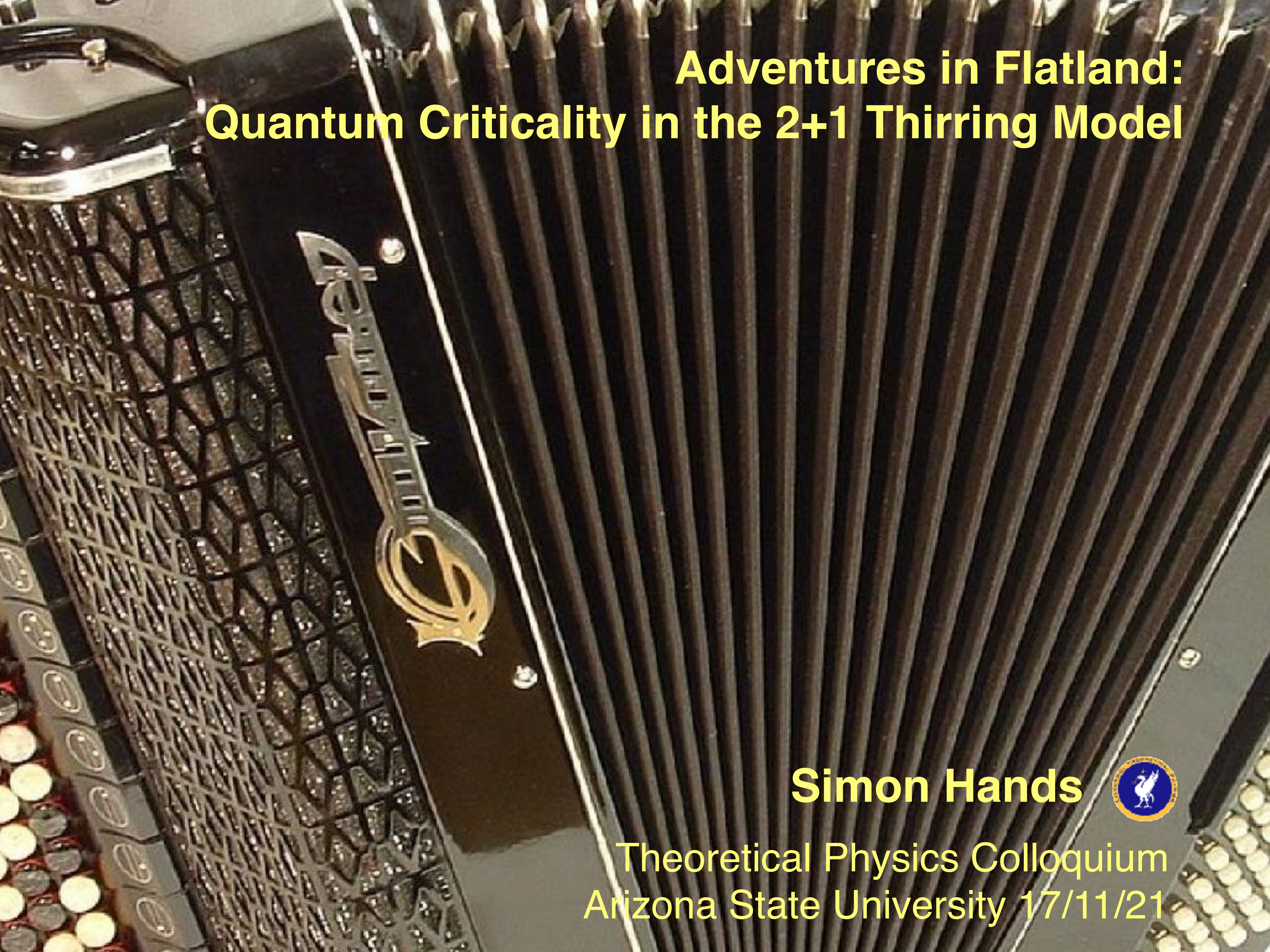


Adventures in Flatland: Quantum Criticality in the 2+1 Thirring Model

Simon Hands

Theoretical Physics Colloquium
Arizona State University 17/11/21

Quantum field theories of particle physics are (for the most part) weakly-interacting at short distances: we have a reasonable idea of how to calculate with them.

But this is not the only possibility…

Inspired by condensed-matter systems eg. **graphene**, I will

- discuss QFTs of relativistic fermions in 2+1d exhibiting **critical behaviour**, signalling a new strongly-interacting QFT
- argue that to characterise the **Quantum Critical Point** it is crucial to capture relevant global symmetries accurately
- present simulation results showing that different lattice discretisations tell very different stories…

The Thirring Model in 2+1d

$$\mathscr{L} = \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i + \frac{g^2}{2N}(\bar{\psi}_i\gamma_\mu\psi_i)^2$$

Covariant quantum field theory of
N flavors of interacting fermion in 2+1 dimensions.
Fermions are spinor fields $\psi, \bar{\psi}$ acted on by 4x4 Dirac matrices γ_μ

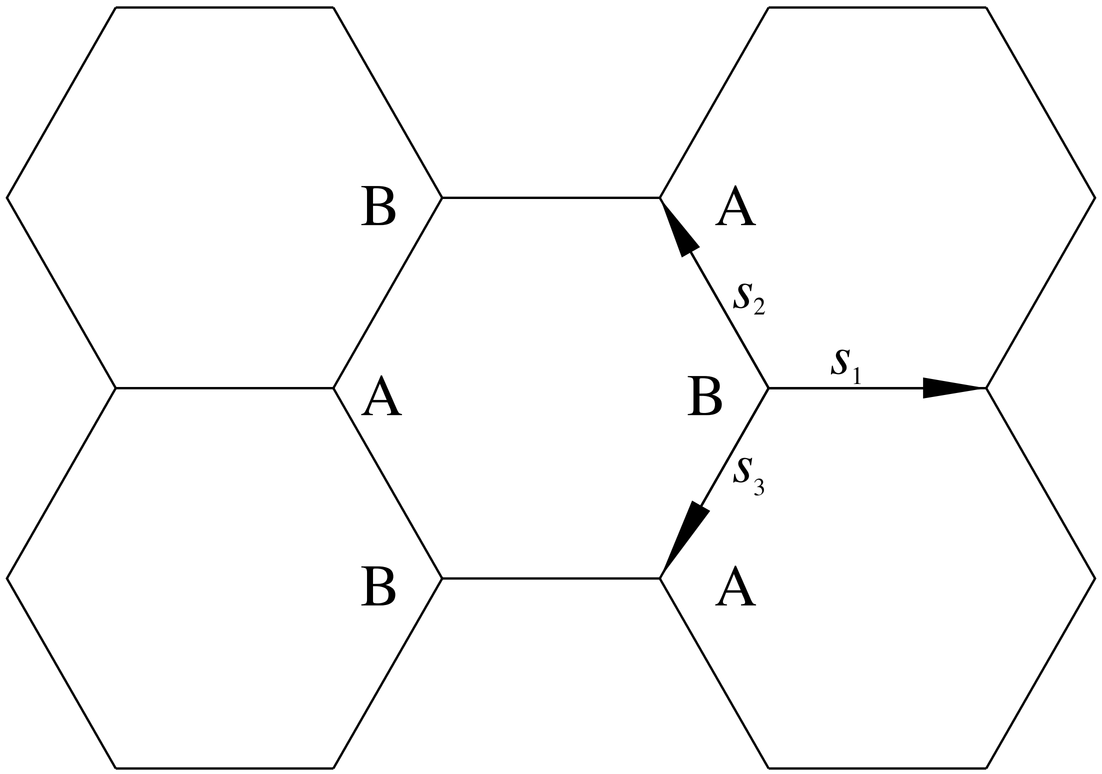
Interaction between conserved currents:
like charges ***repel***, opposite charges ***attract***

$$\partial\!\!\!/ \equiv \partial_\mu\gamma_\mu \quad \mu = 0, 1, 2 \qquad i = 1,\ldots,N$$

$$\mathrm{tr}(\gamma_\mu\gamma_\mu) = 4 \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$$

$$\mu, \nu = 0,1,2,3$$

Many applications of "Flatland fermions" in condensed matter physics

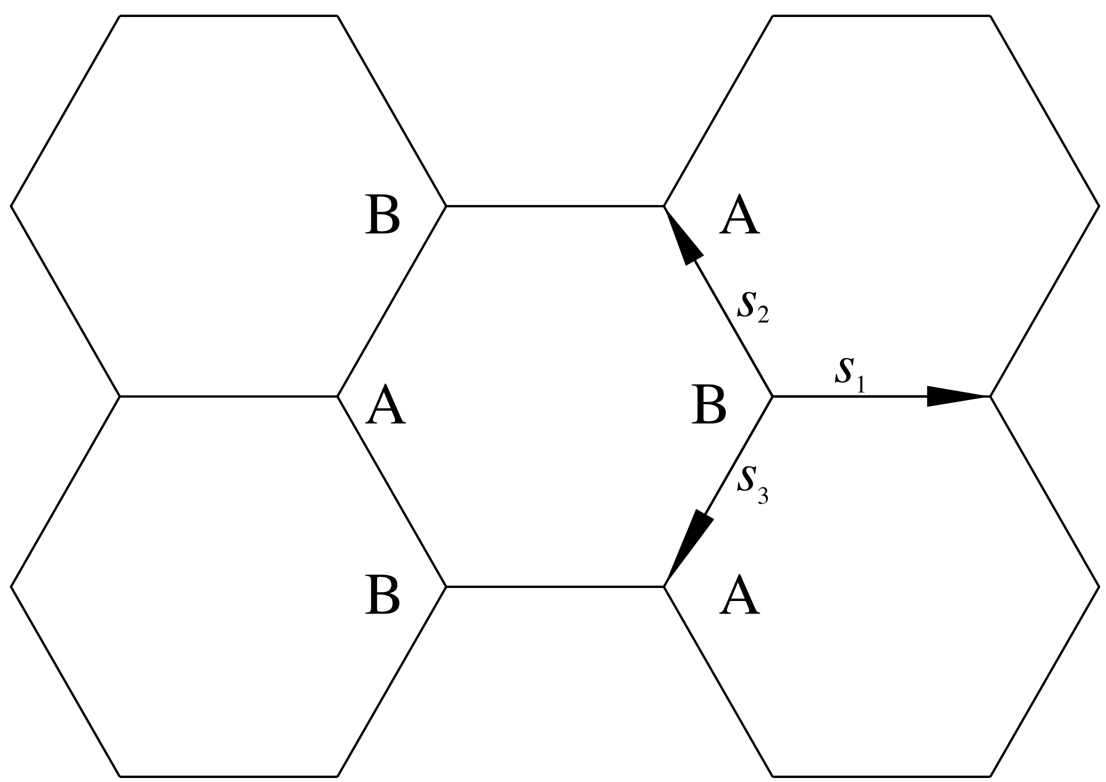


- Nodal fermions in *d*-wave superconductors
- Spin liquids in Heisenberg AFM
- surface states of topological insulators
- graphene

