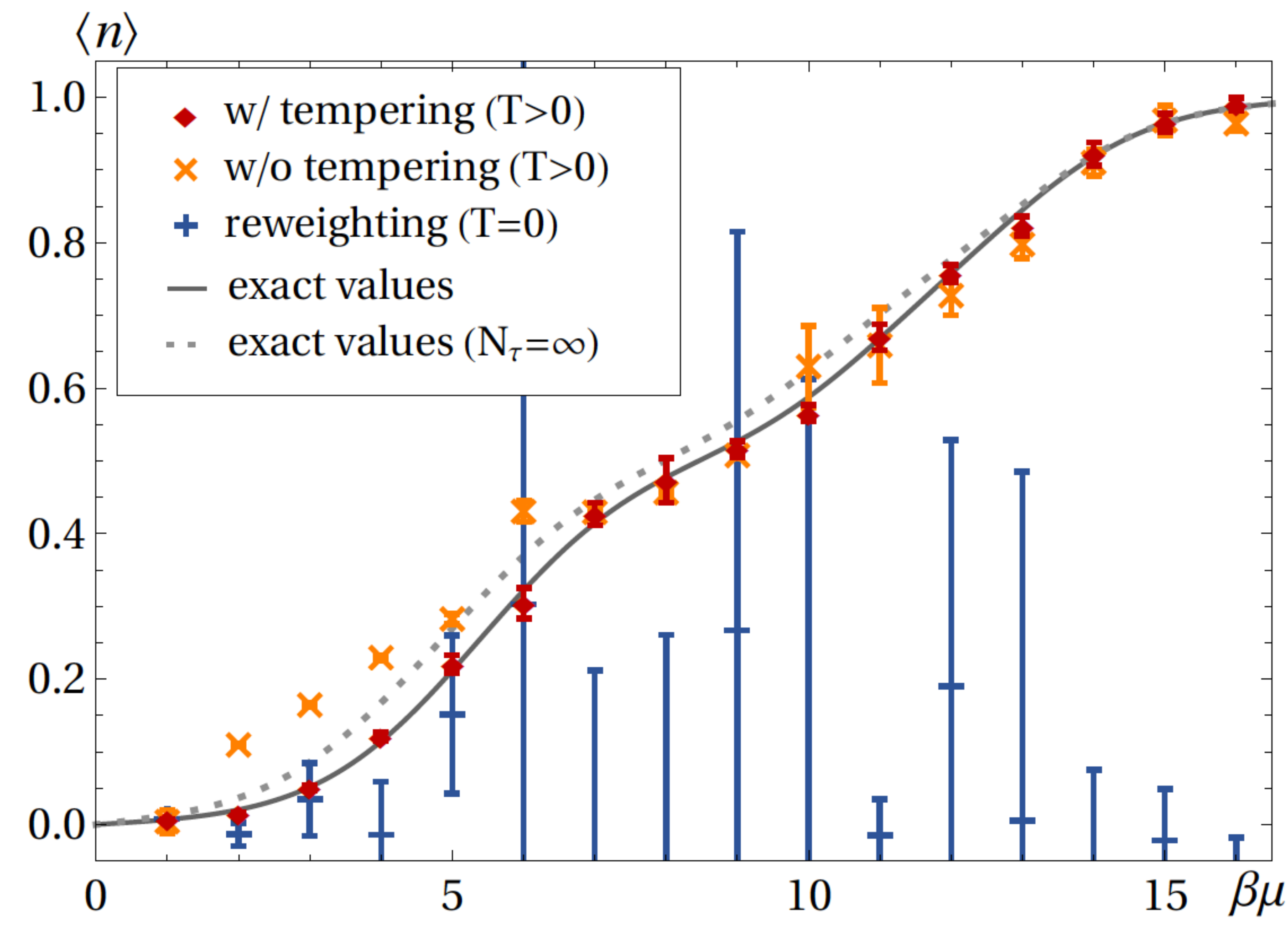
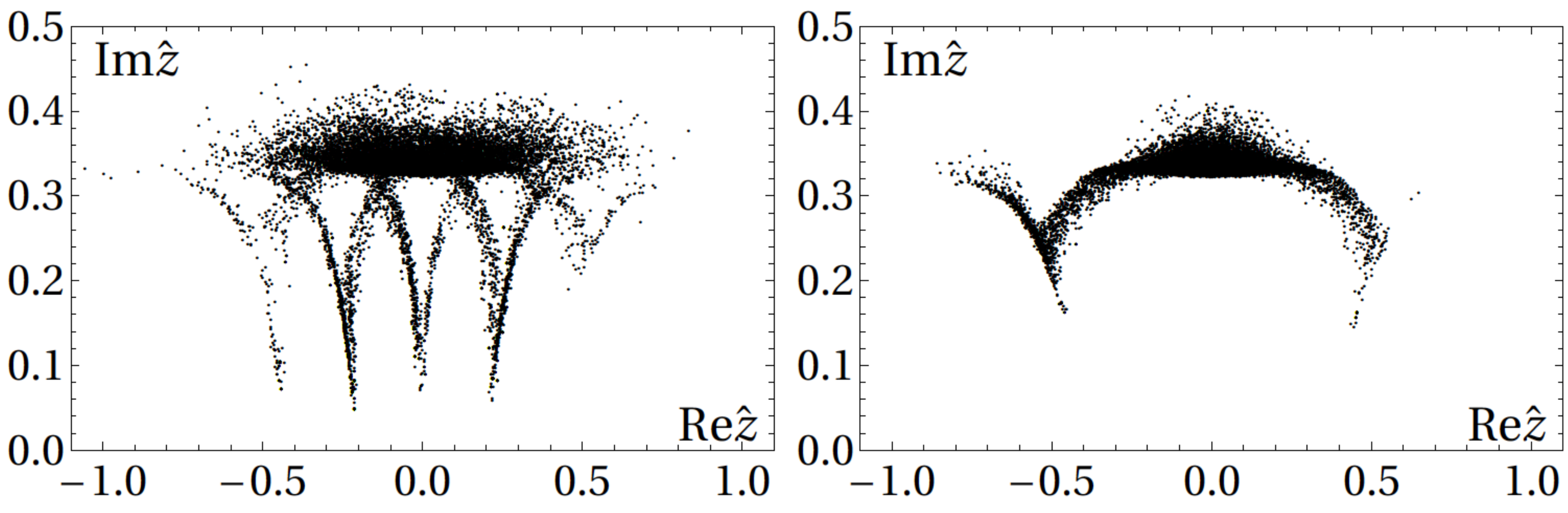




