

# A Primer on Tsallis Statistics for Nuclear and Particle Physics

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Introduction

Tsallis Nonextensive Entropy

Tsallis Type I

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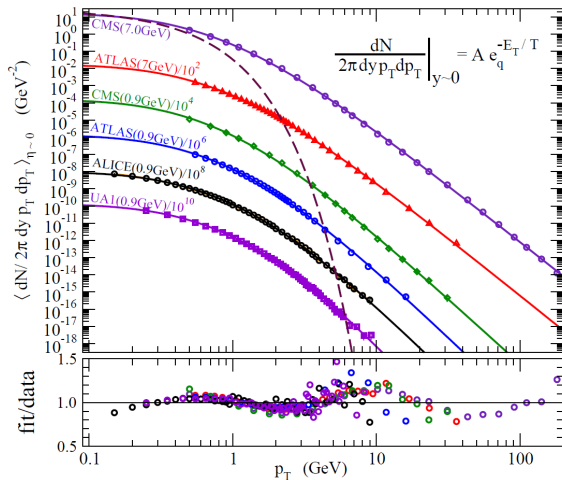
Boltzmann Equation

High Energy Collision Phenomenology

Conclusion

In 1988 Constantino Tsallis entertained the possibility of ensembles where the entropy took a nonextensive form involving a parameter  $q$  which reduced to the usual entropy in the limit  $q = 1$ . Since then the so-called Tsallis statistics has been applied to many areas of the natural and social sciences. I will give a concise introduction to the topic meant especially for those interested in its relation to high energy proton-proton, proton-nucleus, and nucleus-nucleus collisions.

# Introduction



Three-parameter fits to  $pp$  collisions with  $A(1 + (q - 1)\beta m_T)^{-1/(q-1)}$ .

## Possible Generalization of Boltzmann–Gibbs Statistics

Constantino Tsallis<sup>1</sup>

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With the use of a quantity normally scaled in multifractals, a generalized form is postulated for entropy, namely  $S_q \equiv k[1 - \sum_{i=1}^W p_i^q]/(q-1)$ , where  $q \in \mathbb{R}$  characterizes the generalization and  $\{p_i\}$  are the probabilities associated with  $W$  (microscopic) configurations ( $W \in \mathbb{N}$ ). The main properties associated with this entropy are established, particularly those corresponding to the microcanonical and canonical ensembles. The Boltzmann–Gibbs statistics is recovered as the  $q \rightarrow 1$  limit.

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**KEY WORDS:** Generalized statistics; entropy; multifractals; statistical ensembles.

9,000 citations to this paper according to Google Scholar.  
1,500 articles in the arXiv with Tsallis in the abstract.

In statistical mechanics the entropy is defined to be

$$S = - \sum_i p_i \ln p_i$$

where  $i$  labels an eigenstate of the Hamiltonian  $H$ , and  $p_i$  is the probability of that state being occupied within the ensemble. In 1988 Tsallis entertained the possibility of ensembles where the entropy instead had the form

$$S = \frac{1}{q-1} \sum_i (p_i - p_i^q)$$

where  $q$  is a real parameter. This has the property that  $S(q \rightarrow 1)$  yields the normal expression for the entropy.

If there are two independent systems  $A$  and  $B$  such that the probabilities multiply  $p_i \rightarrow p_{Ai}p_{Bj}$  and the probabilities are normalized

$$\sum_i p_{Ai} = \sum_j p_{Aj} = 1$$

then the total entropy is not the sum of the entropies of each system. Instead

$$S_{A+B} = S_A + S_B + (1 - q)S_A S_B$$

meaning that the entropy as defined is not an extensive quantity. Hence  $q$  is referred to as the nonextensivity parameter. The average of an operator must also be defined to complete the statistics of the ensemble. Tsallis and collaborators proposed three.

Type I assumes that probabilities are normalized in the usual way

$$\sum_i p_i = 1$$

as are averages of an operator  $\mathcal{O}$  that commutes with  $H$

$$\langle \mathcal{O} \rangle = \sum_i p_i \mathcal{O}_i$$

where  $\mathcal{O}_i = \langle i | \mathcal{O} | i \rangle$ . The  $|i\rangle$  are eigenstates of  $H$  such that  $H|i\rangle = E_i|i\rangle$ . The probabilities are found by maximizing the entropy at fixed energy  $E = \langle H \rangle$  subject to the normalization condition.

$$\frac{\delta}{\delta p_k} \left[ S - \beta E - \alpha \sum_i p_i \right] = 0$$

Here  $\alpha$  and  $\beta$  are Lagrange multipliers.