Relativity in Graphene

P.R. Wallace, Phys. Rev. **71** (1947) 622

The electronic properties of graphene were first studied theoretically almost 75 years ago



Rules of the Dance

On each carbon atom there can reside 0, 1 or 2 electrons which can hop between sites

$$H = -t \sum_{\mathbf{r}\in\mathbf{B}} \sum_{i=1}^{3} b^\dagger(\mathbf{r})a(\mathbf{r}+\mathbf{s}_i) + a^\dagger(\mathbf{r}+\mathbf{s}_i)b(\mathbf{r})$$

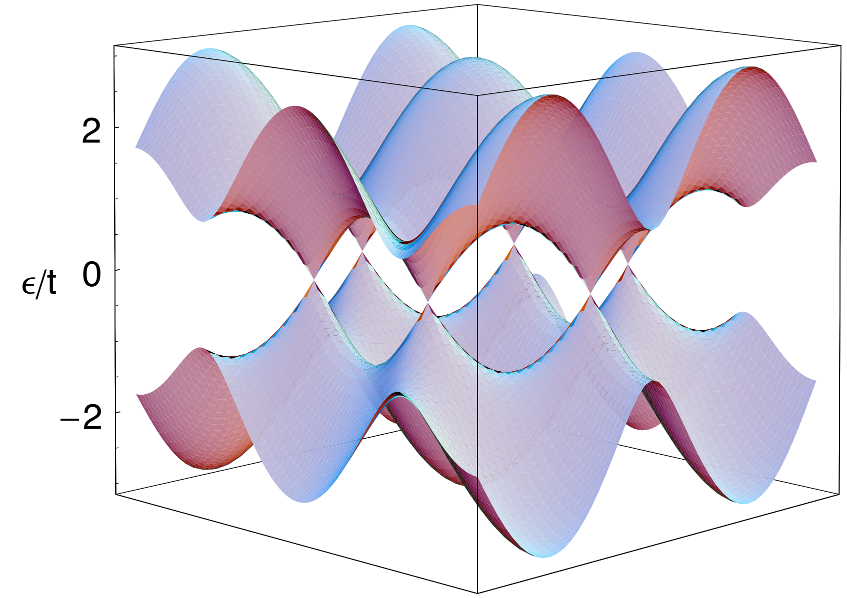
"tight-binding" Hamiltonian

describes hopping of electrons in π-orbitals from A to B sublattices and *vice versa*

In momentum space $$H = \sum_{\vec{k}} \left(\Phi(\vec{k})a^\dagger(\vec{k})b(\vec{k}) + \Phi^*(\vec{k})b^\dagger(\vec{k})a(\vec{k}) \right)$$

with $\Phi(\vec{k}) = -t\left[e^{ik_x l} + 2\cos\left(\frac{\sqrt{3}k_y l}{2}\right) e^{-i\frac{k_x l}{2}} \right]$

Define states $|\vec{k}_{\pm}\rangle = (\sqrt{2})^{-1}[a^{\dagger}(\vec{k}) \pm b^{\dagger}(\vec{k})]|0\rangle$



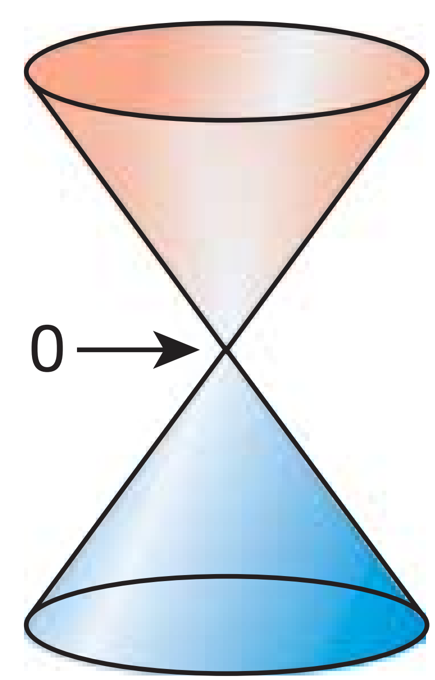
$$\Rightarrow \langle\vec{k}_{\pm}|H|\vec{k}_{\pm}\rangle = \pm(\Phi(\vec{k}) + \Phi^{*}(\vec{k})) \equiv \pm E(\vec{k})$$

Energy spectrum is symmetric about $E = 0$

Half-filling (neutral "undoped" graphene) has zero energy at **Dirac points** at corners of first Brillouin Zone:

Two independent Dirac points

$$\Phi(\vec{k}) = 0 \;\Rightarrow\; \vec{k} = \vec{K}_{\pm} = (0, \pm\frac{4\pi}{3\sqrt{3}l})$$



Taylor expand @ Dirac point

$$\Phi(\vec{K}_{\pm} + \vec{p}) = \pm v_F[p_y \mp i p_x] + O(p^2)$$

the pitch of the cone is the "Fermi velocity"

$$v_F = \frac{3}{2}tl$$

Define modified operators $a_\pm(\vec{p}) = a(\vec{K}_\pm + \vec{p})$ etc.

Now combine them into a "4-spinor" $\Psi = (\, b_+, a_+, a_-, b_-)^{tr}$

$$\Rightarrow H \simeq v_F \sum_{\vec{p}} \Psi^\dagger(\vec{p}) \begin{pmatrix} & p_y + ip_x & & \\ p_y - ip_x & & & \\ & & & -p_y - ip_x \\ & & -p_y + ip_x & \end{pmatrix} \Psi(\vec{p})$$

$$= v_F \sum_{\vec{p}} \Psi^\dagger(\vec{p}) \vec{\alpha}.\vec{p}\, \Psi(\vec{p})$$

Dirac Hamiltonian

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

ie. low-energy excitations are massless fermions with **Fermi velocity** $v_F = \frac{3}{2} t l \approx \frac{1}{300} c$

For monolayer graphene the number of flavors $N = 2$

(2 C atoms/cell × 2 Dirac points/zone × 2 spins = 2 flavors × 4 spinor)

Inter-electron interactions: an effective field theory

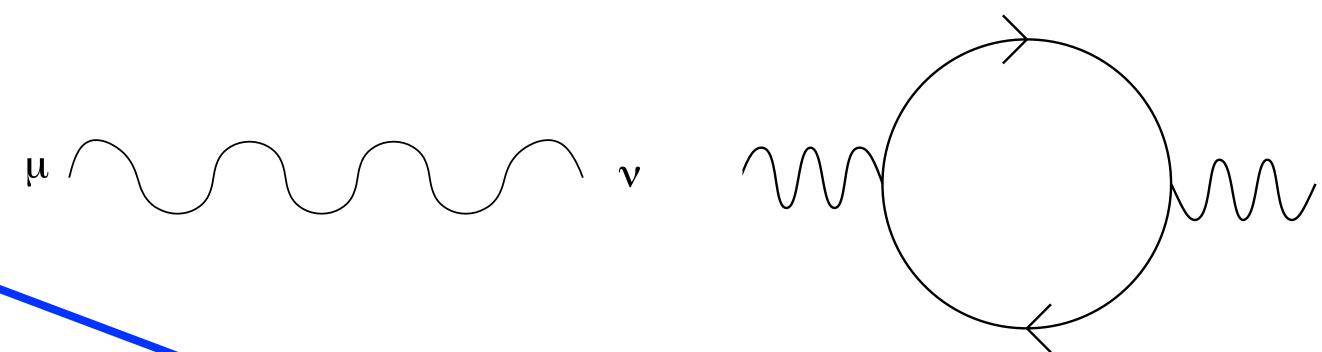
fermions live on two-dimensional "braneworld" & interact with photons living in the 3*d* bulk

$$S = \sum_{a=1}^{N_f} \int dx_0 d^2x (\bar{\psi}_a \gamma_0 \partial_0 \psi_a + v_F \bar{\psi}_a \vec{\gamma}.\vec{\nabla}\psi_a + iV\bar{\psi}_a \gamma_0 \psi_a) + \frac{1}{2e^2}\int dx_0 d^3x (\partial_i V)^2,$$

"instantaneous" Coulomb potential

since $v_F \ll c$ - unscreened since $\varrho(E=0)=0$

classical 3d Coulomb $\propto \frac{1}{r}$



V-propagator (large-N_f): $D(p) = \left(\frac{2|\vec{p}|}{e^2} + \frac{N_f}{8} \frac{|\vec{p}|^2}{(p_0^2 + v_F^2 |\vec{p}|^2)^{\frac{1}{2}}} \right)^{-1}$

quantum screening due to virtual electron-hole pairs $\propto \frac{1}{r}$

$$\lambda = \frac{e^2 N_f}{16\varepsilon\varepsilon_0 \hbar v_F} \simeq \frac{1.4 N_f}{\varepsilon}$$