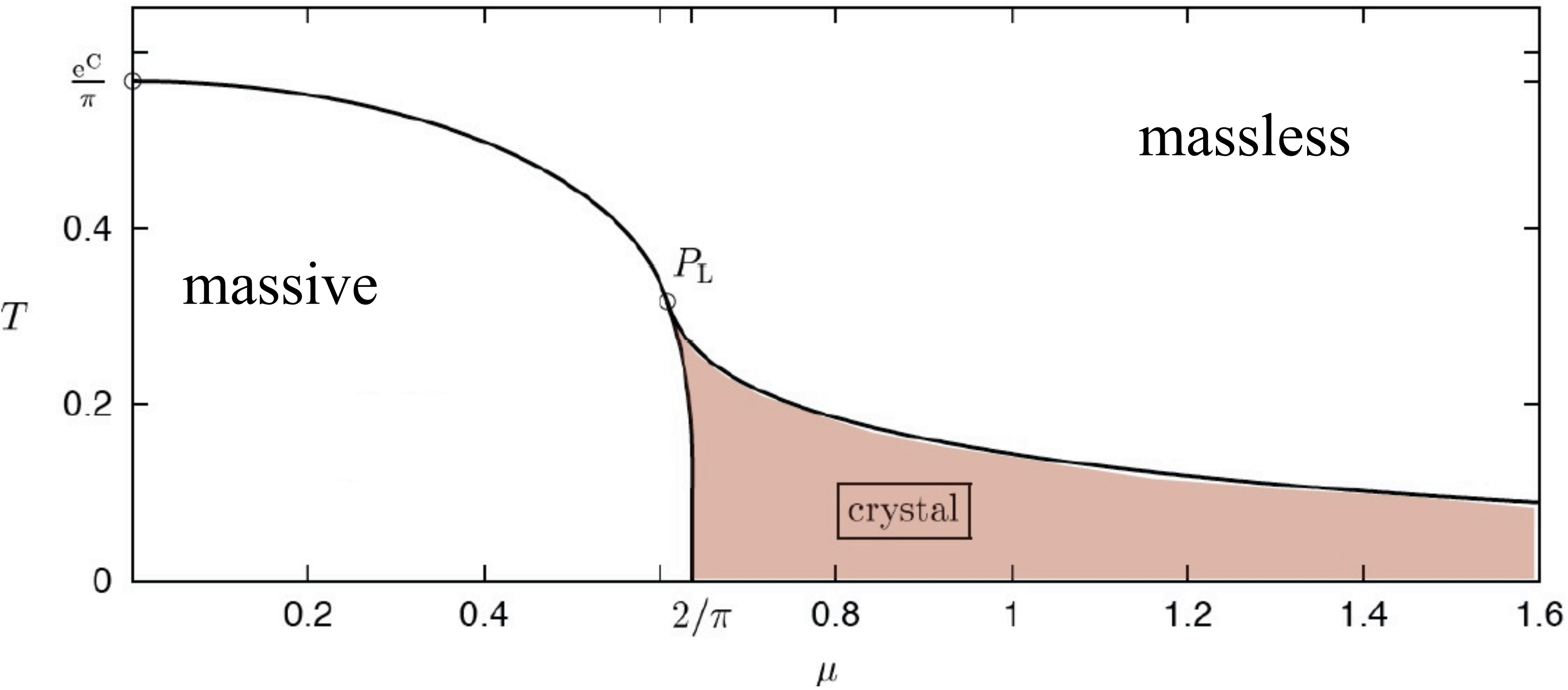
with tempering

without tempering

Phase Transitions in 2d Gross-Neveu Model

$$\mathcal{L}_{\rm Gross-Neveu} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g^2}{2} \left(\bar{\psi}_a \psi_a \right)^2$$

- asymptotically free; dynamical mass; chiral symmetry; model for QCD
- large N_f chiral symmetry breaking phase transition
- physics = (relativistic) Peierls dimerization instability in 1+1 dim.



chiral symmetry breaking condensate

$$\sigma(x;T,\mu) \equiv \langle \bar{\psi}\psi \rangle (x;T,\mu)$$

develops crystalline phases

(Thies et al)

saddles solve an inhomogeneous gap equation

$$\sigma(x;T,\mu) = \frac{\delta}{\delta\sigma(x;T,\mu)} \ln\det\left(i\,\partial\!\!\!/ - \sigma(x;T,\mu)\right)$$

Basar, GD, Thies, 0903.1868

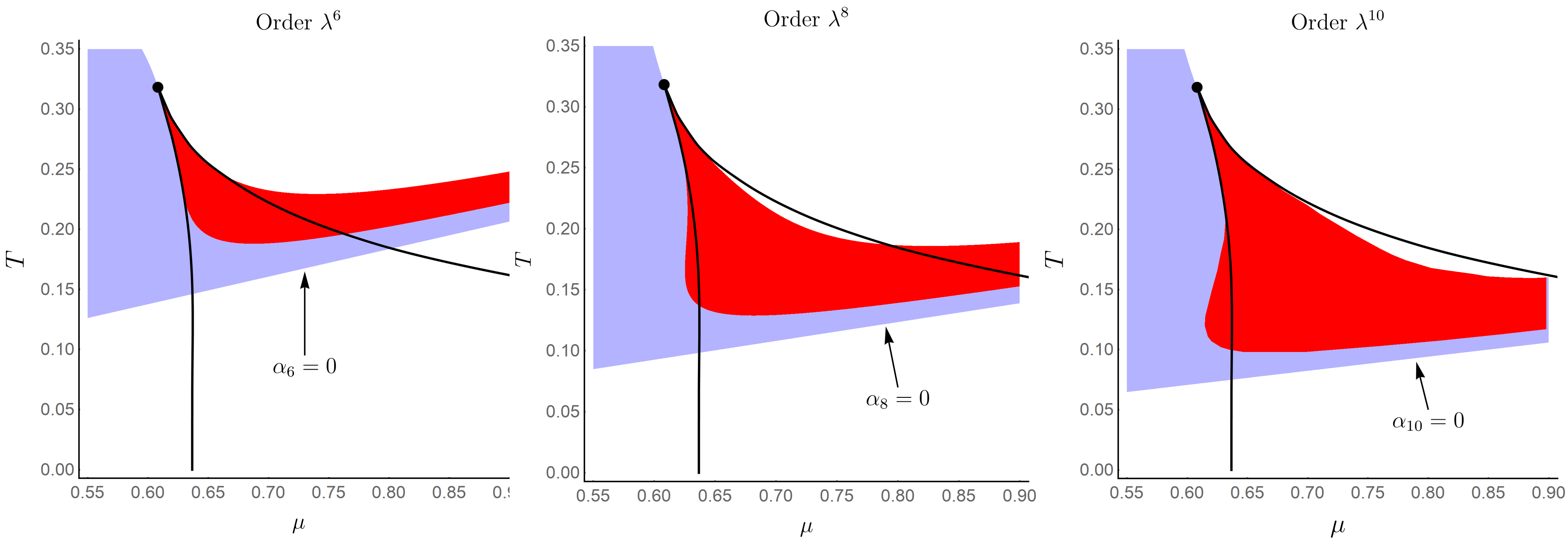
Phase Transitions in Gross-Neveu Model

Basar, GD, Thies, 0903.1868

- thermodynamic potential

$$\Psi[\sigma; T, \mu] = \sum_n \alpha_n(T,\mu) f_n[\sigma(x,T,\mu)] = \alpha_0 + \alpha_2\,\sigma^2 + \alpha_4\left(\sigma^4 + (\sigma')^2\right) + \ldots$$