The resulting distribution is

$$p_k = \left[ \frac{1 + \alpha(1 - q)}{q} \right]^{\frac{1}{q-1}} \left[ 1 + \frac{(1 - q)\beta E_k}{1 + \alpha(1 - q)} \right]^{\frac{1}{q-1}}$$

There is an upper limit on the allowed values of energy  $E_k$  if  $q > 1$  and a lower limit if  $q < 1$ . The distribution is not invariant under an overall shift in energy. Normally, only energy differences matter. For independent systems  $A$  and  $B$  the energy is additive

$$E_{A+B} = E_A + E_B$$

even though the entropy is not.

Type II assumes that probabilities are normalized in the usual way

$$\sum_i p_i = 1$$

but that averages of an operator  $\mathcal{O}$  are computed in an unconventional way

$$\langle \mathcal{O} \rangle = \sum_i p_i^q \mathcal{O}_i$$

This leads to the probability distribution

$$p_k = \left[ \frac{1 + \alpha(1 - q)}{q} \right]^{\frac{1}{q-1}} [1 + (q - 1)\beta E_k]^{-\frac{1}{q-1}}$$

There is a lower limit on the allowed values of energy  $E_k$  if  $q > 1$  and an upper limit if  $q < 1$ . The distribution is not invariant under an overall shift in energy. This choice has the strange feature that  $\langle 1 \rangle \neq 1$  in general.

For independent systems  $A$  and  $B$  the energy is not additive but satisfies

$$E_{A+B} = E_A + E_B + (1 - q)(S_A E_B + S_B E_A)$$

This is rather odd since the assumption was that  $H = H_A + H_B$  with no interaction between the two systems. A redefinition of the entropy should not introduce interactions between the systems.

Type III assumes that probabilities are normalized in the usual way

$$\sum_i p_i = 1$$

but that averages of an operator  $\mathcal{O}$  are computed in another unconventional way

$$\langle \mathcal{O} \rangle = \frac{\sum_i p_i^q \mathcal{O}_i}{\sum_j p_j^q}$$

This leads to the probability distribution

$$p_k = \left[ \frac{1 + \alpha(1 - q)}{q} \right]^{\frac{1}{q-1}} \left[ 1 + \frac{(q-1)\beta(E_k - E)}{1 + (1-q)S} \right]^{-\frac{1}{q-1}}$$

This distribution is invariant under an overall shift in energy; only energy differences matter. Furthermore the energies of two independent systems add, namely

$$E_{A+B} = E_A + E_B$$

Note that the entropy  $S$  and average energy  $E$  enter explicitly and nontrivially in the probability distribution.

## Observations on the Three Types

A notable property of all three types is that in the limit  $q \rightarrow 1$  the probability becomes the Boltzmann one

$$p_i = \frac{1}{Z} e^{-\beta E_i}$$

with

$$Z = \sum_j e^{-\beta E_j}$$

Keep in mind that  $E_i$  is the total energy of the system in the quantum state  $|i\rangle$ . All of the discussion so far concerns the canonical ensemble where conserved charges, such as baryon number, electric charge, strangeness, and so on are fixed. One may introduce Lagrange multipliers, or chemical potentials, for each conserved charge in the same way as  $\beta$  is the Lagrange multiplier for the conserved energy. The calculation of the probabilities proceeds in a similar manner.

D. H. Zanette and M. A. Montemurro showed that for any observed probability distribution function  $p(X_i)$  of the variable  $X$  one can always find a constraint together with the Tsallis entropy which gives that distribution. For example, if one takes  $\mathcal{O} = \phi(H)$  with Type III then

$$p(E_k) = \left[ \frac{1 + \alpha(1 - q)}{q} \right]^{\frac{1}{q-1}} \left[ 1 + \frac{(q - 1)\beta_\phi(\phi(E_k) - \langle \phi \rangle)}{1 + (1 - q)S} \right]^{-\frac{1}{q-1}}$$

where  $\beta_\phi$  is the Lagrange multiplier. This can be solved for  $\phi(E_k)$  in terms of  $p(E_k)$ .