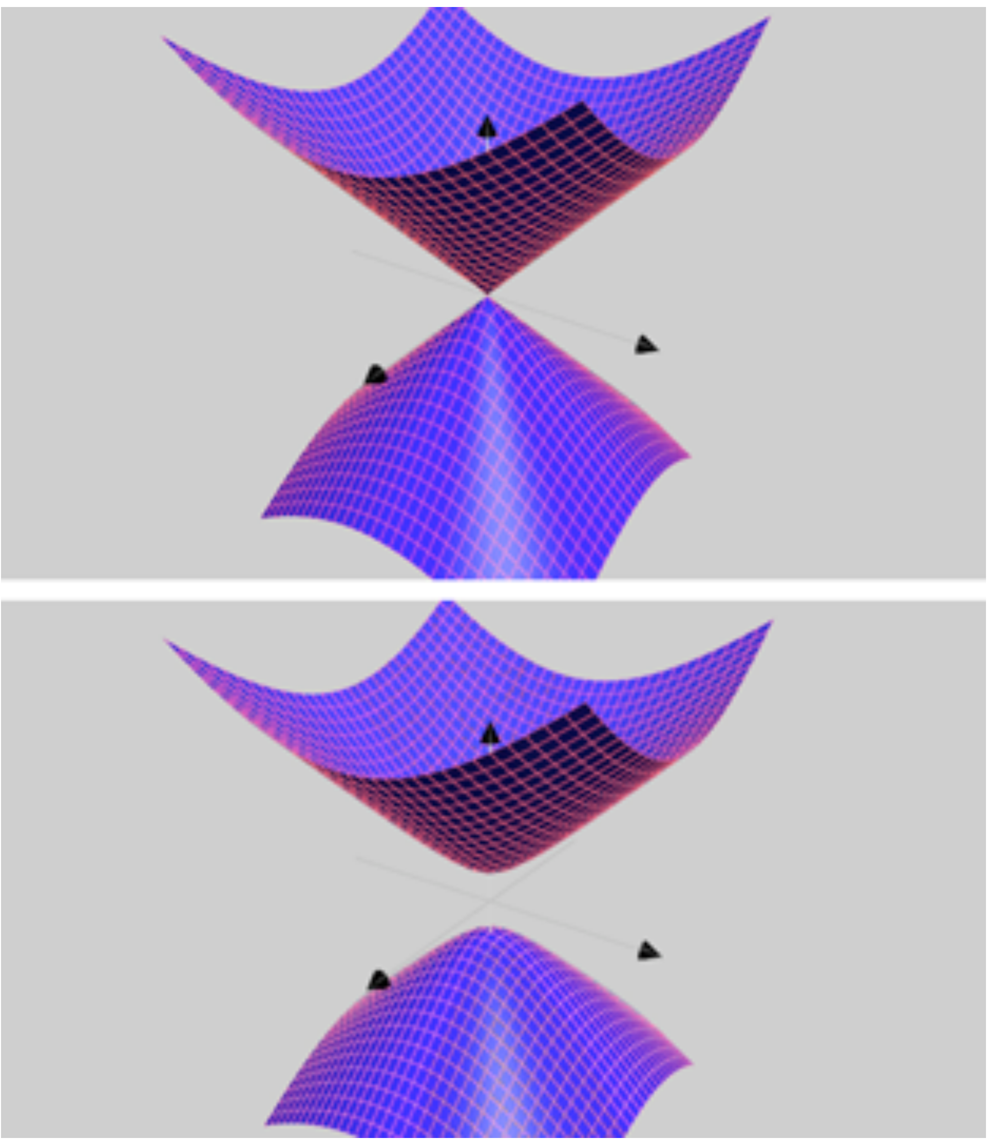
(i) parametrises quantum vs. classical

(ii) depends on dielectric properties of substrate

For sufficiently large g^2, or sufficiently small N, the Fock vacuum is conceivably disrupted by a particle-hole **bilinear condensate**

$$\langle\bar{\psi}\psi\rangle \equiv \frac{\partial \ln Z}{\partial m} \neq 0$$

resulting in a dynamically-generated mass gap at the Dirac point



Cf. chiral symmetry breaking in QCD

Hypothesis:

the semimetal-insulator transition at $g_c^2(N)$ defines a **Quantum Critical Point**: whose universal properties characterise low-energy excitations of graphene

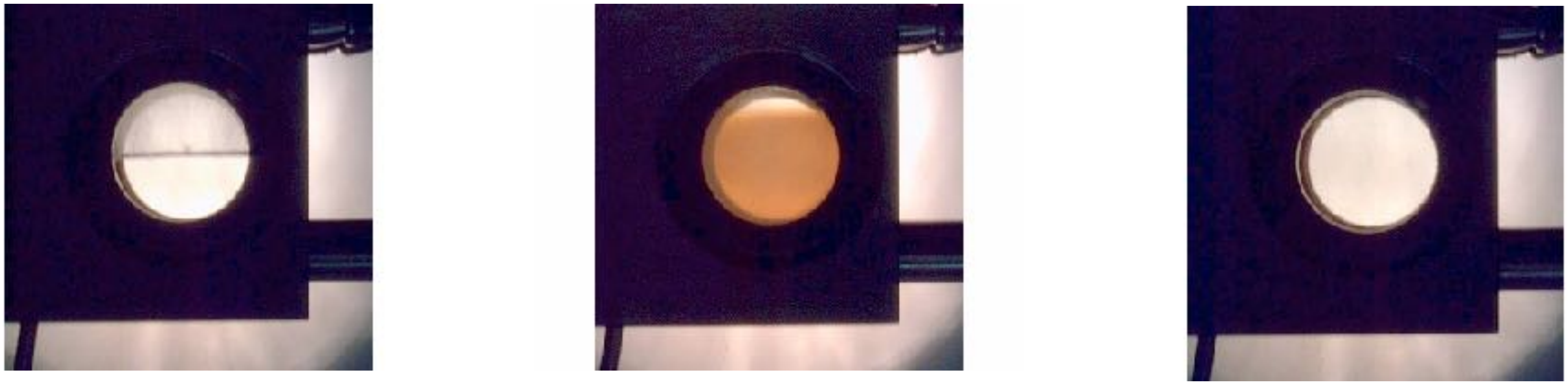
D.T. Son, Phys. Rev. B**75** (2007) 235423

Corresponds to a new strongly-interacting QFT...

...a priori there are no small dimensionless parameters 😮

What happens at a critical point?

$T < T_c$ $\qquad$ $T \simeq T_c$ $\qquad$ $T > T_c$



Three views of NaF_6

Inter-particle correlations scale $\propto e^{-x/\xi}$

the correlation length $\xi \to \infty$ as $T \to T_c$

droplet size $\longrightarrow$ $$\frac{\xi}{a} \sim \frac{\Lambda}{\mu}$$ $\longleftarrow$ UV cutoff

molecular scale (a)

scale of interesting physics, eg. particle mass (μ)

critical phenomena ⇔ continuum QFT

Continuum Symmetries in $d = 2+1$

$$\mathcal{S} = \int d^3x \; \bar\Psi(\gamma_\mu\partial_\mu)\Psi \; + \; m\bar\Psi\Psi$$

For m=0 S is invariant under global U(2N) symmetry generated by

(i) $\Psi \mapsto e^{i\alpha}\Psi\,; \qquad \bar\Psi \mapsto \bar\Psi e^{-i\alpha},$ (ii) $\Psi \mapsto e^{i\alpha\gamma_5}\Psi\,; \quad \bar\Psi \mapsto \bar\Psi e^{i\alpha\gamma_5}$

(iii) $\Psi \mapsto e^{\alpha\gamma_3\gamma_5}\Psi\,; \quad \bar\Psi \mapsto \bar\Psi e^{-\alpha\gamma_3\gamma_5}$ (iv) $\Psi \mapsto e^{i\alpha\gamma_3}\Psi\,; \quad \bar\Psi \mapsto \bar\Psi e^{i\alpha\gamma_3}$

For $m\neq0$, γ_3 (iv) and γ_5 (ii) rotations are no longer symmetries

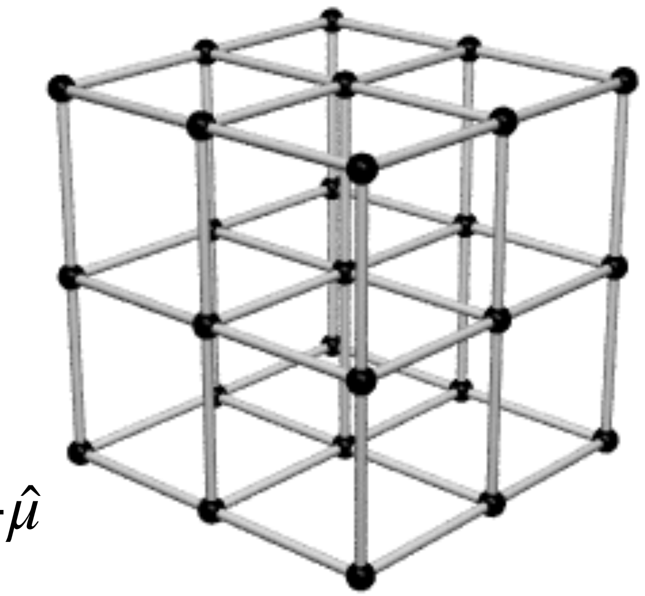
$\Rightarrow$ U(2N) → U(N)⊗U(N)

Mass term $m\bar\Psi\Psi$ is **hermitian** & invariant under **parity** $x_\mu \mapsto -x_\mu$

Two physically equivalent **antihermitian** *twisted* or *Kekulé* mass terms: $im_3\bar\Psi\gamma_3\Psi; \quad im_5\bar\Psi\gamma_5\Psi$

The *Haldane* mass $m_{35}\bar\Psi\gamma_3\gamma_5\Psi$ is *not* parity-invariant

Numerical Lattice Approach



Early work used staggered fermions

$$S_{latt} = \frac{1}{2}\sum_{x,\mu i} \bar{\chi}^i_x \eta_{\mu x}(1 + \underbrace{iA_{\mu x}}) \chi^i_{x+\hat{\mu}} - \bar{\chi}^i_x \eta_{\mu x}(1 - \underbrace{iA_{\mu x-\hat{\mu}}}) \chi^i_{x-\hat{\mu}}$$

non-unitary link fields

$$+ m \sum_{xi} \bar{\chi}^i_x \chi^i_x + \frac{N}{4g^2}\sum_{x\mu} A^2_{\mu x}$$

vector auxiliary $A_{\mu x}$ defined on **link** *between* x and $x+\mu$

symmetry breaking resulting from gap generation: $\mathrm{U(N)} \otimes \mathrm{U(N)} \rightarrow \mathrm{U(N)}$