- expansion about tricritical point = Ginzburg-Landau expansion (divergent)
- mKdV hierarchy
- successive orders of GL expansion "reveal" crystal phase



- all orders gives full crystal phase … but T=0 critical point is difficult

Phase Transitions in Gross-Neveu Model

- density expansion has non-perturbative terms: "trans-series"
- high-density expansion at T=0: convergent; radius gives μ_c

$$\mathcal{E}(\rho) \sim \frac{\pi}{2}\,\rho^2\left(1 - \frac{1}{32(\pi\rho)^4} + \frac{3}{8192(\pi\rho)^8} - \dots\right)$$

- low-density expansion at T=0: (non-perturbative trans-series)

$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{-k/\rho}}{\rho^{k-2}}\mathcal{F}_{k-1}(\rho)$$

- T=0 quantum phase transition

$$\mu_{\text{critical}} = \frac{2}{\pi} \quad \leftrightarrow \quad \rho = 0$$

Resurgence and Large N Phase Transitions in Matrix Models

3rd order phase transition in Gross-Witten-Wadia unitary matrix model

$$Z(t,N) = \int_{U(N)} DU \; \exp\left[\frac{N}{t}\mathrm{tr}\left(U + U^\dagger\right)\right]$$

Gross-Witten, 1980
Wadia, 1980
Marino, 2008

Z depends on two parameters: ’t Hooft coupling t, and matrix size N

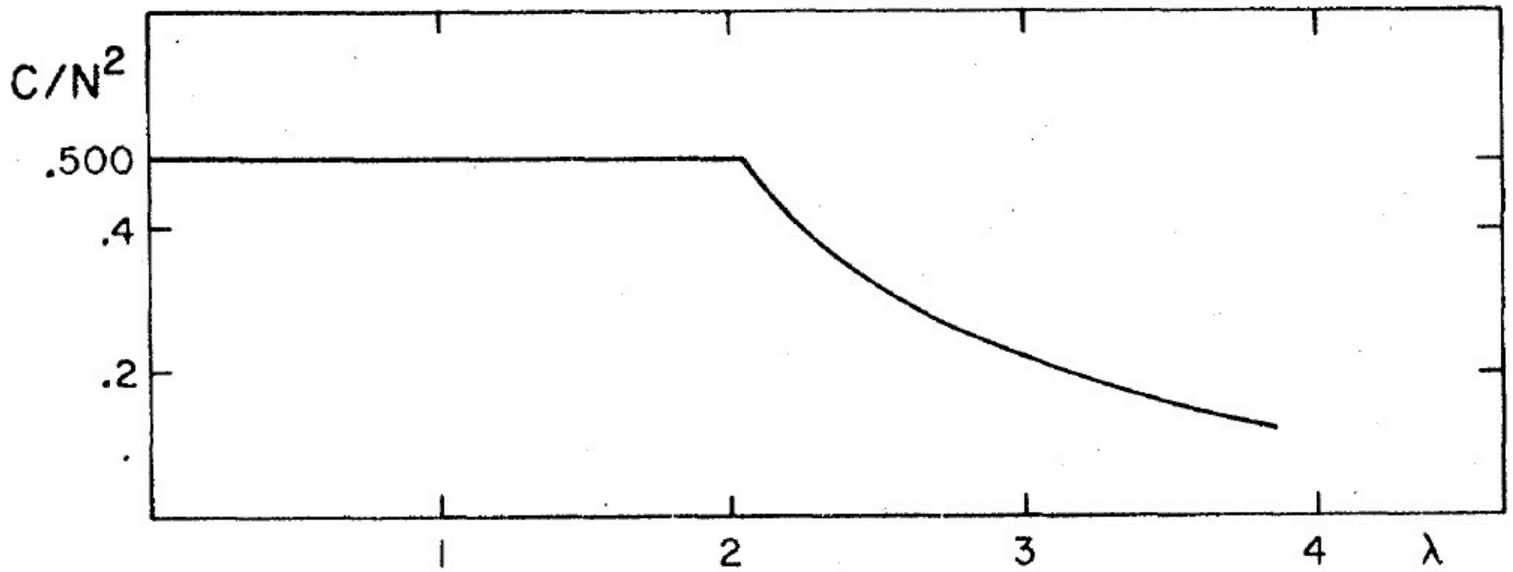


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

$$Z(t,N) = \det\left[I_{j-k}\left(\frac{N}{t}\right)\right]_{j,k=1,\ldots N}$$

phase transition in the
“thermodynamic” large N limit

Resurgence in Matrix Models at Large N

"order parameter" $\Delta(t,N) \equiv \langle \det U \rangle$ satisfies a Painleve III equation

Ahmed & GD, 1710.01812

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

N appears only as a parameter: perfect for large N asymptotics

$$\Delta(t,N) \sim \sum_n \frac{a_n(t)}{N^n} + e^{-N\,S(t)}\sum_n \frac{b_n(t)}{N^n} + e^{-2N\,S(t)}\sum_n \frac{c_n(t)}{N^n} + \dots$$

large N instanton contributions: generated from ODE

e.g. $a_0(t) = \sqrt{1-t}$

all physical observables inherit this large N trans-series structure

Resurgence in Matrix Models at Large N

ODE $\Rightarrow$ large N weak coupling trans-series:

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i\,\sigma_{\rm weak}}{2\sqrt{2\pi N}}\,\frac{t\,e^{-N S_{\rm weak}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

weak coupling large N action:

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\mathrm{arctanh}\left(\sqrt{1-t}\right)$$

Resurgence in Matrix Models at Large N

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“one-instanton" fluctuations:

$$\sum_{n=0}^\infty \frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{1}{N} + \dots$$

