Hydrodynamization and attractors in rapidly expanding fluids

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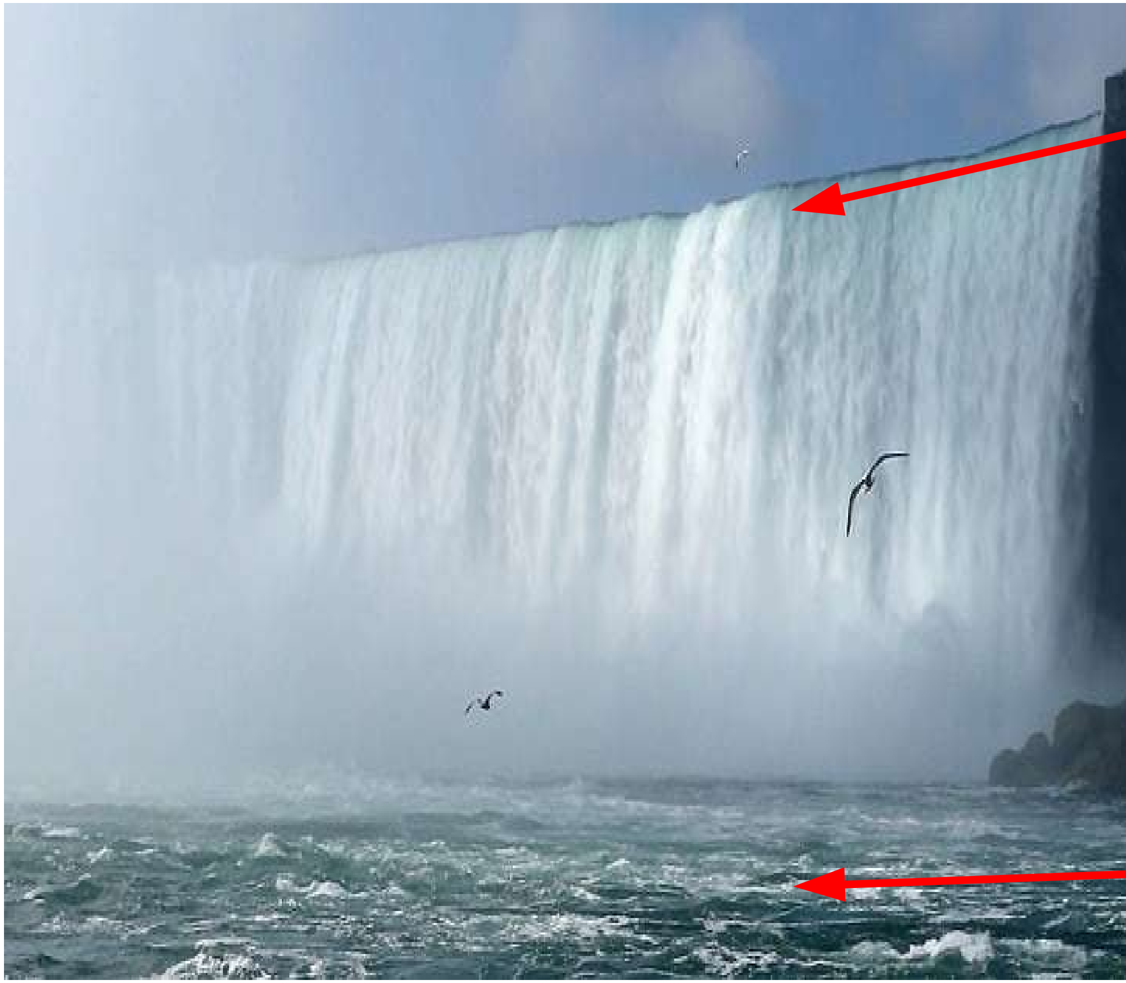
Special Theoretical Physics Seminar

NC STATE UNIVERSITY

BEST COLLABORATION



Far-from-equilibrium

?

Equilibrium

Far-from-equilibrium

↓ **?**

Hydrodynamics

Today: Attractors in kinetic theory and fluid dynamics out of equilibrium

Far-from-equilibrium

?