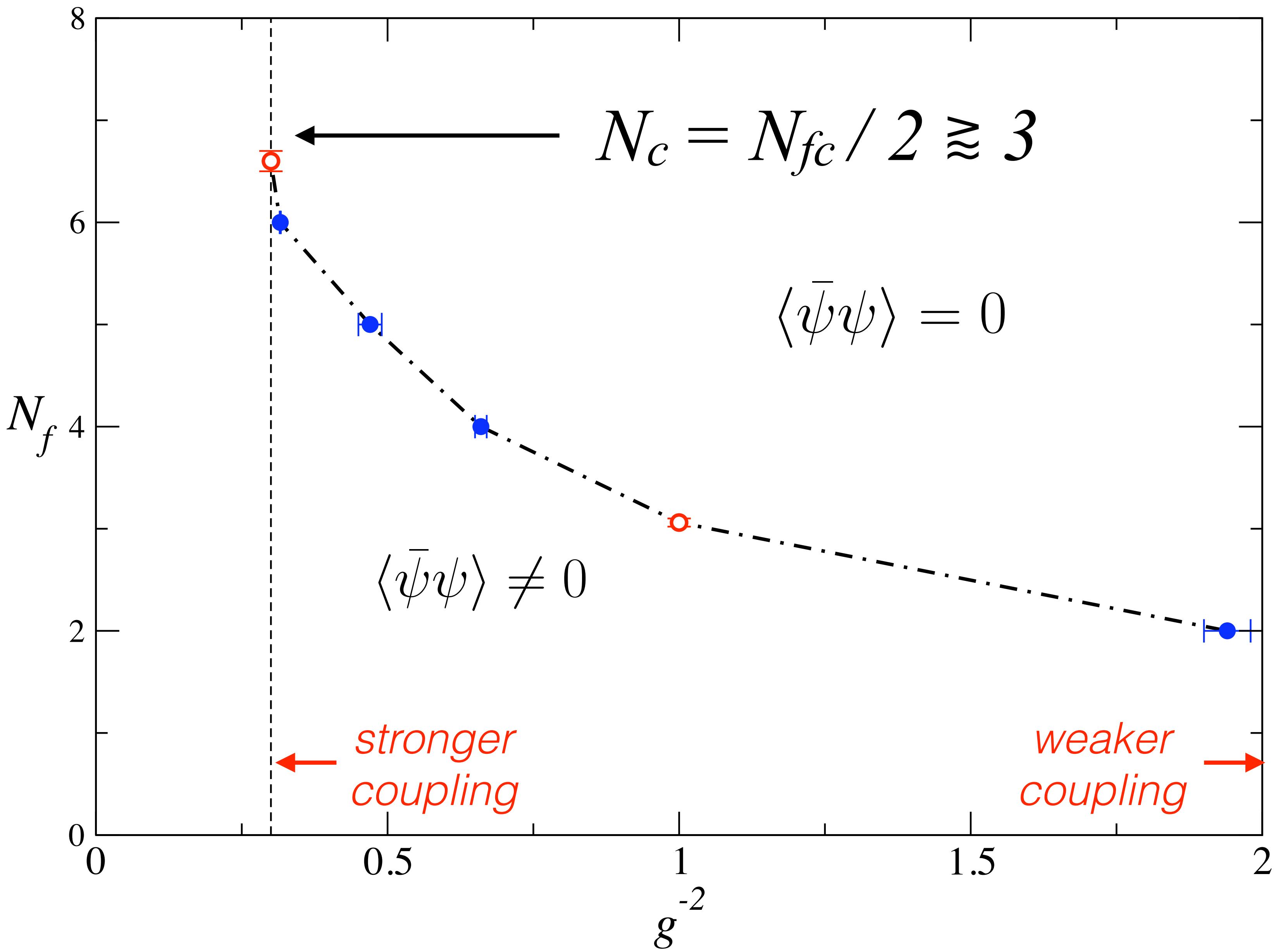
In **weak coupling** continuum limit

$\mathrm{U(2N}_f)$ symmetry is recovered, with $\mathrm{N}_f = 2\mathrm{N}$

this is an instance of "lattice fermion doubling"

no such expectation **in general** at a **QCP**

Phase diagram of the Staggered Thirring Model



Find symmetry broken phase (gapped, insulating) for small N, large g^2

Christofi, SJH, Strouthos PR**D75** (2007) 101701

Two Old Conjectures…

1. In leading order large-N, in the limit $g^2 \to \infty$, the interaction between conserved currents is mediated by a vector boson with mass $$\frac{M_V}{m} = \sqrt{\frac{6\pi}{mg^2}} \to 0$$ SJH, PRD **51** (1995) 5816

Thirring QCP in strong coupling limit equivalent to IR limit of QED_3?

2. An asymptotically-free theory like QED_3 is constrained by the following inequality: $$f_{IR} \leq f_{UV}$$

Here $f = -\frac{90}{\pi^2 T^4} \times$ (thermodynamic free energy density)

Appelquist, Cohen, Schmaltz PRD **60** (1999) 045003

f can be related to # degrees of freedom, be they massless fermions or Goldstone bosons

Prediction for QED3 from $f_{IR} \leq f_{UV}$

Staggered fermions U(N)⊗U(N)→U(N):

$$N^2 \leq \frac{3}{4} \times \underbrace{2^d \times N}_{\text{\# fermion dofs}} \Rightarrow N_c \leq 6$$

Goldstone bosons

fermion dofs

Continuum fermions U(2N)→U(N)⊗U(N):

$$2N^2 \leq \frac{3}{4} \times 4N \Rightarrow N_c \leq \frac{3}{2}$$

Big disparity !

NB F-theorem prediction: $N_c < 4.4$

Giombi, Klebanov, Tarnopolsky JPA**49** (2016) 135403

This motivates studies using
fermions transforming with the correct symmetry

Staggered Fermion Bag Algorithm with minimal $N_f = 2$ (N=1)

Chandrasekharan & Li, PRL **108** (2012) 140404; PRD**88** (2013) 021701

Thirring Model: $\nu=0.85(1)$, $\eta=0.65(1)$, $\eta_\psi=0.37(1)$ $(N_f < N_{fc} \approx 7)$

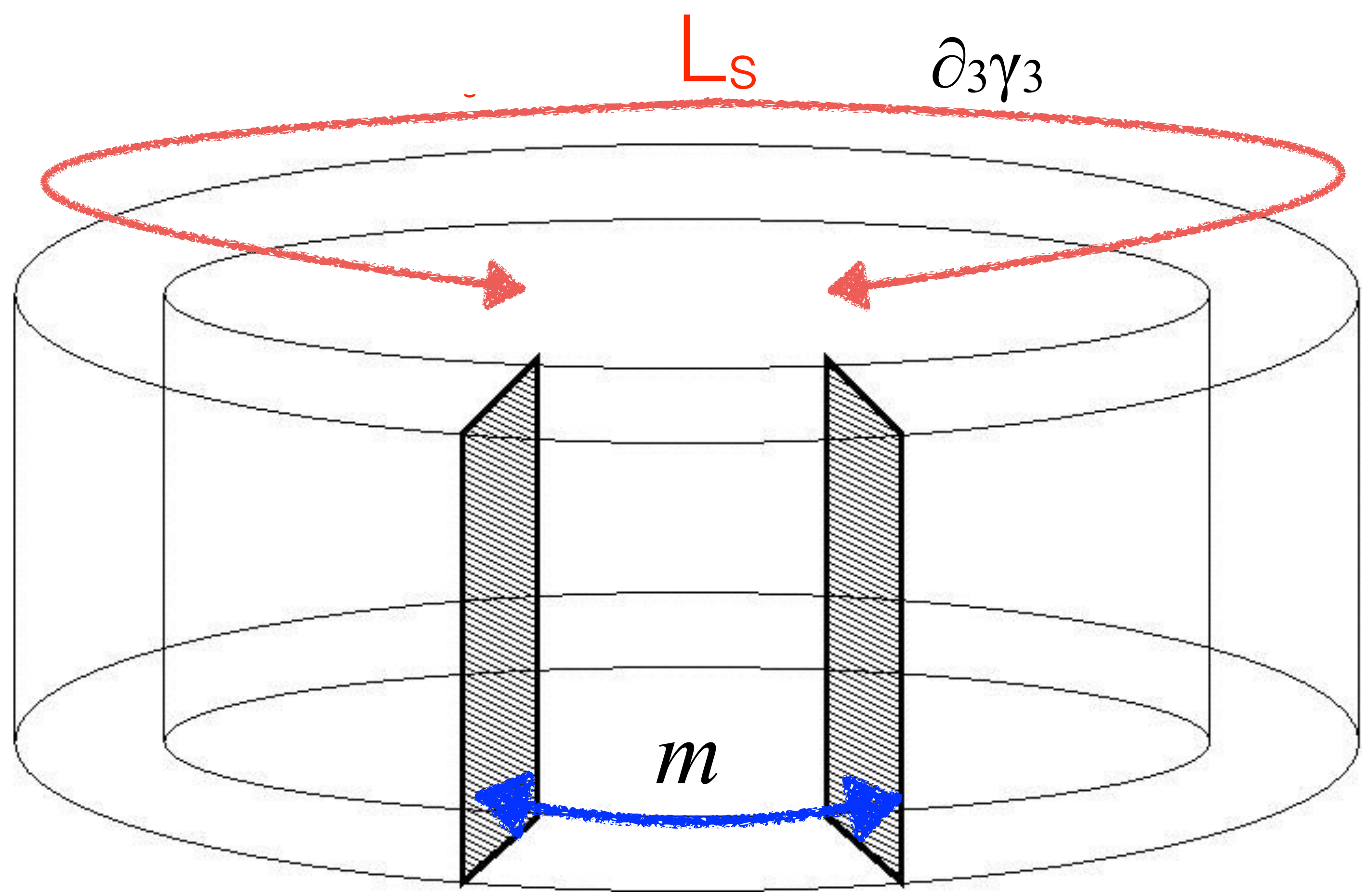
U(1) GN Model: $\nu=0.849(8)$, $\eta=0.633(8)$, $\eta_\psi=0.373(3)$ $(N_f \to \infty: \nu=\eta=1)$

Interactions between staggered fields $\chi, \bar{\chi}$ spread over elementary cubes.
Only difference between Thirring & GN is body-diagonal term

Staggered fermions not reproducing expected distinction between Thirring and GN QCPs

… so we need "better" lattice fermions?

Domain Wall Fermions



$$\mathscr{L} = \bar{\Psi}(x,s) D_{DWF} \Psi(y,s')$$

Fermions propagate freely along a fictitious third direction of extent L_s with open boundaries

Basic idea as $L_s \rightarrow \infty$:

- zero-modes of D_{DWF} localised on walls are $\pm$ eigenmodes of γ_3
- Modes propagating in bulk can be decoupled (with cunning)

"Physical" fields in 2+1d target space

$$\psi(x) = P_- \Psi(x,1) + P_+ \Psi(x, L_s);$$
$$\bar{\psi}(x) = \bar{\Psi}(x, L_s) P_- + \bar{\Psi}(x,1) P_+;$$

with projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_3)$

Bottom Up View…

in DWF approach we simulate
2+1+1d fermions

Desiderata…

- Modes localised on walls carry U(2N)-invariant physics
- Fermion doublers don't contribute to normalisable modes
- Bulk modes can be made to decouple

Claim…

It appears to work for….