Resurgence in Matrix Models at Large N

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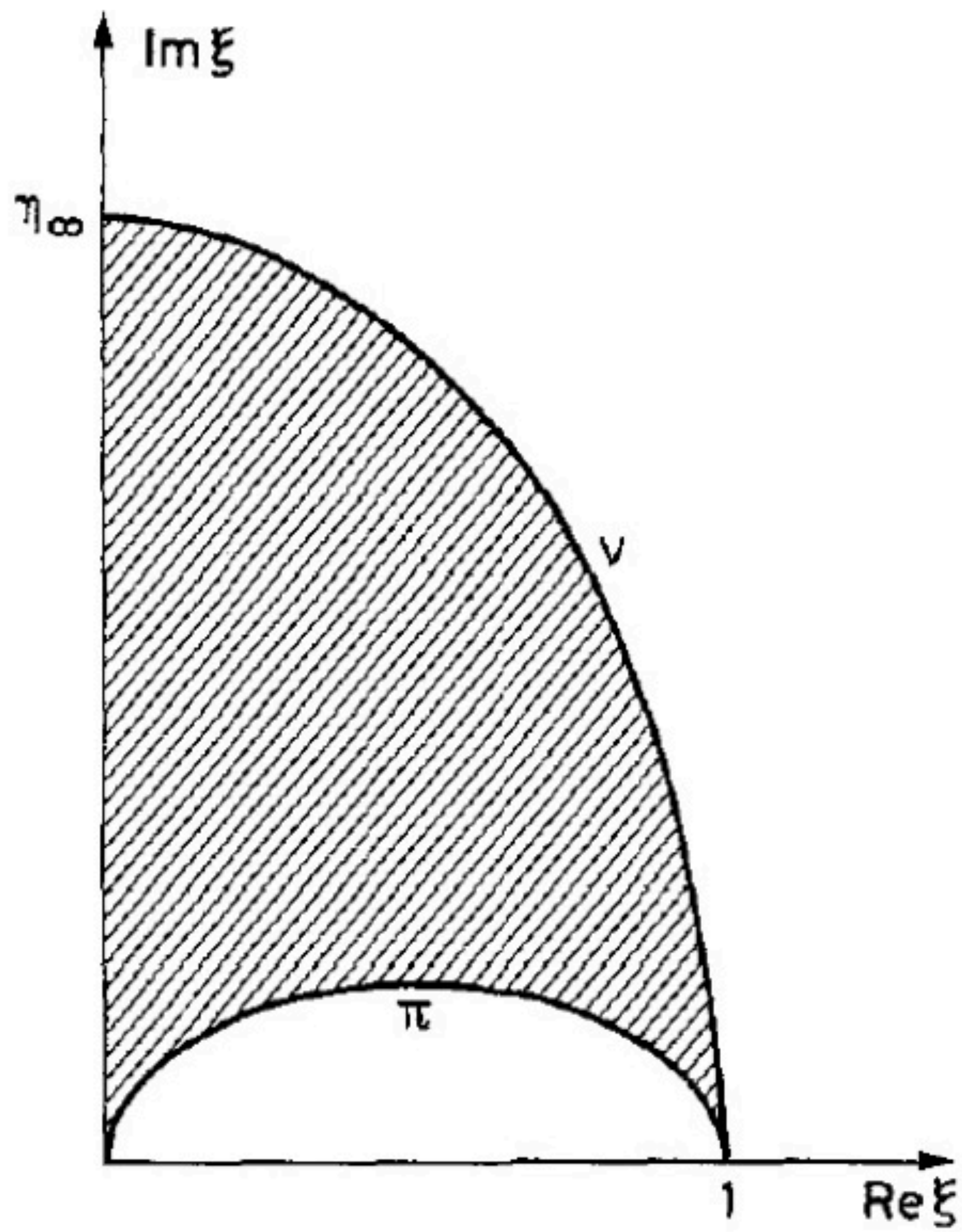
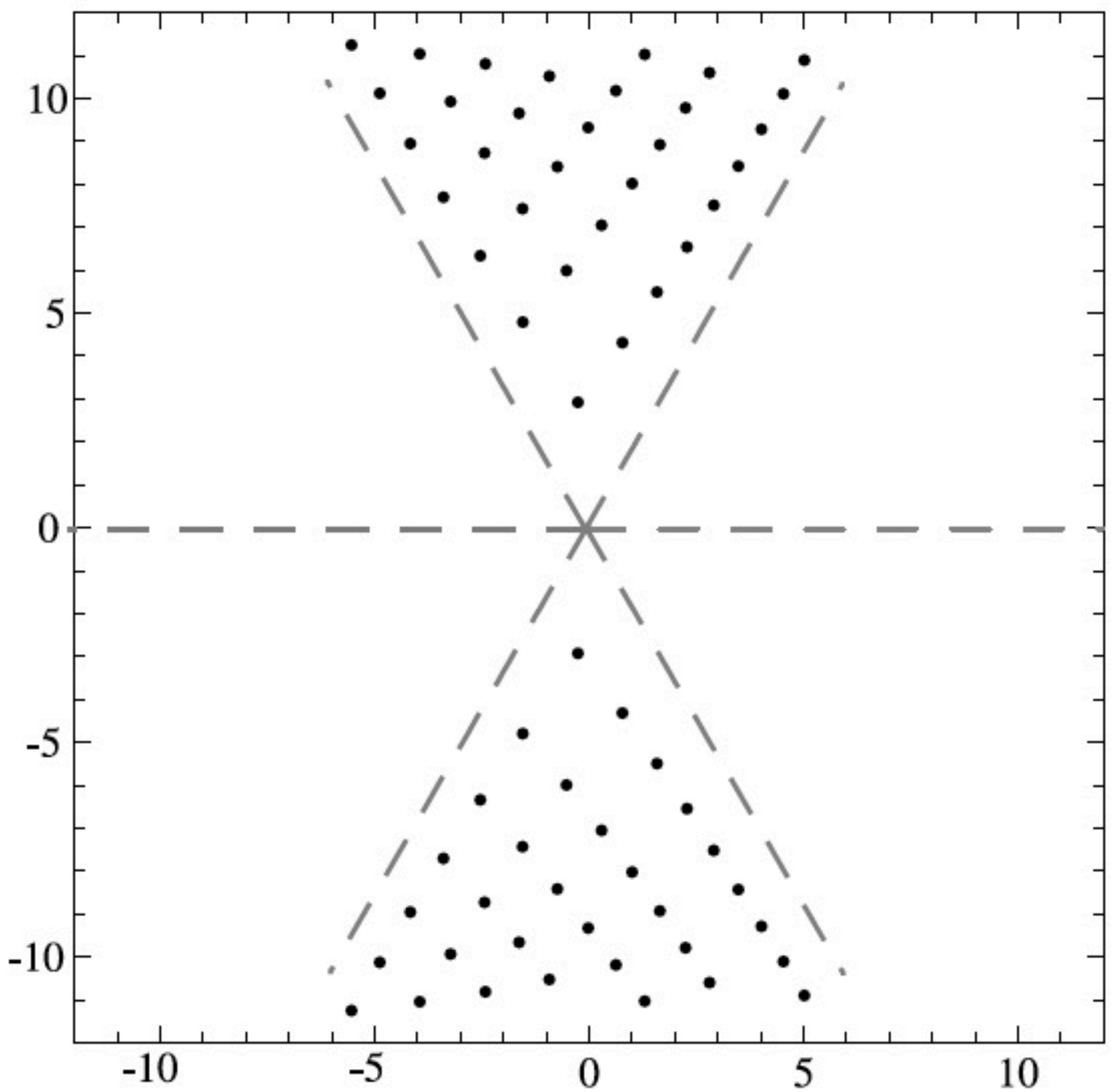
resurgence: large-order growth of “perturbative coefficients”:

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right]$$

Lee-Yang view of Large N Phase Transitions in Matrix Models

Lee-Yang: complex zeros of Z(t, N) pinch the real t axis at the phase transition, in the thermodynamic (large N) limit





Lee-Yang zeros near t=1 transition can be recovered from large t expansion

Resurgence in 2d Lattice Ising Model

diagonal correlation functions: $$C(s, N) = \langle \sigma_{0,0}\, \sigma_{N,N} \rangle (s)$$

C(s, N) = tau function for Painleve VI equation (Jimbo, Miwa)

C(s, N) has a trans-series expansion: convergent about T=0, T= ∞

scaling limit: PVI → PIII as $N \to \infty \;\&\; T \to T_c$ (McCoy et al)

convergent conformal block expansions at low T and high T: (Lisovyy et al, 2012, 2013 …)

$$\tau(s) \sim \sum_{n=-\infty}^{\infty} \rho^n \, C(\vec{\theta}, \sigma + n)\, \mathcal{B}(\vec{\theta}, \sigma + n; s)$$

$$\mathcal{B}(\vec{\theta}, \sigma; s) \propto s^{\sigma^2} \sum_{\lambda, \mu \in \mathcal{Y}} \mathcal{B}_{\lambda,\mu}(\vec{\theta}, \sigma)\, s^{|\lambda| + |\mu|}$$

resurgence also for convergent expansions GD, 1901.02076

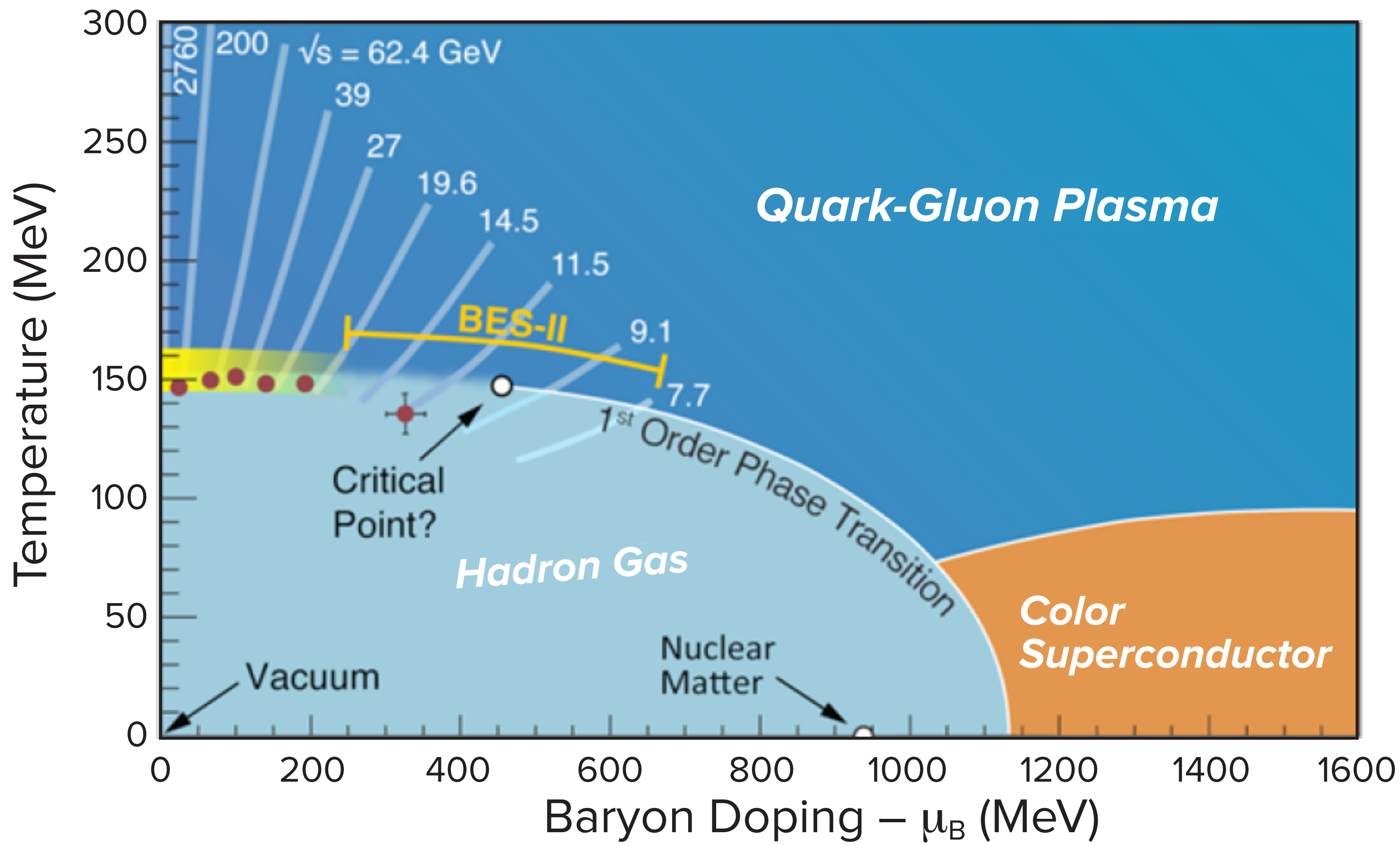
Resurgent Extrapolation

Costin, GD: 1904.11593, 2003.07451
2009.01962

- often, perturbation theory/asymptotics is the ONLY thing we can do

- question: how much global information can be decoded from a FINITE number of perturbative coefficients ?