Hydrodynamics

Hydrodynamics:
one theory to rule them all

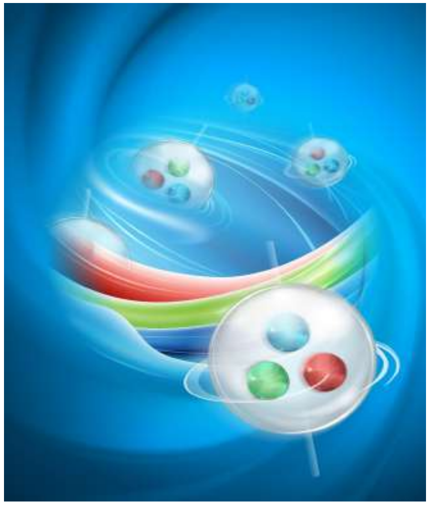
Water

Ketchup

Olive oil

Coffee

Quark-Gluon Plasma



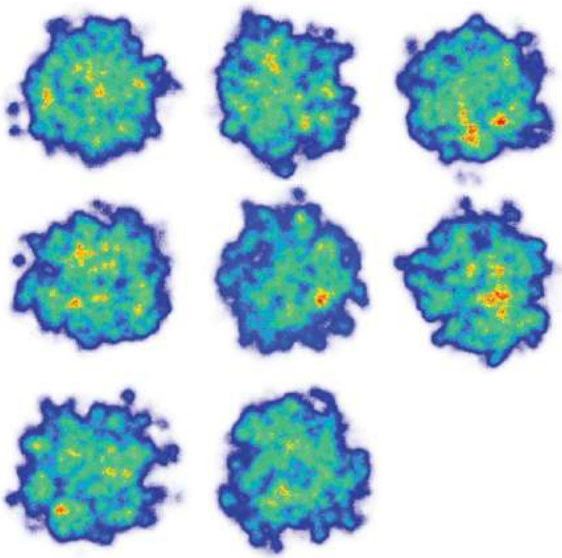
$T \sim 10^{12}\,K$

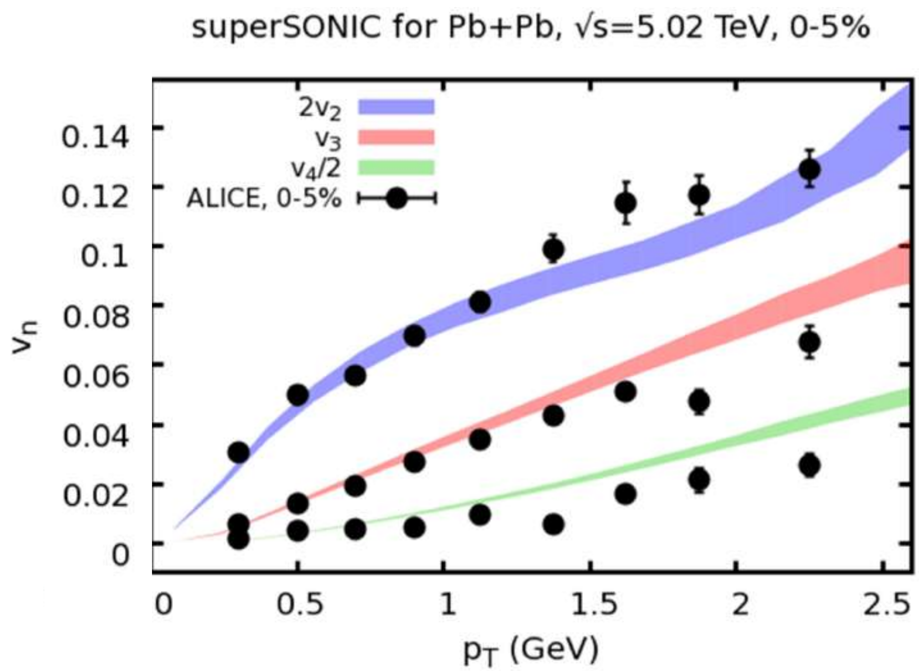
New discoveries:
Nearly
Perfect Fluids

Ultracold atoms

$T \sim 10^{-7}\,K$