- carefully-chosen "domain wall height" M
- smooth gauge field background

Top Down View…

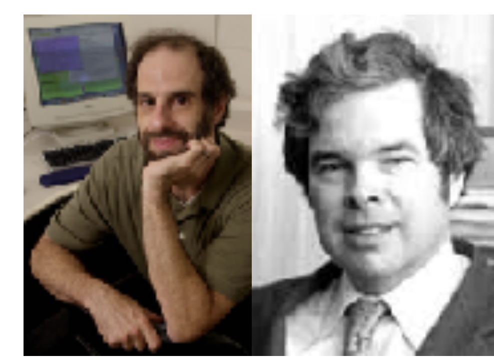
write the fermion bilinear: $\bar{\psi} M \psi = \bar{\psi}(D + m)\psi$

Then U(2N) symmetry can be re-expressed:

$$\{\gamma_3, D\} = \{\gamma_5, D\} = [\gamma_3\gamma_5, D] = 0$$

There is no regular lattice discretisation respecting these relations, while simultaneously describing unitary, local dynamics of a single Dirac fermion species

Nielsen & Ninomiya,1981



The closest we can get is articulated by the **Ginsparg-Wilson** relations:

$$\{\gamma_3, D\} = 2D\gamma_3 D \qquad \{\gamma_5, D\} = 2D\gamma_5 D \qquad [\gamma_3\gamma_5, D] = 0$$

RHS is $O(aD)$, so U(2N) recovered in long-wavelength limit if D local

By construction GW is satisfied by the 2+1d *overlap* operator

$$D_{ov} = \frac{1}{2}\left[(1+m_h) + (1-m_h)\frac{A}{\sqrt{A^\dagger A}}\right]$$

with, eg. $A \equiv [2+(D_W - M)]^{-1}[D_W - M]; \quad D_W \text{ local}; \; Ma = O(1)$

$$\gamma_3 A \gamma_3 = \gamma_5 A \gamma_5 = A^\dagger$$

locality of D_{ov} not manifest but confirmed numerically

SJH, Mesiti, Worthy PRD **102** (2020) 094502

DWF provide a regularisation of overlap with a *local* kernel in 2+1+1d 🙂

ie. $\frac{\det D_{\text{DWF}}(m_i)}{\det D_{\text{DWF}}(m_h = 1)} = \det D_{L_s}(m_i)$ with $\lim_{L_s \to \infty} D_{L_s} = D_{ov}$

SJH PLB **754** (2016) 264

Formulational issues

By analogy with QCD, formulate auxiliary field $A_\mu(x)$ throughout bulk and 3-static ie. $\partial_3 A_\mu = 0$:

$\mathcal{S} = \bar\Psi \mathcal{D} \Psi = \bar\Psi D_W \Psi + \bar\Psi D_3 \Psi + m_i S_i$ with

$$\begin{aligned} D_W &= \gamma_\mu D_\mu - (\hat{D}^2 + M); \\ D_3 &= \gamma_3 \partial_3 - \hat{\partial}_3^2, \end{aligned}$$

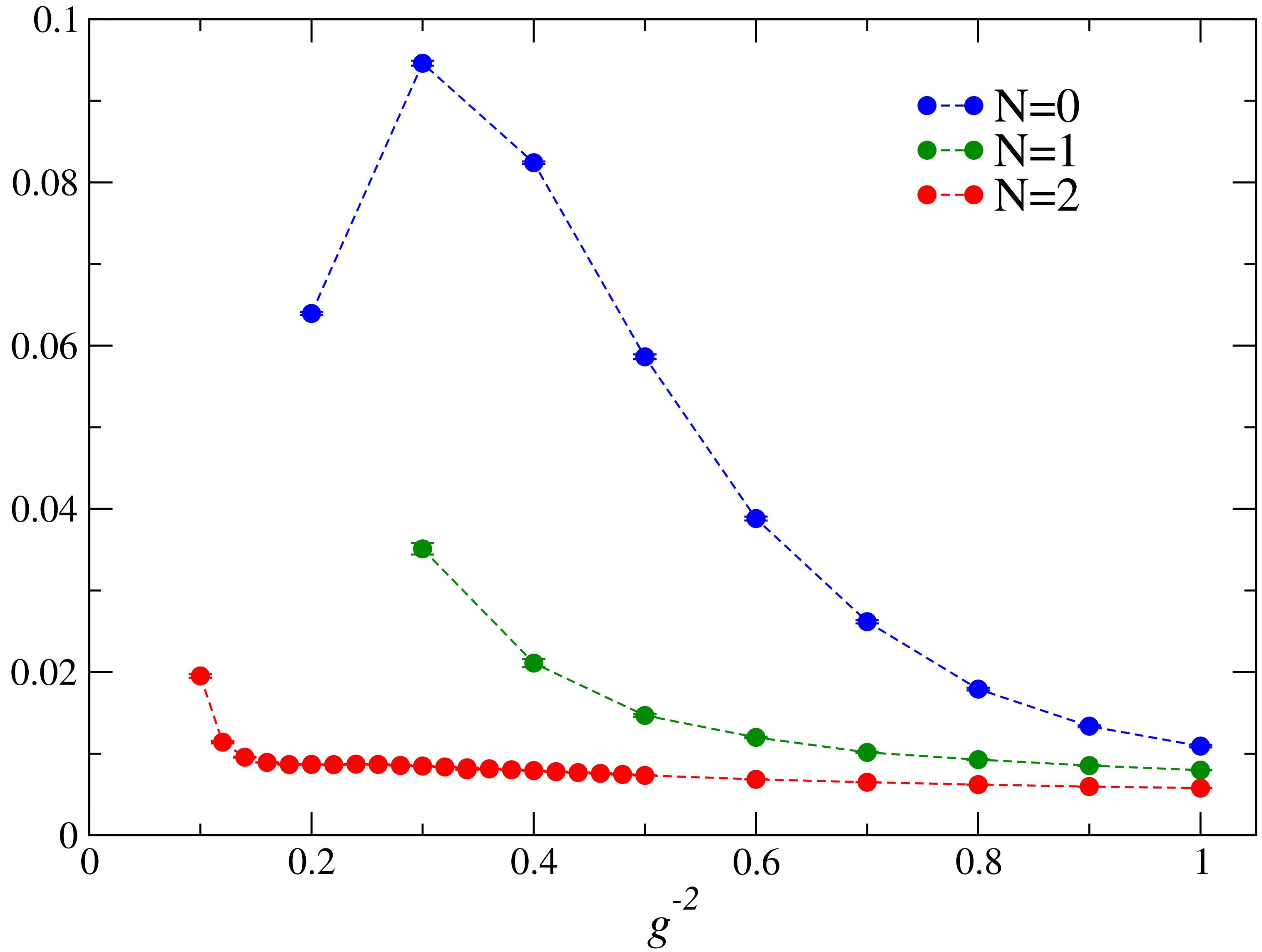
NB $D_\mu \propto (1 + iA_\mu)$, not e^{iA_μ}, ie. links are ***non-compact*** and ***non-unitary***

$[\partial_3, D_\mu] = [\partial_3, \hat{D}^2] = 0$ but $[\partial_3, \hat{\partial}_3^2] \neq 0$ on walls

obstruction to proving det$\mathscr{D} > 0$

RHMC with measure $\sqrt{\mathbf{det}(\mathscr{D}^\dagger \mathscr{D})}$ **for** $N = 1$

Exploratory Results with fixed $L_s = 16$

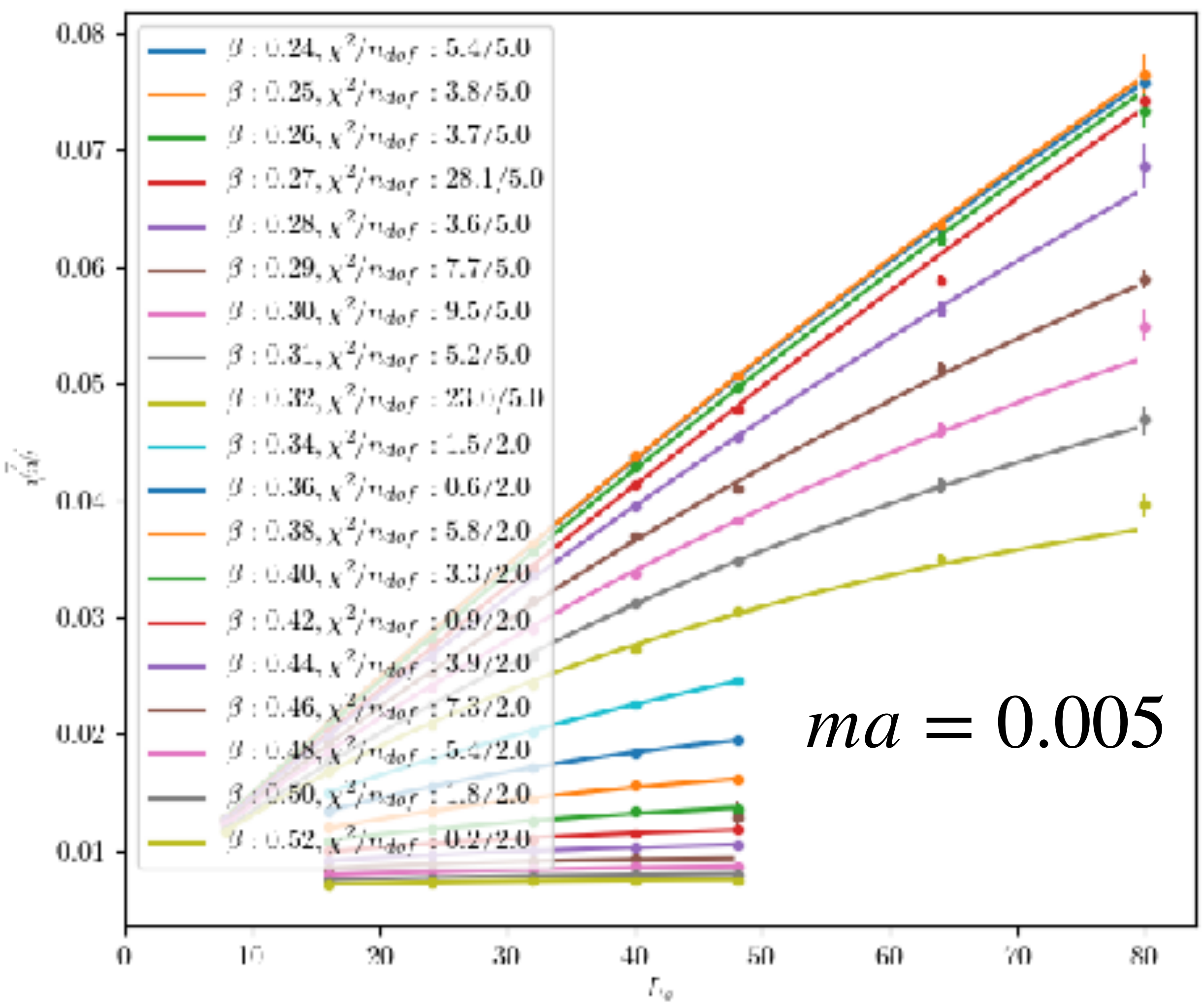


Bilinear condensate $\langle i\bar{\psi}\gamma_3\psi\rangle$ for $N = 0,1,2\ldots$
hierarchy consistent with $1 < N_c < 2$