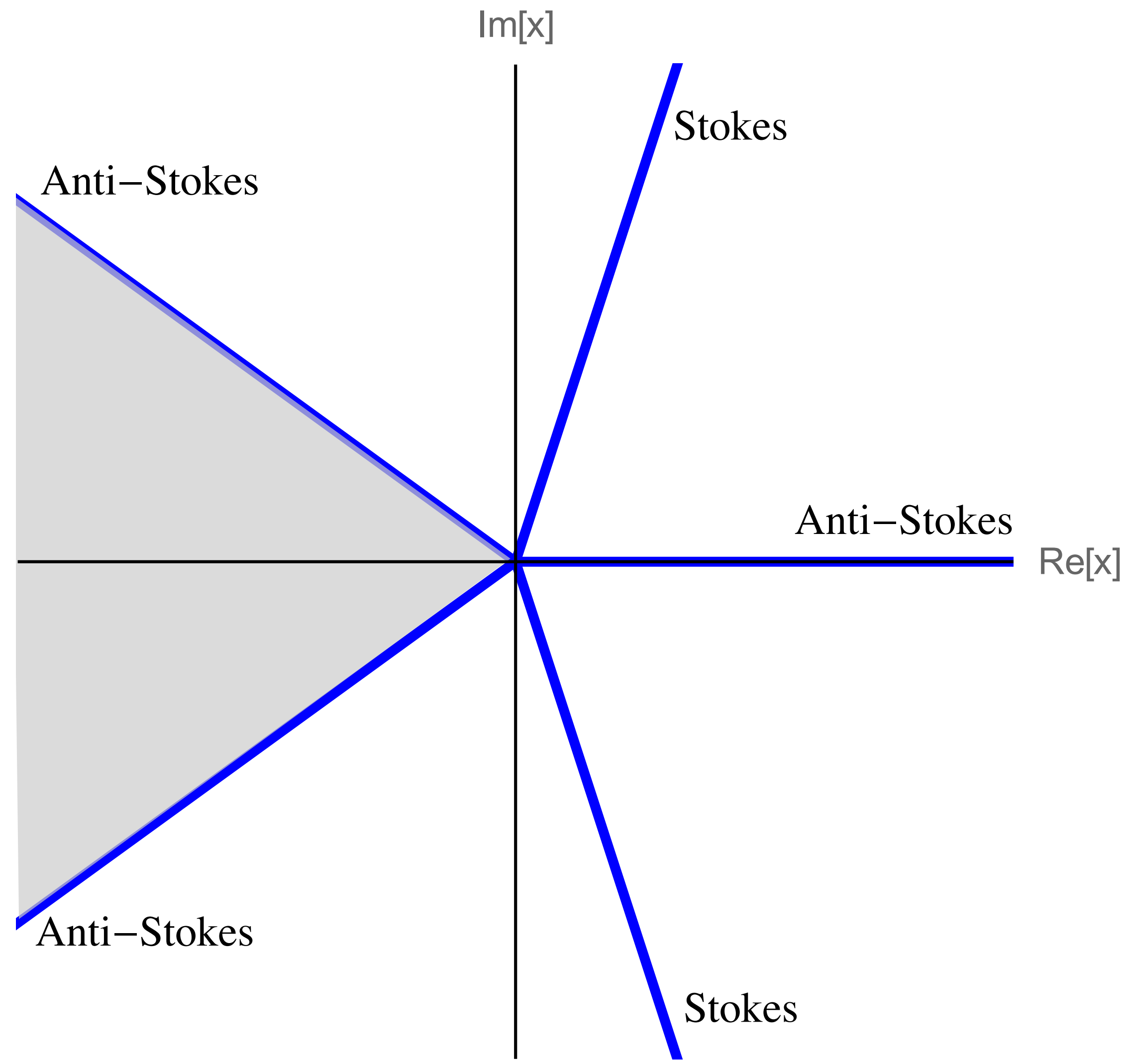
- how much "perturbative" information is required to detect, and to probe the properties of, a phase transition, possibly at a distant point ?



Resurgent Extrapolation

- case-study: Painleve I equation $$y''(x) = 6\, y^2(x) - x$$

pole expansion



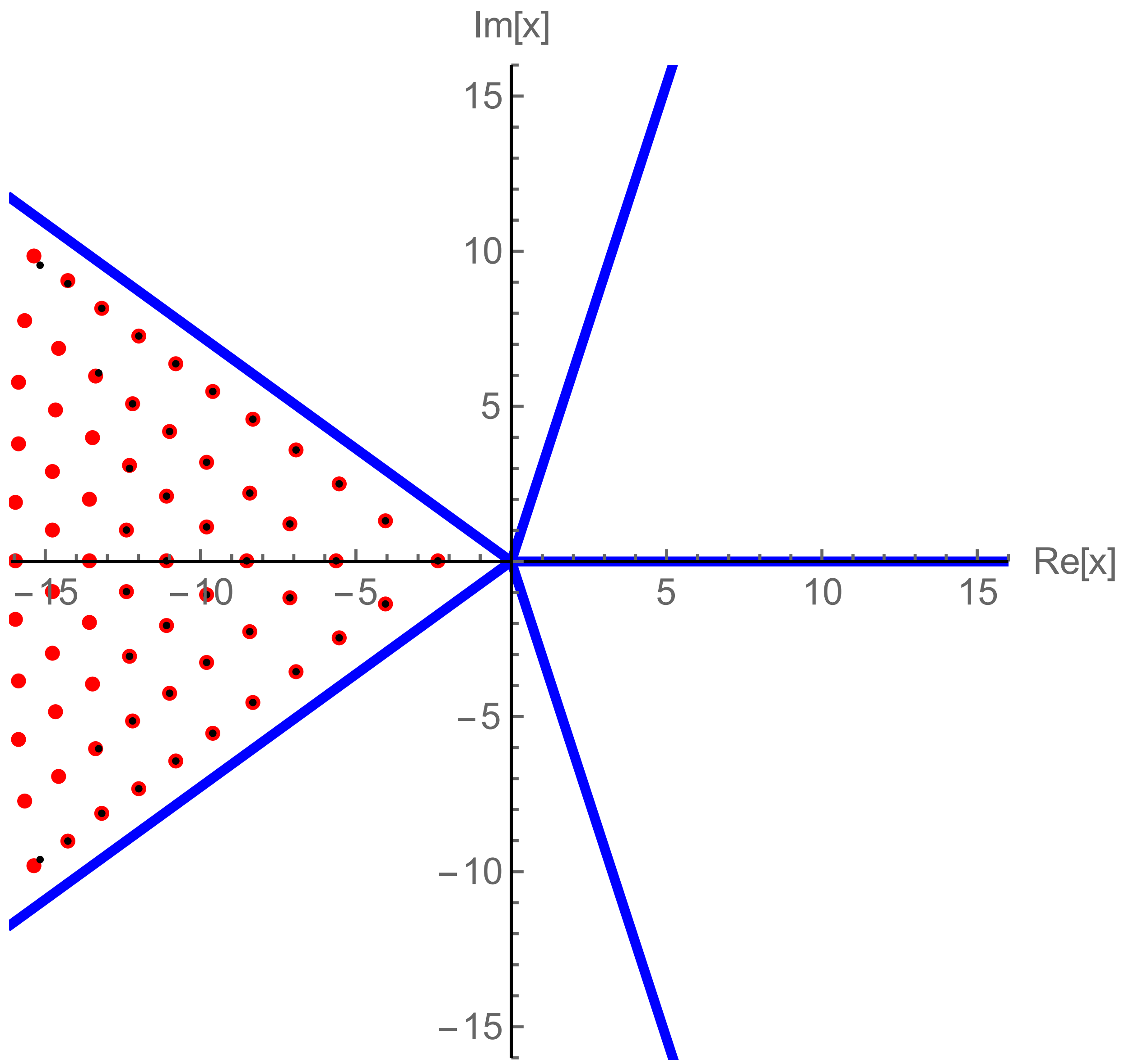
asymptotic expansion

- Painleve I equation has 5 sectors in the complex x plane, separated by phase transitions
- *tritronquée* solution: poles only in shaded region
- suppose we expand about x=+infty to finite order N: how much do these coefficients “know” about the other sectors?