## Microcanonical Ensemble

The fundamental hypothesis of equilibrium statistical mechanics is that all quantum states with energies between  $E$  and  $E + \Delta$ , with  $\Delta \ll E$ , are equally likely. The number of such states is

$$\Omega(E) = \sum_n \theta(E_n - E)\theta(E + \Delta - E_n)$$

and the probabilities for  $E_n$  in this range are

$$p(E_n) = \frac{1}{\Omega(E)}$$

and zero otherwise. This is the microcanonical ensemble. The standard entropy is

$$S(E) = \ln \Omega(E)$$

whereas the Tsallis entropy is

$$S(E) = \frac{1}{q-1} \left[ 1 - \left( \frac{1}{\Omega(E)} \right)^{(q-1)} \right]$$

## Microcanonical Ensemble

We emphasize that the difference is in the definition of the entropy. The temperature associated with the standard definition of entropy is

$$\frac{1}{T} = \frac{dS}{dE} = \frac{1}{\Omega} \frac{d\Omega}{dE}$$

whereas using the Tsallis entropy

$$\frac{1}{T} = \frac{dS}{dE} = \frac{1}{\Omega^q} \frac{d\Omega}{dE}$$

Consider the canonical ensemble which follows from the microcanonical ensemble. A closed system is divided into parts  $A$  and  $B$  with total energy  $E$ . The probability for part  $A$  to be in the state  $n$  with energy  $E_{An}$  is

$$\begin{aligned} p(E_{An}) &= \frac{1}{\Omega_{A+B}(E)} \sum_m \theta(E_{An} + E_{Bm} - E) \theta(E + \Delta - E_{An} - E_{Bm}) \\ &= \frac{\Omega_B(E - E_{An})}{\Omega_{A+B}(E)} \end{aligned}$$



## Microcanonical Ensemble

Let part  $A$  be much smaller than part  $B$ . Then a Taylor series expansion yields

$$\begin{aligned} S_B(E - \langle E_A \rangle + \langle E_A \rangle - E_{A_n}) &= S_B(E - \langle E_A \rangle) + \frac{dS_B(E - \langle E_A \rangle)}{d(E - \langle E_A \rangle)} \epsilon + \dots \\ &= S_B(E - \langle E_A \rangle) + \beta \epsilon + \dots \end{aligned}$$

where  $\epsilon = \langle E_A \rangle - E_{A_n}$  and  $\beta$  is the inverse temperature associated with the major part  $B$ . With the usual definition of entropy this leads to the Boltzmann distribution

$$p(E_{A_n}) = \frac{\Omega_B(E - \langle E_A \rangle)}{\Omega_{A+B}(E)} e^{\beta \epsilon} = \frac{1}{Z} e^{-\beta(E_{A_n} - \langle E_A \rangle)}$$

Whereas with the Tsallis definition of the entropy it leads to

$$\begin{aligned} p(E_{A_n}) &= \frac{1}{\Omega_{A+B}(E)} [1 + (1 - q)S_A]^{-\frac{1}{q-1}} \left[ 1 + \frac{(q-1)\beta(E_{A_n} - \langle E_A \rangle)}{1 + (1 - q)S_A} \right]^{-\frac{1}{q-1}} \\ &= \frac{1}{Z} \left[ 1 + \frac{(q-1)\beta(E_{A_n} - \langle E_A \rangle)}{1 + (1 - q)S_A} \right]^{-\frac{1}{q-1}} \end{aligned}$$

The latter is exactly the Tsallis Type III distribution. It is the one consistent with the fundamental hypothesis of equilibrium statistical mechanics.

## Single Particle Distributions

Consider one fermion degree of freedom with Type III. There is one quantum state which can be unoccupied with zero energy, or it can have one occupant with energy  $\omega$ . The probabilities are

$$\begin{aligned}p_0 &= Z^{-1} [1 + (1 - q)(S + \beta E)]^{-\frac{1}{q-1}} \\p_1 &= Z^{-1} [1 + (1 - q)(S + \beta E - \beta\omega)]^{-\frac{1}{q-1}}\end{aligned}$$

The number operator  $\hat{N}$  commutes with the Hamiltonian  $H = \hat{N}\omega$ . Its average is

$$\langle \hat{N} \rangle = \frac{p_1^q}{p_0^q + p_1^q}$$

and not  $p_1/(p_0 + p_1)$  as one would have expected.