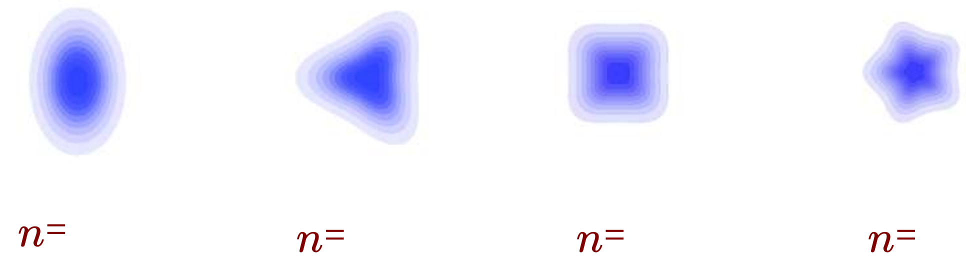
Fluidity in Heavy Ions





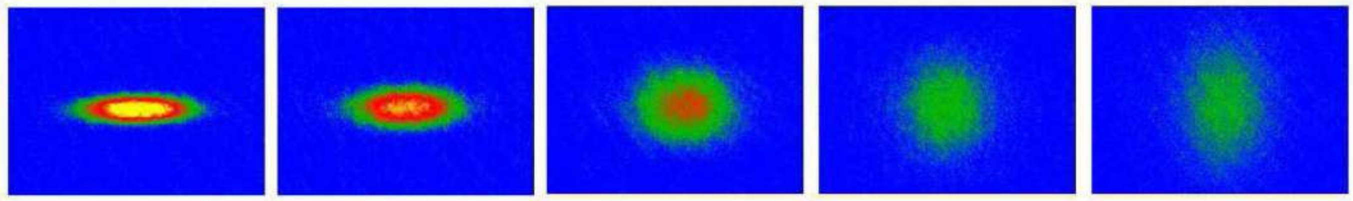
Weller & Romatschke (2017)

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right)$$

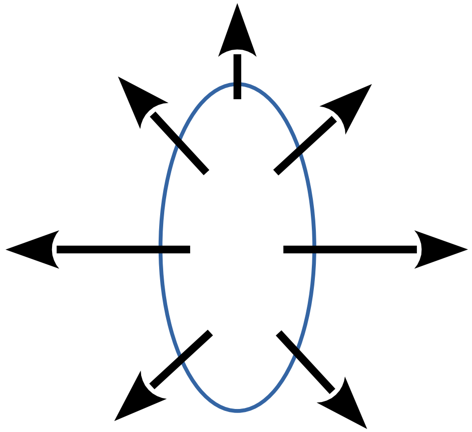


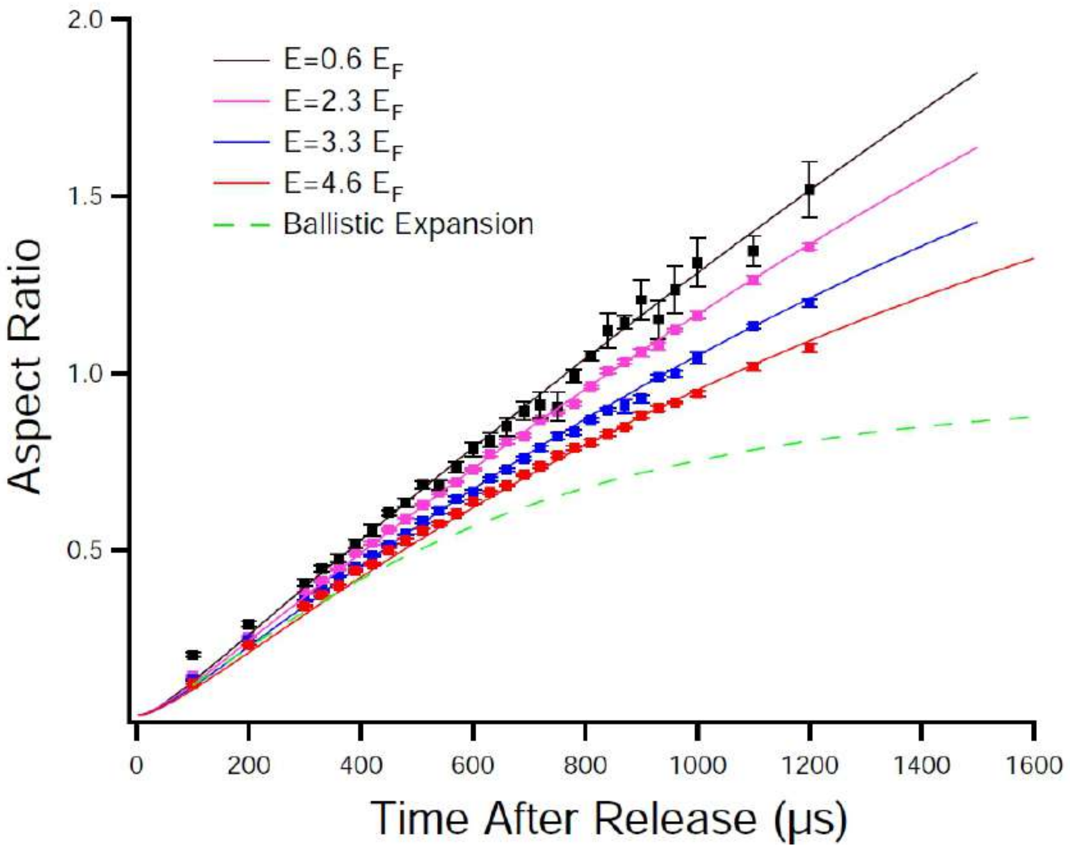
v_n provides information of the initial spatial geometry of the collision