But to achieve U(2) symmetry we need $L_s \to \infty\ldots$

Stress-testing DWF...

$$\langle\bar{\psi}\psi\rangle_\infty - \langle\bar{\psi}\psi\rangle_{L_s} = A(\beta, m)e^{-\Delta(\beta,m)L_s}$$



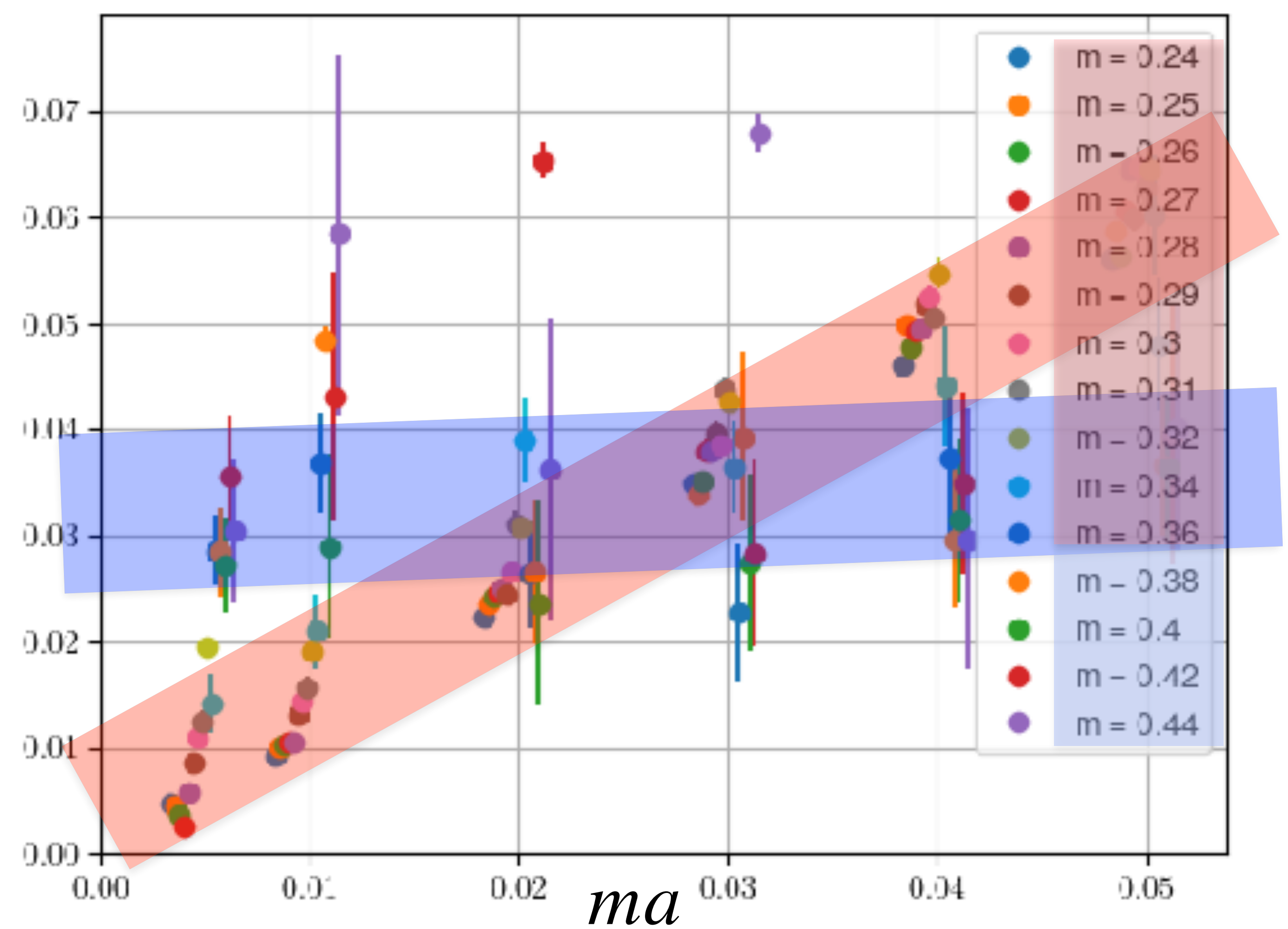
Have $L_s = 8, 16, \ldots, 48$
currently accumulating $L_s = 64, 80$

$L_s \to \infty$ not yet under control
at lightest masses, strongest couplings

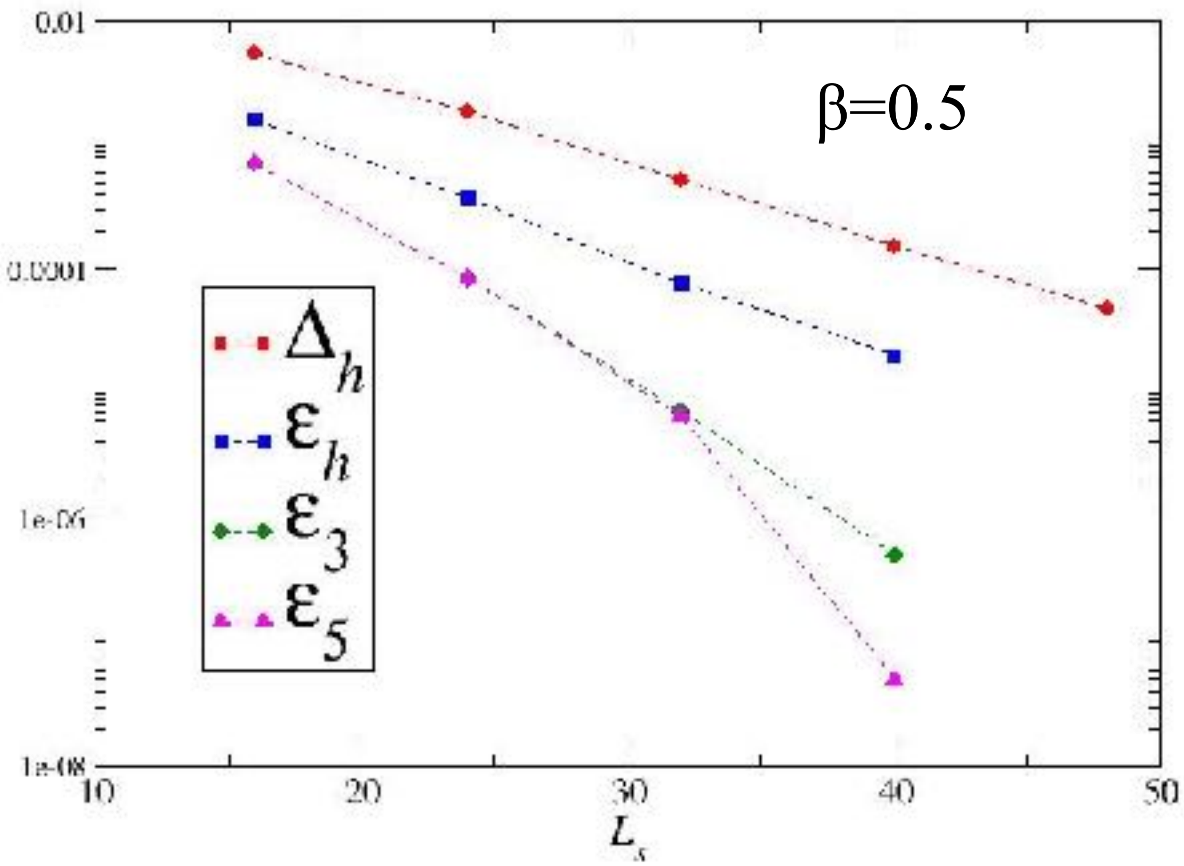
Decay constant $\Delta(\beta, m)$:

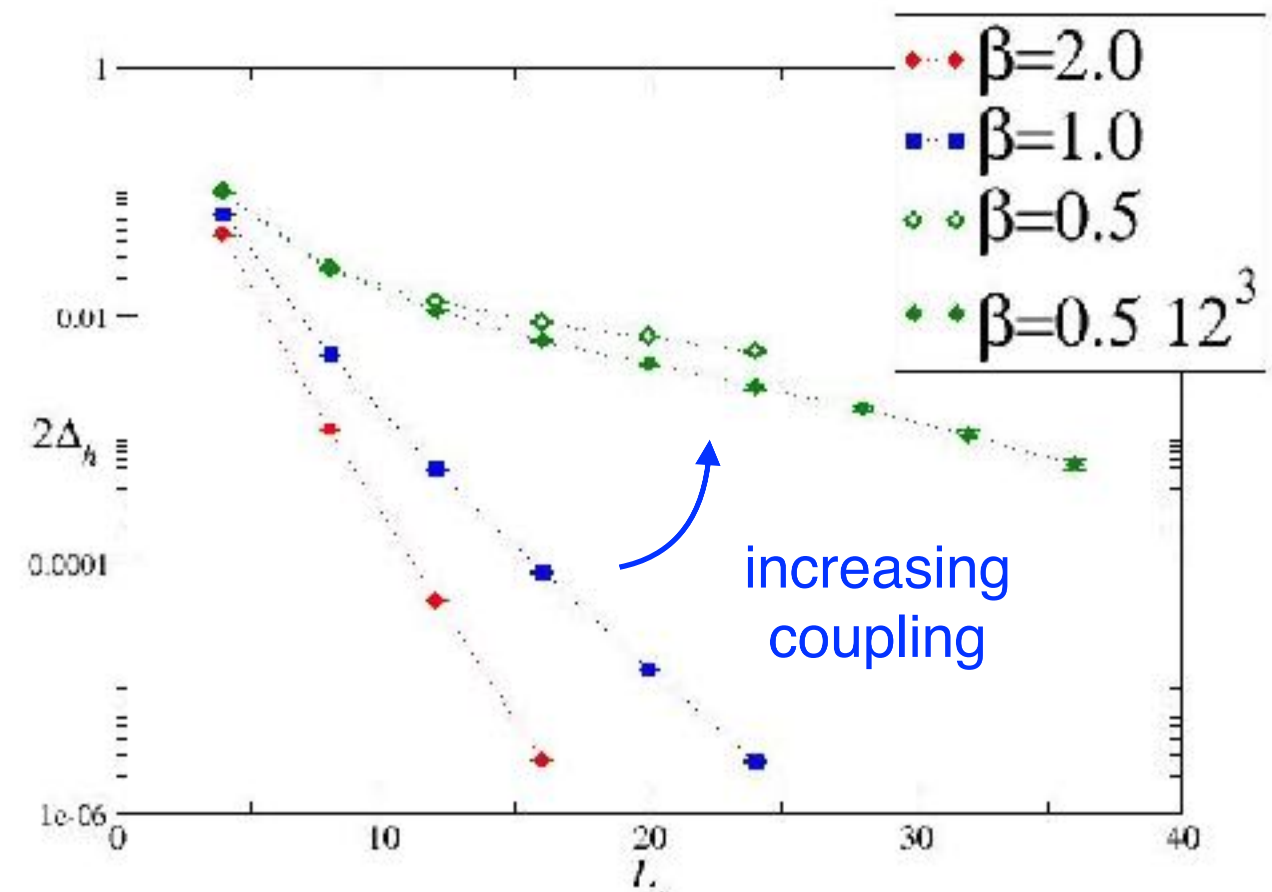
$\sim \propto m^0$ at weak coupling

$\sim \propto m$ at strong coupling



Bilinear Condensates in Quenched QED_3 on $24^3\times L_s$…





Define main *residual*: $i\langle\bar\Psi(1)\gamma_3\Psi(L_s)\rangle = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_s} + i\Delta_h(L_s)$

real — imaginary

$$\frac{1}{2}\langle\bar\psi\psi\rangle_{L_s} = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_S\to\infty} + \Delta_h(L_s) + \epsilon_h(L_s);$$
$$\frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_s} = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_3(L_s);$$
$$\frac{i}{2}\langle\bar\psi\gamma_5\psi\rangle_{L_s} = \frac{i}{2}\langle\bar\psi\gamma_3\psi\rangle_{L_S\to\infty} + \epsilon_5(L_s).$$