Explicitly

$$\langle \hat{N} \rangle = \frac{1}{[1 + (q - 1)\beta^*\omega]^{\frac{q}{q-1}} + 1}$$

and

$$E \equiv \langle E \rangle = \frac{\omega}{[1 + (q - 1)\beta^*\omega]^{\frac{q}{q-1}} + 1}$$

where

$$\beta^* = \frac{\beta}{1 + (1 - q)(S + \beta E)}$$

There is a self-consistency condition on these results. The classical limit is when  $p_1 \ll p_0$  in which case

$$\langle \hat{N} \rangle \rightarrow [1 + (q - 1)\beta^*\omega]^{-\frac{q}{q-1}}$$

and  $\beta^* \rightarrow \beta$ . This limit corresponds to  $\beta\omega \gg 1$  and  $q > 1$ .

## Single Particle Distributions

Next consider an arbitrarily large number of independent states with a set of quantum numbers and/or momenta labeled by  $\alpha$ . The Hamiltonian is  $H = \sum_{\alpha} \hat{N}_{\alpha} \omega_{\alpha}$  with number operators  $\hat{N}_{\alpha}$ . The probabilities for these independent states factorize and are written as

$$\begin{aligned} p_{\alpha 0} &= Z_{\alpha}^{-1} [1 + (1 - q)(S_{\alpha} + \beta E_{\alpha})]^{-\frac{1}{q-1}} \\ p_{\alpha 1} &= Z_{\alpha}^{-1} [1 + (1 - q)(S_{\alpha} + \beta E_{\alpha} - \beta \omega_{\alpha})]^{-\frac{1}{q-1}} \end{aligned}$$

with the averages for each state being

$$\begin{aligned} N_{\alpha} &= \frac{1}{[1 + (q - 1)\beta_{\alpha}^* \omega_{\alpha}]^{\frac{q}{q-1}} + 1} \\ E_{\alpha} &= \frac{\omega_{\alpha}}{[1 + (q - 1)\beta_{\alpha}^* \omega_{\alpha}]^{\frac{q}{q-1}} + 1} \end{aligned}$$

where

$$\beta_{\alpha}^* = \frac{\beta}{1 + (1 - q)(S_{\alpha} + \beta E_{\alpha})}$$

## Single Particle Distributions

The total energy is

$$E = \sum_{\alpha} E_{\alpha}$$

and the total number of particles is

$$N = \sum_{\alpha} N_{\alpha}$$

Note that there is a common temperature  $T = 1/\beta$  but that  $\beta_{\alpha}^*$  is state-dependent. For a gas of particles

$$\omega_{\alpha} \rightarrow \omega(k)$$

where  $k$  is the momentum and

$$\sum_{\alpha} \rightarrow (2s + 1) \int \frac{d^3x d^3k}{(2\pi)^3}$$

with  $s$  the spin of the fermions.

## Single Particle Distributions

Any number of bosons may occupy a given quantum state. This means that for bosons

$$p_n = Z^{-1} [1 + (1 - q)(S + \beta E - n\beta\omega)]^{-\frac{1}{q-1}}$$

The average of the number operator is

$$\langle \hat{N} \rangle = \frac{\sum_{n=1}^{\infty} n p_n^q}{\sum_{m=0}^{\infty} p_m^q} = \frac{\sum_n n [1 + (q - 1)\beta^* \omega n]^{-\frac{q}{q-1}}}{\sum_m [1 + (q - 1)\beta^* \omega m]^{-\frac{q}{q-1}}}$$

where  $\beta^*$  is again calculated self-consistently. This expression cannot be evaluated in closed form for arbitrary  $q$ . Several methods exist for evaluating it numerically. In the limit of classical statistics (small occupation probabilities) it reduces to the same expression as the fermions did, and in the limit  $q \rightarrow 1$  it reduces to the usual Bose-Einstein distribution

$$\langle \hat{N} \rangle = \frac{1}{e^{\beta\omega} - 1}$$

One might have guessed that it was equal to

$$\frac{1}{[1 + (q - 1)\beta^*\omega]^{\frac{q}{q-1}} - 1}$$

but it is not. The extension to a gas of bosons follows in the same way as for fermions.

It is worth pointing here out that Tsallis's definition of entropy, which leads to so-called "Tsallis statistics", was not expected to describe a system of noninteracting particles even according to him.