Fluidity in Cold Atoms



Aspect ratio measures pressures anisotropies



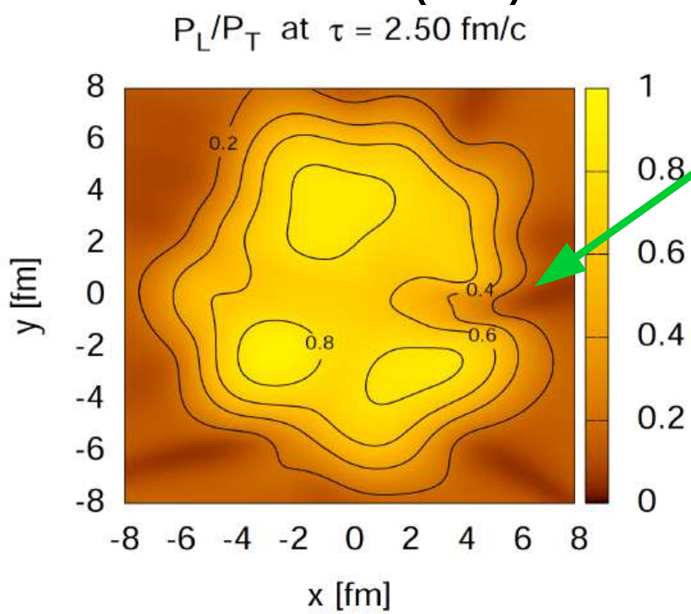


Cao et. al (2010)

Size of the hydrodynamical gradients

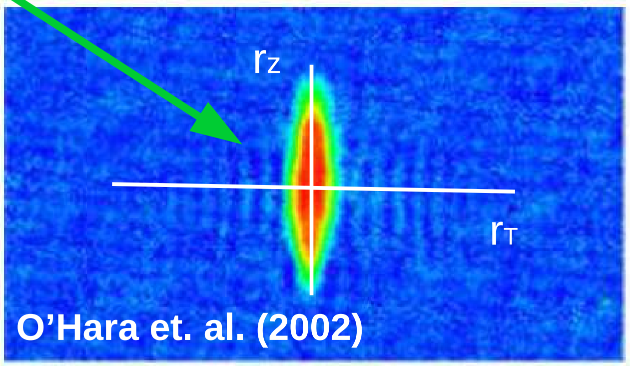
Heavy Ion Collisions

Martinez et. al. (2012)



Pressure anisotropies are not small

Cold Atoms



Paradox:

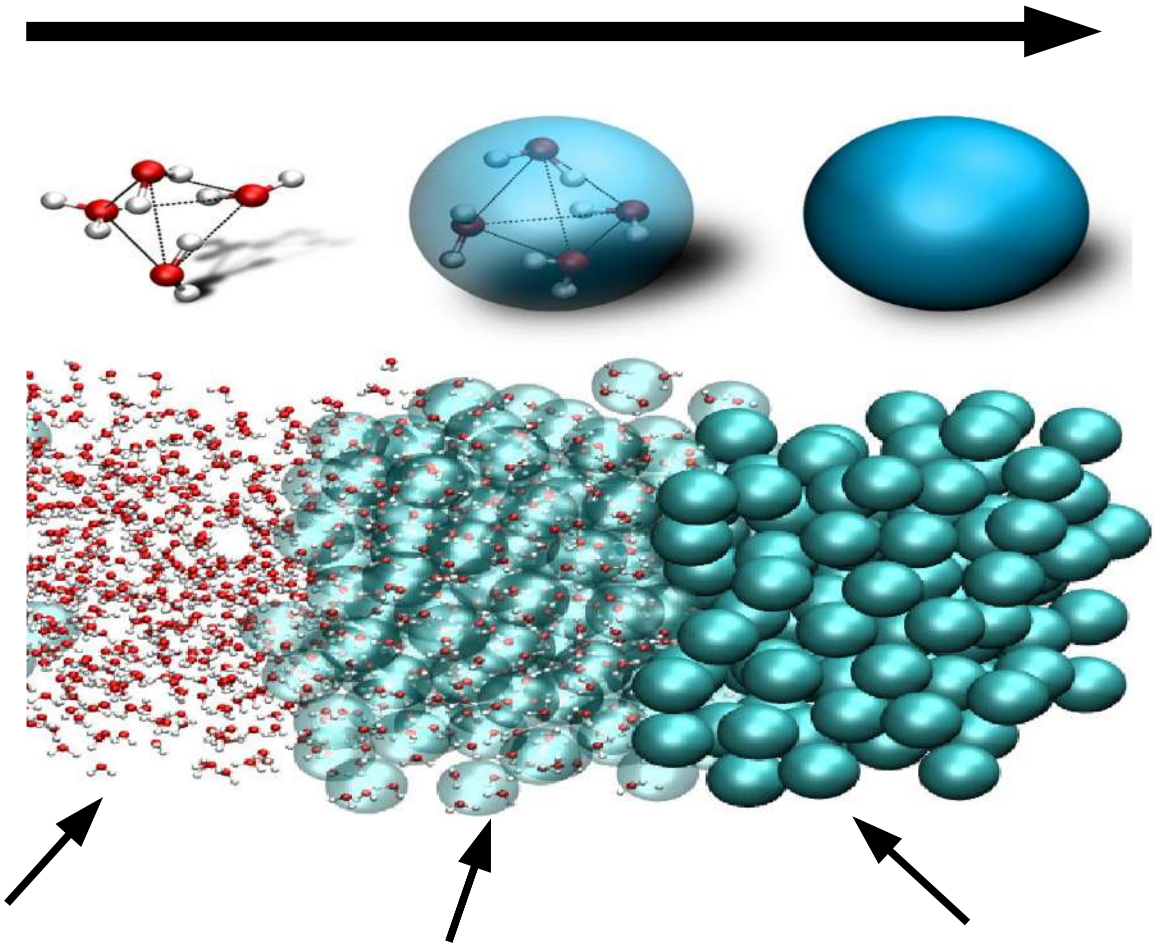
Hydrodynamics provides a good description despite large gradients…. Why?

Introductory textbook: Hydrodynamics works as far as there is a hierarchy of scales