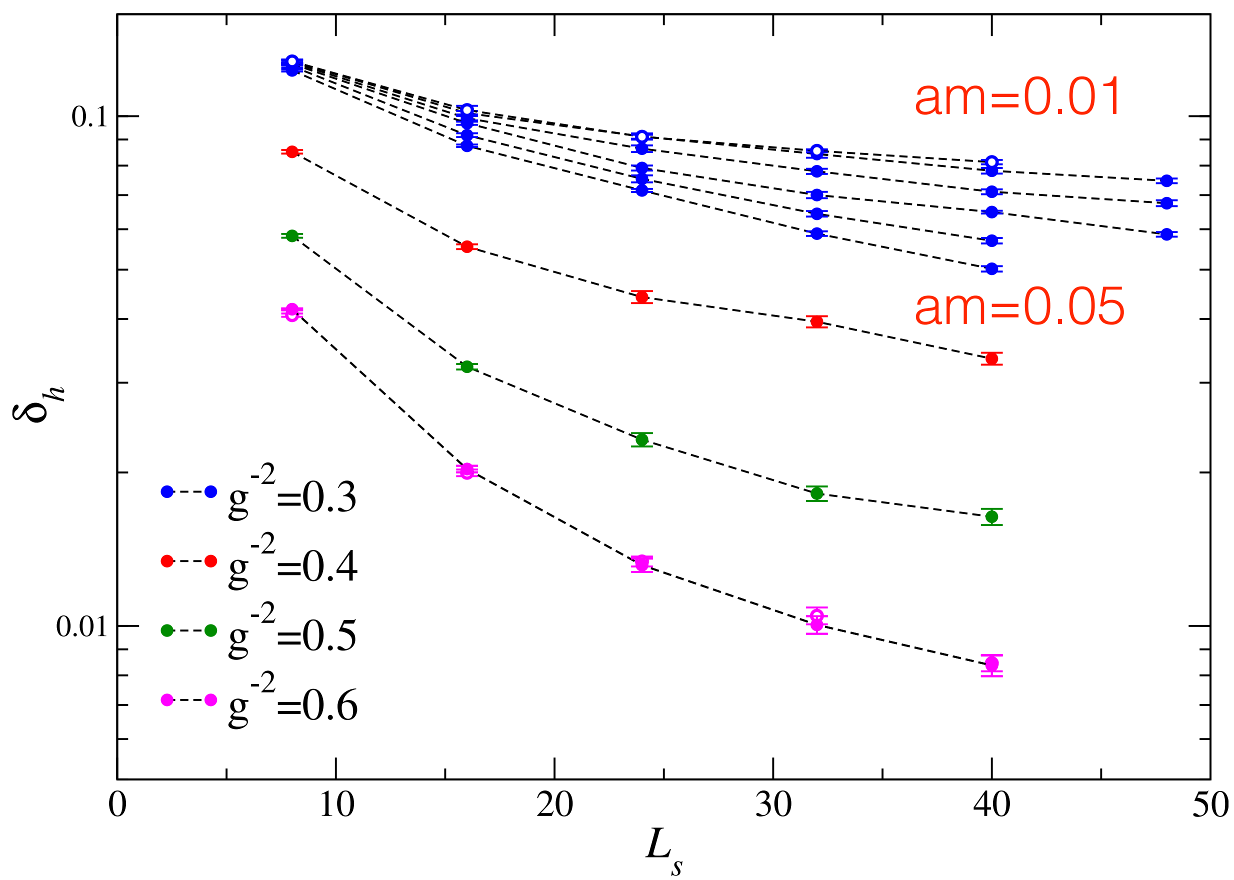
U(2) symmetry restored

$\Leftrightarrow \Delta_h\to 0$

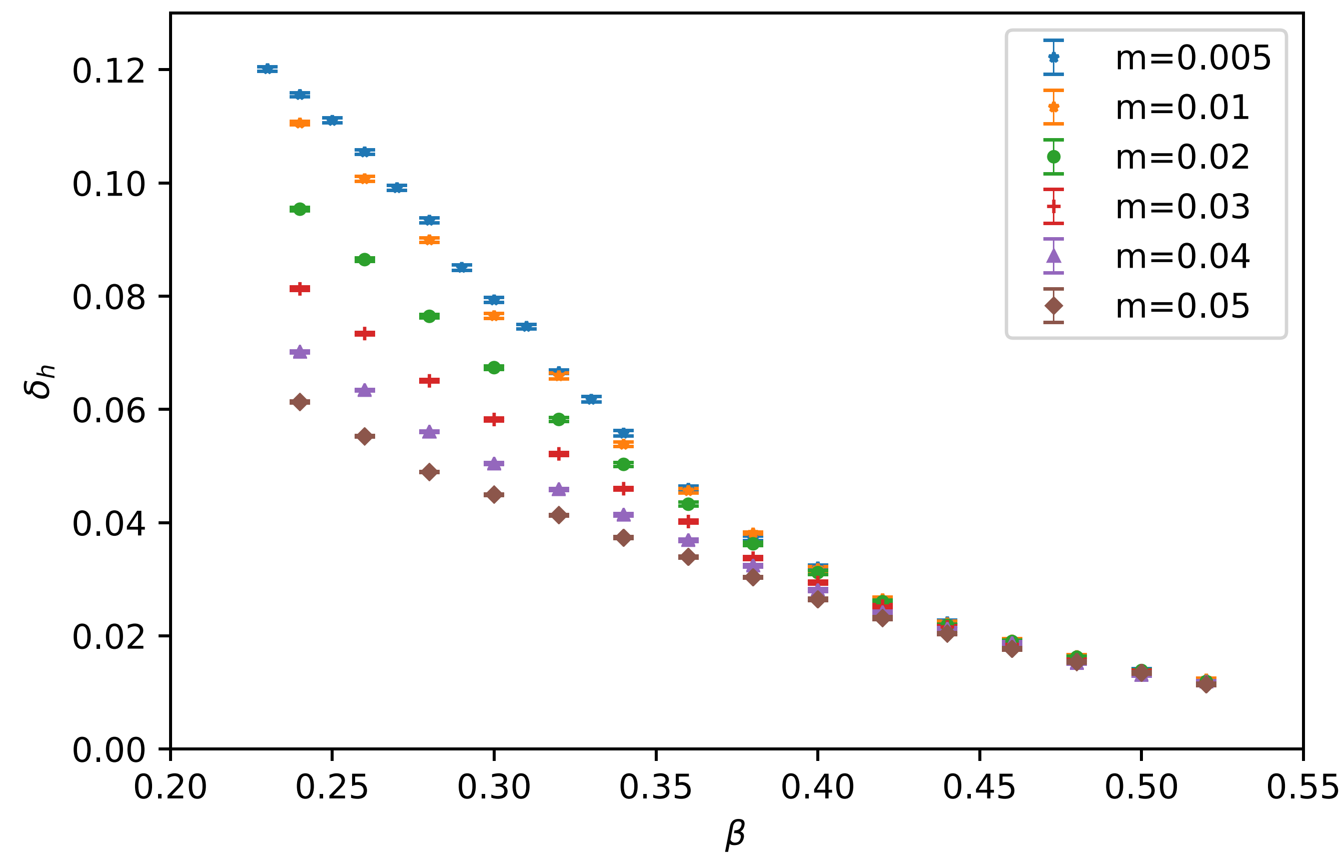
- exponentially suppressed as $L_s\to\infty$ ✓
- hierarchy: $\Delta_h > \varepsilon_h > \varepsilon_3 \equiv \varepsilon_5$ ✓✓