## Boltzmann Equation

Consider the Boltzmann equation for the reaction  $a + b \rightarrow c + d$  and its inverse.

$$\frac{df_a}{dt} = \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} \left\{ \frac{1}{1 + \delta_{cd}} W(c+d \rightarrow a+b) f_c f_d (1 + (-1)^{2s_a} f_a) (1 + (-1)^{2s_b} f_b) \right. \\ \left. - \frac{1}{1 + \delta_{ab}} W(a + b \rightarrow c + d) f_a f_b (1 + (-1)^{2s_c} f_c) (1 + (-1)^{2s_d} f_d) \right\}$$

This includes Pauli suppression for fermions and Bose enhancement for bosons in the final state. The  $s_i$  is the spin of particle  $i$ . The coefficients take into account the possibility that the particles in the initial state are identical. There is a gain term and a loss term. Microscopic physics originating in quantum mechanics or quantum field theory says that

$$\frac{1}{1 + \delta_{ab}} W(a + b \rightarrow c + d) = \frac{1}{1 + \delta_{cd}} W(c + d \rightarrow a + b)$$

The  $W$ 's are proportional to the square of a dimensional scattering amplitude  $|\mathcal{M}|^2$  which in turn is proportional to the differential cross section  $d\sigma/d\Omega$  in the center of momentum frame. They do not depend on the distribution functions  $f$ . In addition the  $W$ 's are proportional to an energy-momentum conserving  $\delta$  function  $\delta(p_a + p_b - p_c - p_d)$ .



## Boltzmann Equation

In perturbation theory, relevant to the Boltzmann equation, the equilibrium distributions are

$$f = \frac{1}{\exp(\beta\omega) - (-1)^{2s}}$$

In equilibrium  $df_a/dt = 0$ . This leads to the condition  $\omega_a + \omega_b = \omega_c + \omega_d$ , in other words energy conservation. This is also true in the limit of classical statistics.

On the other hand, consider using one of the Tsallis distributions. We choose the limit of classical statistics for simplicity. It is

$$f = [1 + (q - 1)\beta\omega]^{-\frac{q}{q-1}}$$

In order that  $df_a/dt = 0$  requires

$$\omega_a + \omega_b + (q - 1)\beta\omega_a\omega_b = \omega_c + \omega_d + (q - 1)\beta\omega_c\omega_d$$

This is inconsistent with energy conservation and the  $\delta$  function constraint unless  $q = 1$  which is conventional statistical mechanics. One can only imagine what happens when a collision involves  $m$  particles in the initial state and  $n$  particles in the final state. Thus the aforementioned Tsallis distribution cannot be time independent.

## Boltzmann Equation

To address this problem it has been *postulated* by A. Lavagno that the Boltzmann equation be modified so that the factor  $f_a f_b$  (with classical statistics) be replaced with

$$h_q[f_a, f_b] \equiv (f_a^{1-q} + f_b^{1-q} - 1)^{\frac{1}{1-q}}$$

and that the equilibrium distribution function be

$$f_{\text{eq}} = [1 + (q - 1)\beta\omega]^{\frac{1}{1-q}}$$

By doing this it is easy to see that  $\omega_a + \omega_b = \omega_c + \omega_d$ . There is a mention that the “molecular chaos hypothesis” used to derive the original Boltzmann equation is not valid for Tsallis statistics but no proof or quantitative reasoning was given. It would seem that this postulated modification is not consistent with the definition and physical interpretation of the  $W$ 's.

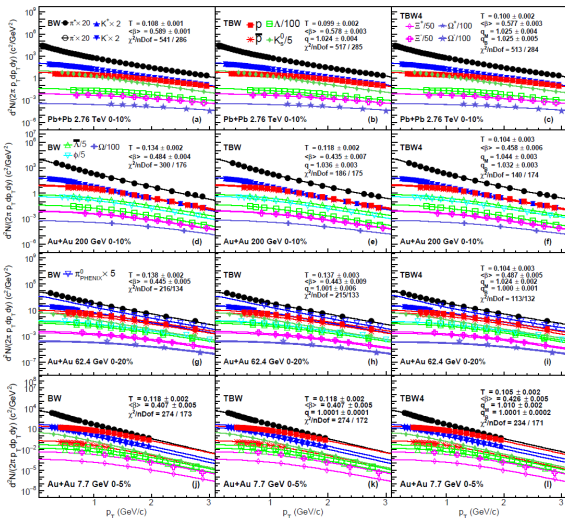
Tsallis-like or Tsallis-inspired distributions have been used many times to parametrize particle spectra in high-energy proton-proton collisions. This has been done by the STAR and PHENIX collaborations at RHIC and the ALICE and CMS collaborations at the LHC. Generally they are of the form