Resurgent Extrapolation

Costin, GD: 1904.11593, 2009.01962

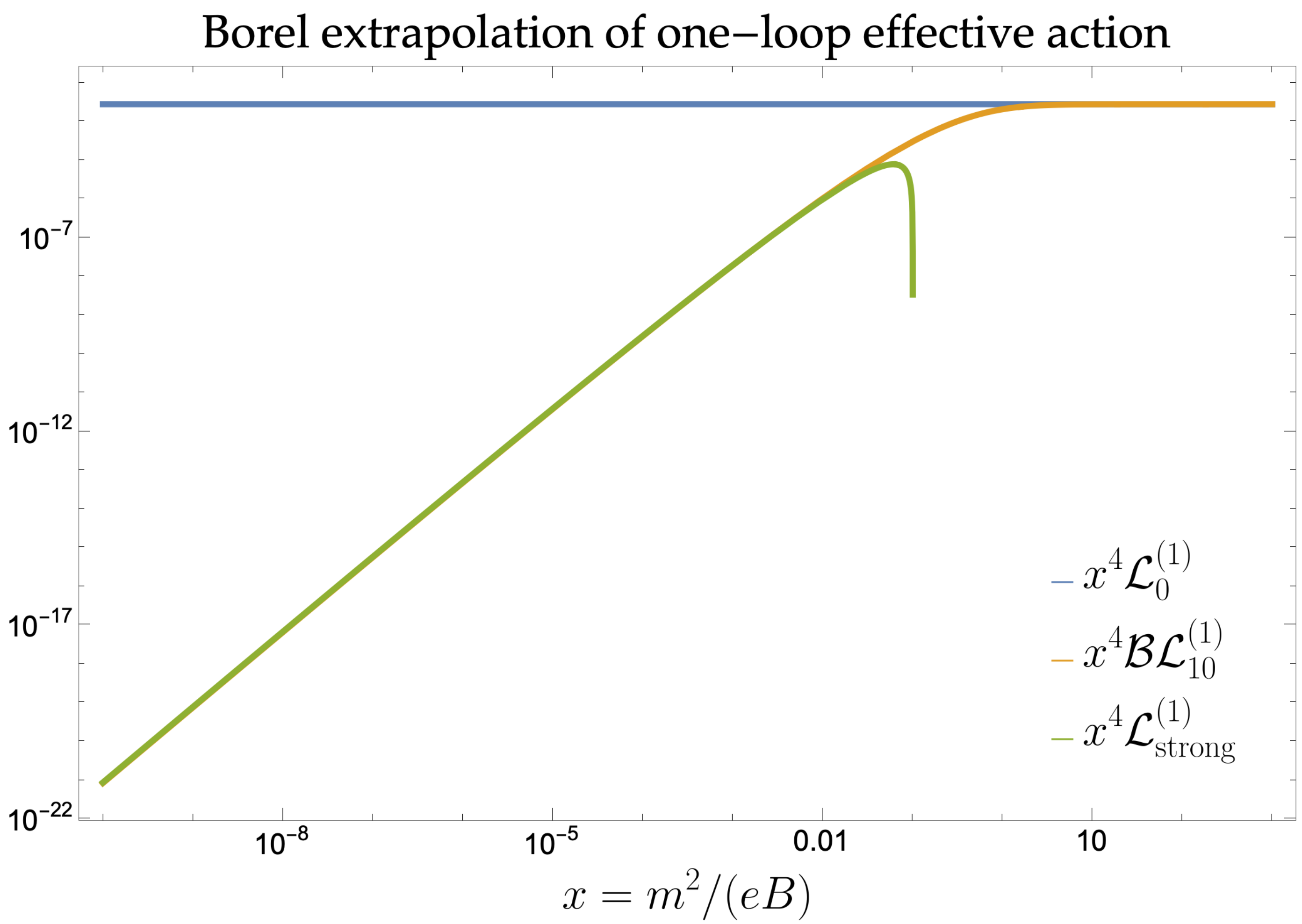
- there is an optimal way to extrapolate
- Pade-Borel + uniformizing maps: extreme precision



- extrapolate across Stokes transitions, and also onto higher Riemann sheets

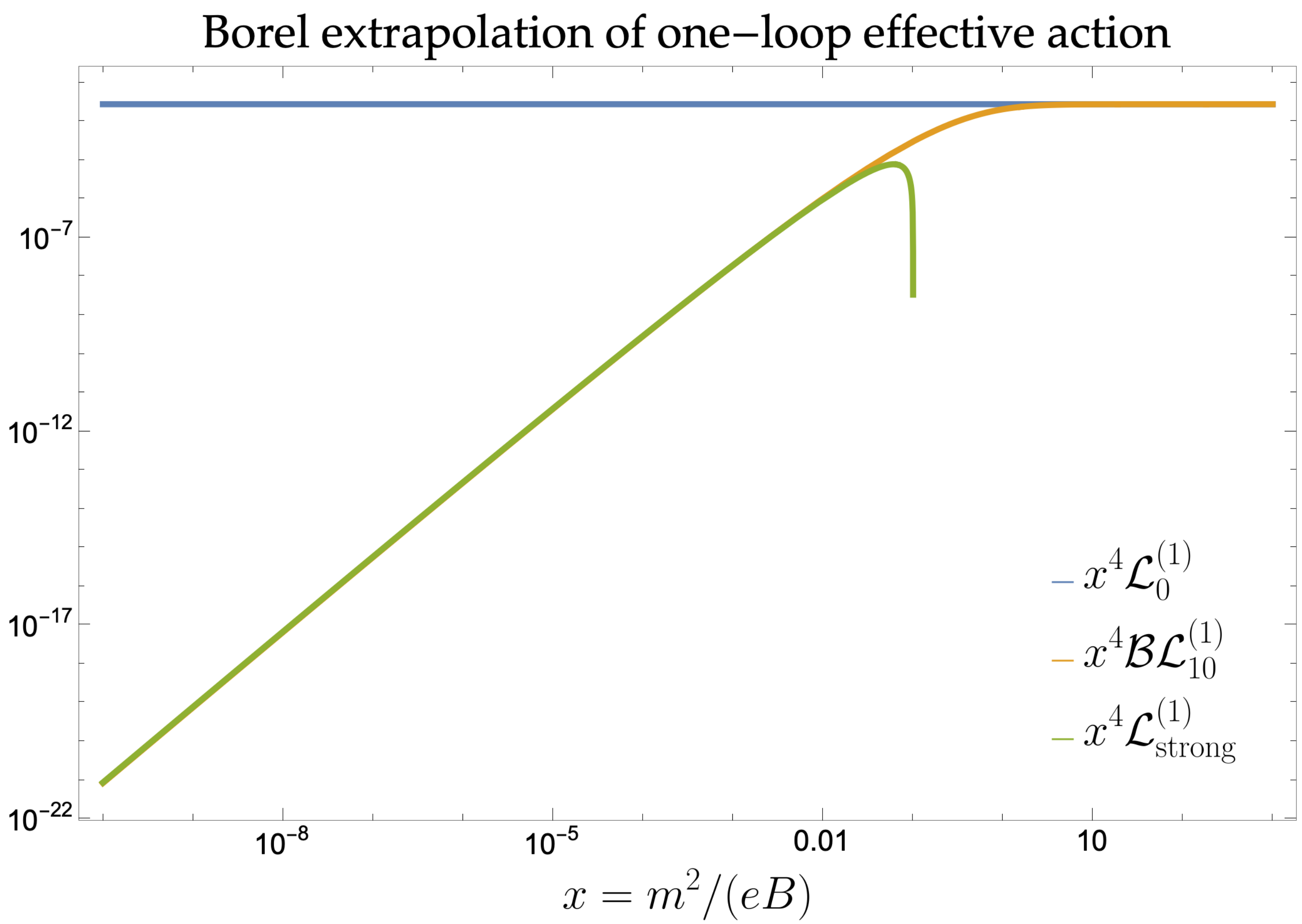
- resurgent extrapolation can decode global behavior from surprisingly little input data from some other regime