SJH JHEP **09**(2015)047,
PLB **754** (2016) 264

U(2) symmetry restoration requires residual $\delta_h \to 0$



on $12^3 \times L_s \to \infty$



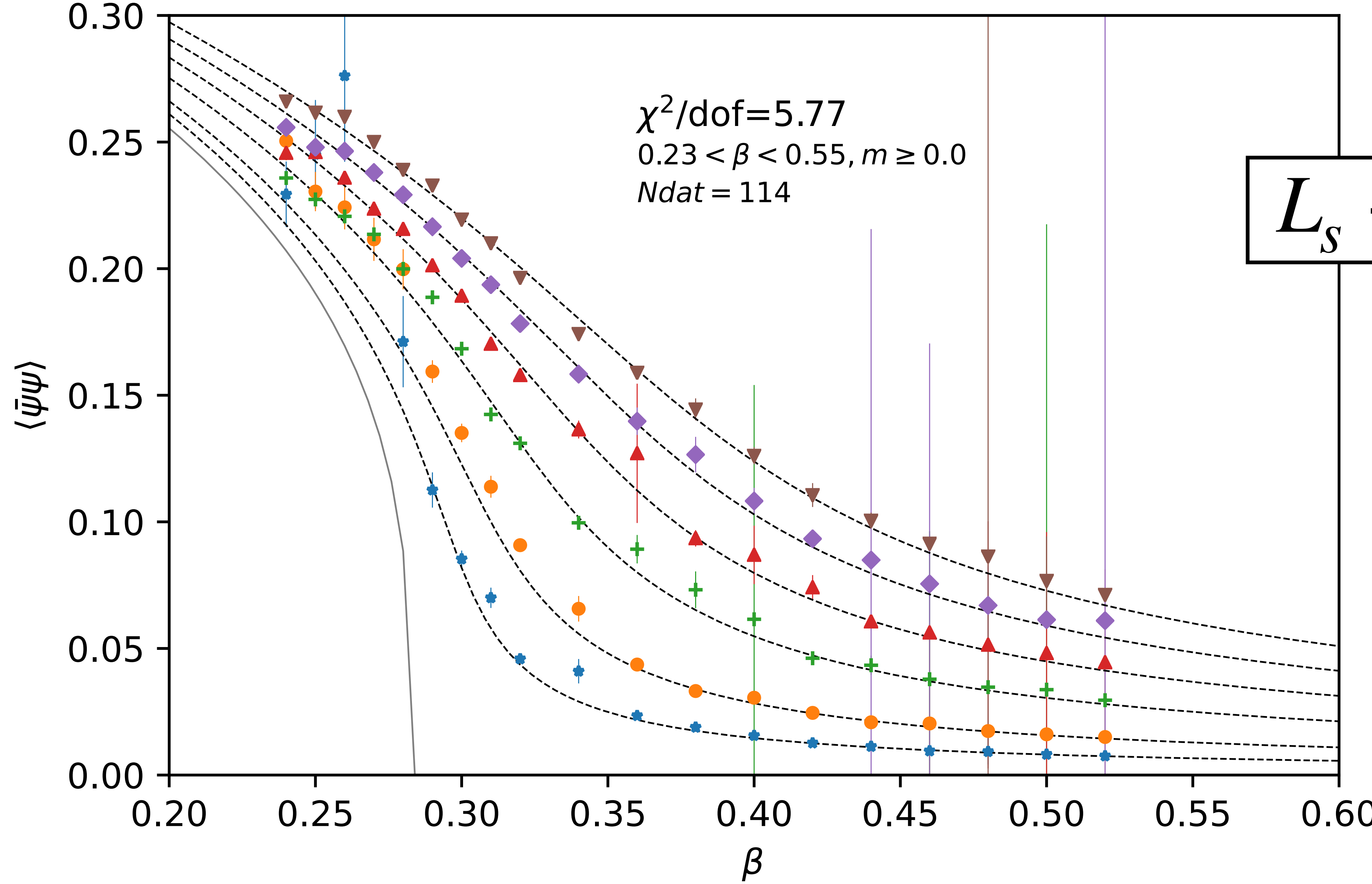
as a function of $\beta \equiv ag^{-2}$ on $16^3 \times 48$

Qualitatively different at strong and weak coupling,
and *slow*…

$16^3 \times L_s = 48$, am=0.005, $ag^{-2}=0.3$:

RHMC Hamiltonian step requires ~20k solver iterations (recall non-unitary links)

Latest data extrapolated with $L_s = 64{,}80$ and straddling β_c



$L_s \to \infty$

$$m = A(g^{-2} - g_c^{-2})\langle\bar{\psi}\psi\rangle^{\delta-1/\beta} + B\langle\bar{\psi}\psi\rangle^{\delta}$$

Fit to renormalisation group-inspired equation of state suggests QCP exists for N=1

SJH, Mesiti, Worthy arXiv:2110.03944

Preliminary EoS fit in $L_s \to \infty$ limit

$1 < N_c < 2$

Critical parameters $\begin{cases} \beta_c \equiv g_c^{-2} = 0.283(1) \\ \delta = 4.17(5) \qquad \beta = 0.320(5) \end{cases}$

hyperscaling ⇒ $\nu = 0.55(1) \qquad \eta = 0.16(1)$

Cf: old result for $N = 1$ staggered fermion

⇔ $N = 1$ **Kähler-Dirac fermion**

≠ $N_f = 2$ Dirac fermions**!**

Del Debbio, SJH, Mehegan
NP**B502** (1997) 269
Christofi, SJH, Strouthos
PR**D75** (2007) 101701
SJH Symmetry **13** (2021) 8

$3 < N_c < 4$

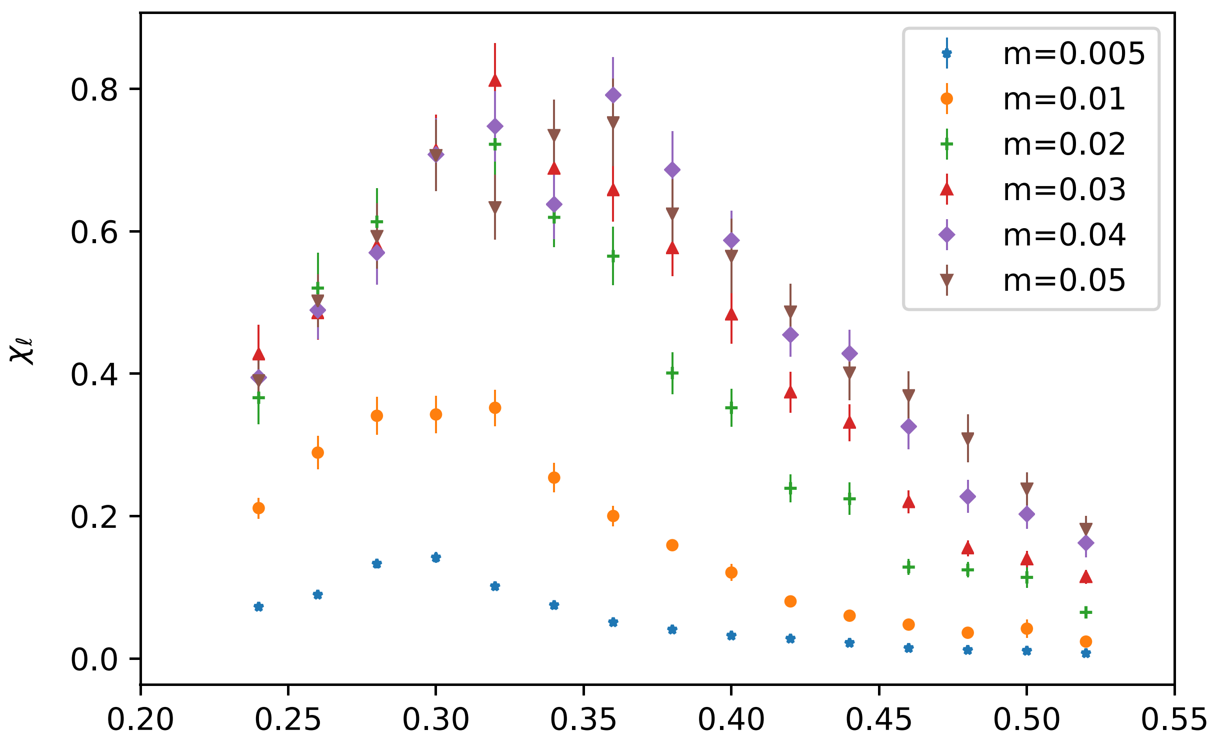
$\delta = 2.75(9) \qquad \beta = 0.57(2)$

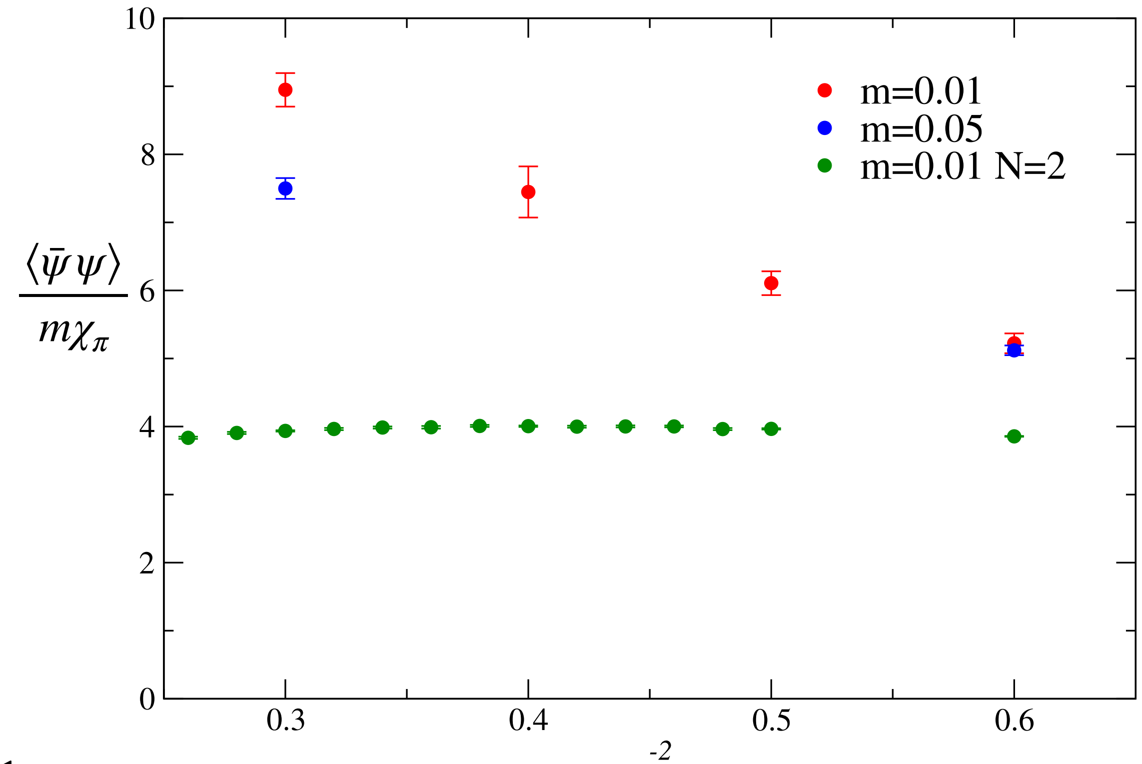
$\nu = 0.71(3) \qquad \eta = 0.60(4)$

Dirac and Kähler-Dirac fermions have distinct QCPs

Funnies….

- Susceptibility $\chi_\ell = \langle(\bar{\psi}\psi)^2\rangle - \langle\bar{\psi}\psi\rangle^2$ shows inverted mass hierarchy





- Axial Ward Identity $\langle\bar{\psi}\psi\rangle/m\chi_\pi = 1$ a long way from being satisfied

Much more work to do…

Summary

- new QCP for Dirac fermions in $2{+}1\mathrm{d}$ with $1 < N_c < 2$
- not everyone agrees: $N_c \approx 0.8$ with SLAC fermions
  Lenz, Wellegehausen & Wipf, PRD**100** (2019) 054501

- Dirac / Kähler-Dirac fermions support distinct QCPs
- need fermion propagator data to access exponent η_ψ
- need meson $\psi\bar{\psi}$ propagators to confirm Goldstones
- need smarter ways to access U(2) limit $L_s \to \infty$

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