$$\frac{dN}{dyd^2p_T} \propto \frac{dN}{dy} \left(1 + \frac{m_T - m}{nT}\right)^{-n} \quad (1)$$

where  $m_T = \sqrt{m^2 + p_T^2}$  is the transverse mass and  $p_T$  is the transverse momentum. This is approximately  $\exp(-m_T/T)$  at low  $p_T$  and  $p_T^{-n}$  at high  $p_T$ . If one makes the identification  $n = 1/(q - 1)$  then one would associate  $T$  with the temperature, whereas if one makes the identification  $n = q/(q - 1)$  then one would associate  $qT$  with the temperature. Comparisons to data taken at RHIC for d + Au, Cu + Cu, Au + Au and at the LHC for p + Pb and Pb + Pb show that  $n$  depends on the colliding systems, beam energy, and particle species. J. Chen *et al.* included hydrodynamic flow into comparisons with Au + Au and Pb + Pb data and associated  $n$  with  $1/(q - 1)$ . The results show that  $q$  also depends on the centrality, or impact parameter. Typically  $q$  ranges between 1.01 and 1.15, which is a very wide range in  $n$ . This implies that  $q$  is not a fundamental or intrinsic quantity to be interpreted in terms of new statistics.

The underlying physics in these collisions is QCD. Low transverse momentum hadrons are produced with the normal kinetic and chemical equilibrium distributions boosted by collective hydrodynamic flow. They essentially exhibit exponential decay in energy. High transverse momentum hadrons exhibit power-law decay because of the asymptotic freedom of QCD. The parton model for scattering of point particles predicts  $n = 4$  and therefore either  $q = 1.25$  or  $q = 4/3$ . However, scale breaking and realistic parton distribution functions in the projectile and target increase  $n$  significantly and so numerically  $q$  turns out to be much closer to 1 for hadrons, although  $n \sim 4.5$  to 5.5 for jets according to C.-Y. Wong *et al.* Essentially the Tsallis distributions have an extra parameter,  $q$ , which controls the transition from exponential to power-law behavior. It is an efficient parametrization of data and/or perturbative QCD based parton models, but should not be considered a fundamental constant.

# High Energy Collision Phenomenology



Blast wave model fits.