Resurgent Extrapolation: Euler-Heisenberg example

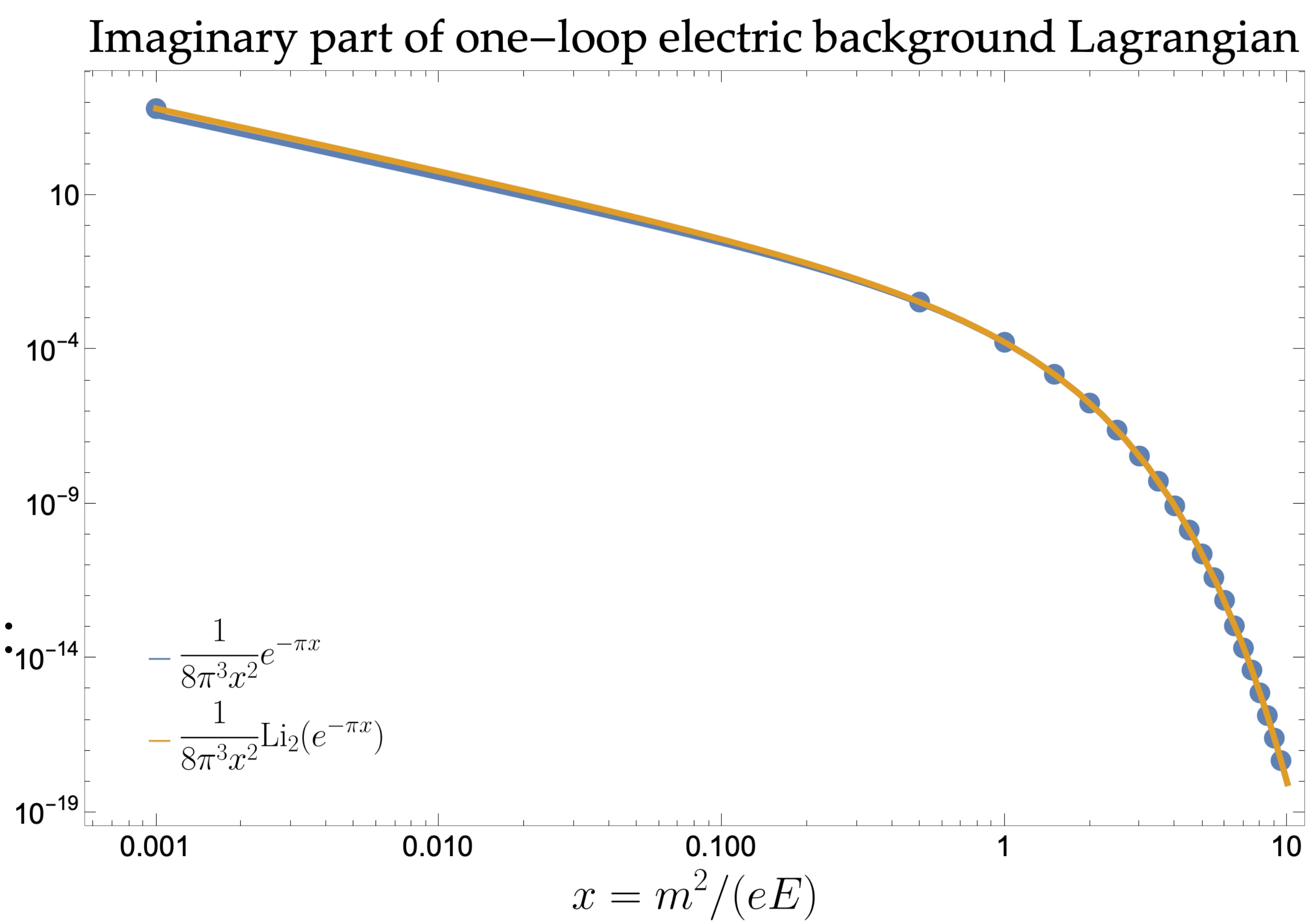


<u>weak to strong</u> magnetic field extrapolation:
12 orders of magnitude from
just 10 weak field coefficients

Resurgent Extrapolation: Euler-Heisenberg example



<u>weak to strong</u> magnetic field extrapolation:
12 orders of magnitude from
just 10 weak field coefficients



<u>magnetic to electric</u> field analytic continuation:
4 orders of magnitude in imaginary part from
just 10 weak magnetic field coefficients

Conclusions

- “resurgence” is based on a new and improved form of asymptotics
- deep(er) connections between perturbative and non-perturbative physics
- recent applications to differential eqs, QM, QFT, string theory, …
- phase transitions from large N in 2-parameter trans-series
- resurgent extrapolation: high-precision extraction of physical information from finite order expansions
- outlook: new theoretical approach to quantum systems in extreme conditions
- outlook: computational access to strongly-coupled systems, phase transitions, particle production, and far-from-equilibrium physics, …