# High Energy Collision Phenomenology

system	$\sqrt{s_{NN}}$ (TeV)	centrality	$\langle\beta\rangle$	$T$ (MeV)	$q_M$	$q_B$	$\chi^2/nDoF$
Pb + Pb	2.76 ( $\pi, K, p$ )	0 – 5%	$0.590 \pm 0.004$	$92 \pm 2$	$1.024 \pm 0.005$	$1.026 \pm 0.006$	246/212
		5 – 10%	$0.588 \pm 0.004$	$91 \pm 2$	$1.030 \pm 0.005$	$1.028 \pm 0.006$	247/212
		10 – 20%	$0.584 \pm 0.004$	$90 \pm 2$	$1.035 \pm 0.005$	$1.029 \pm 0.006$	225/212
		20 – 30%	$0.574 \pm 0.005$	$88 \pm 2$	$1.046 \pm 0.005$	$1.034 \pm 0.006$	191/212
		30 – 40%	$0.557 \pm 0.005$	$84 \pm 2$	$1.061 \pm 0.004$	$1.044 \pm 0.005$	162/212
		40 – 50%	$0.525 \pm 0.006$	$80 \pm 2$	$1.079 \pm 0.003$	$1.060 \pm 0.004$	127/212
		50 – 60%	$0.485 \pm 0.007$	$78 \pm 2$	$1.094 \pm 0.003$	$1.073 \pm 0.004$	117/212
		60 – 70%	$0.441 \pm 0.008$	$74 \pm 2$	$1.107 \pm 0.002$	$1.082 \pm 0.003$	105/212
		70 – 80%	$0.40 \pm 0.01$	$71 \pm 2$	$1.117 \pm 0.002$	$1.088 \pm 0.003$	94/212
		80 – 90%	$0.33 \pm 0.02$	$69 \pm 2$	$1.124 \pm 0.002$	$1.093 \pm 0.003$	85/212
Pb + Pb	2.76 (non – strange)	0 – 10%	$0.601 \pm 0.004$	$80 \pm 3$	$1.037 \pm 0.005$	$1.018 \pm 0.007$	197/142
		10 – 20%	$0.601 \pm 0.004$	$76 \pm 3$	$1.045 \pm 0.005$	$1.015 \pm 0.006$	170/142
		20 – 40%	$0.598 \pm 0.004$	$69 \pm 3$	$1.060 \pm 0.005$	$1.014 \pm 0.006$	106/142
		40 – 60%	$0.559 \pm 0.007$	$64 \pm 3$	$1.090 \pm 0.003$	$1.043 \pm 0.005$	57/142
		60 – 80%	$0.47 \pm 0.01$	$65 \pm 3$	$1.113 \pm 0.002$	$1.075 \pm 0.004$	60/142
Pb + Pb	2.76 (all)	0 – 10%	$0.577 \pm 0.003$	$100 \pm 2$	$1.025 \pm 0.004$	$1.025 \pm 0.005$	513/284
		10 – 20%	$0.570 \pm 0.004$	$98 \pm 2$	$1.034 \pm 0.004$	$1.028 \pm 0.005$	462/284
		20 – 40%	$0.549 \pm 0.004$	$94 \pm 2$	$1.051 \pm 0.004$	$1.039 \pm 0.004$	439/284
		40 – 60%	$0.498 \pm 0.005$	$86 \pm 2$	$1.081 \pm 0.003$	$1.062 \pm 0.003$	273/284
		60 – 80%	$0.421 \pm 0.008$	$77 \pm 2$	$1.108 \pm 0.002$	$1.082 \pm 0.003$	162/282
Pb + Pb	5.02 ( $\pi, K, p$ )	0 – 5%	$0.596 \pm 0.003$	$99 \pm 2$	$1.021 \pm 0.005$	$1.041 \pm 0.006$	274/89
		5 – 10%	$0.596 \pm 0.003$	$95 \pm 2$	$1.028 \pm 0.005$	$1.040 \pm 0.005$	286/89
		10 – 20%	$0.591 \pm 0.003$	$96 \pm 2$	$1.031 \pm 0.005$	$1.040 \pm 0.005$	306/89
		20 – 30%	$0.580 \pm 0.004$	$95 \pm 2$	$1.042 \pm 0.004$	$1.044 \pm 0.005$	267/89
		30 – 40%	$0.565 \pm 0.004$	$91 \pm 2$	$1.058 \pm 0.004$	$1.050 \pm 0.004$	207/89
		40 – 50%	$0.535 \pm 0.005$	$86 \pm 2$	$1.077 \pm 0.003$	$1.064 \pm 0.004$	156/89
		50 – 60%	$0.492 \pm 0.006$	$83 \pm 2$	$1.094 \pm 0.003$	$1.078 \pm 0.003$	128/89
		60 – 70%	$0.447 \pm 0.008$	$75 \pm 2$	$1.112 \pm 0.003$	$1.089 \pm 0.003$	69/89
		70 – 80%	$0.38 \pm 0.01$	$73 \pm 2$	$1.124 \pm 0.002$	$1.099 \pm 0.003$	54/89
		80 – 90%	$0.32 \pm 0.02$	$72 \pm 2$	$1.130 \pm 0.002$	$1.104 \pm 0.003$	51/89

Average flow velocity and temperature. Power-law tails very sensitive to  $1/(q - 1)$ .

## Conclusion

Tsallis and collaborators proposed three types of nonextensive statistical mechanics. They all assumed the same form of a nonextensive entropy but differed in how averages of conserved quantities were defined. It turns out that only Type III is consistent with the fundamental hypothesis of equilibrium statistical mechanics. Everything else follows without ambiguity. Notably there are nontrivial self-consistency conditions to be solved. These self-consistency conditions must be solved mode by mode for single particle distribution functions. In the limit of small occupation probabilities (classical statistics) these conditions simplify and basically are satisfied by the normalization of probabilities. With Type III energies are additive. However, the single particle distribution functions are not time independent solutions of Boltzmann's equation. They are time independent solutions of a modified Boltzmann equation with fractional exponents of the distributions appearing, but this does not appear to be physically consistent with the microscopic derivation of scattering amplitudes. Tsallis-inspired single particle distribution functions provide an efficient parametrization of high energy collision data but have not been derived from QCD. The parameter  $q$  encapsulates a lot of complicated physics in these collisions but is not a fundamental quantity in any sense.