$$Kn = \frac{l_{micro}}{L_{macro}} \ll 1$$

Hydro as an effective theory

Coarse-grained procedure reduces # of degrees of freedom



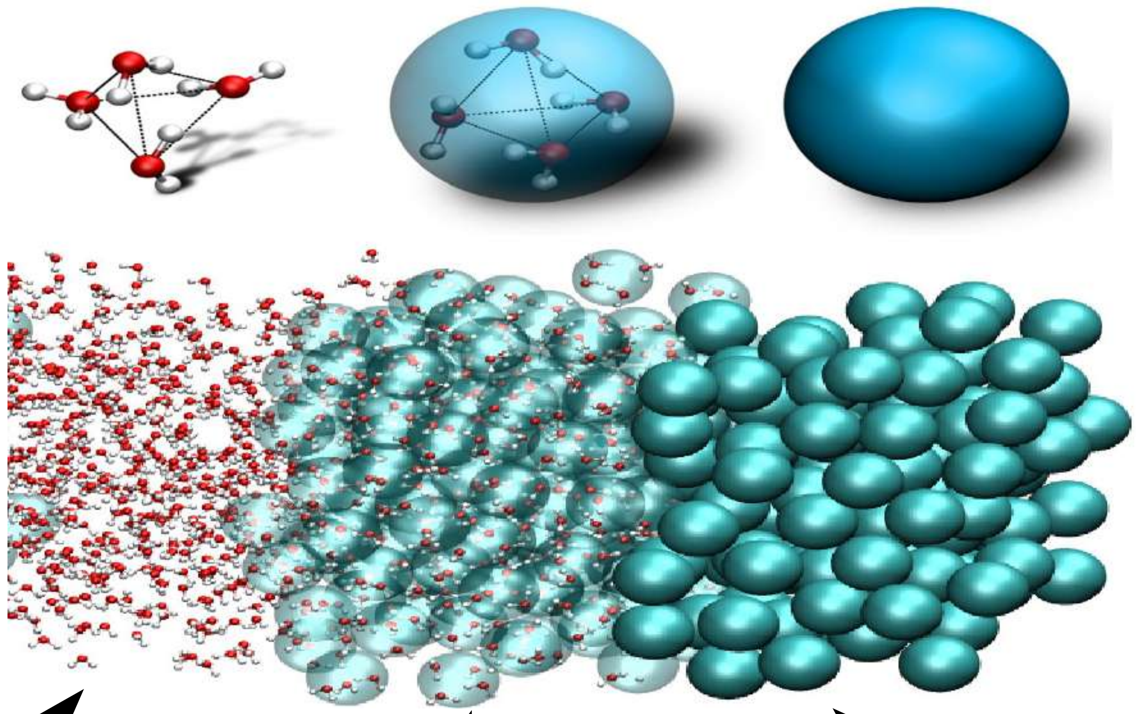
Microscopic:
10^{23} particles

Mesoscopic:
$10^7 - 10^9$ particles

Continuum:
$T, \mu, \mu_i, \epsilon, n, p, \ldots$

Hydro as an effective theory

How does hydrodynamical limit emerges from an underlying microscopic theory?



Microscopic:

10^{23} particles

Mesoscopic:

$10^7 - 10^9$ particles

Continuum:

$T, \mu, \mu_i, \epsilon, n, p, ...$

Kinetic theory: Boltzmann equation

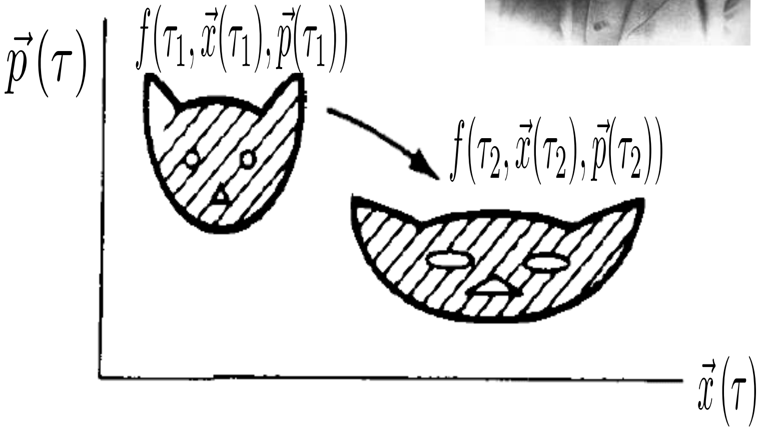
Microscopic dynamics is encoded in the distribution function f(t,**x**,**p**)

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = -\mathcal{C}[f]$$

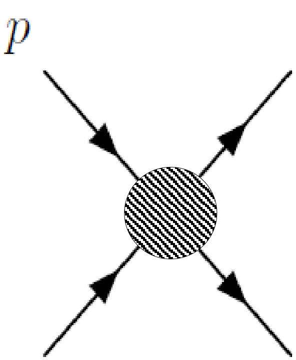
Diffusion

External Force

Particle imbalance

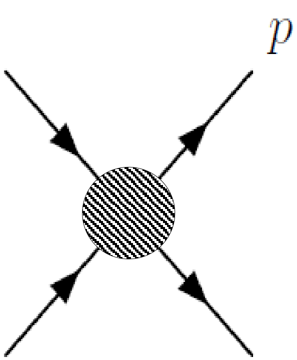


$$\mathcal{C}[f] =$$



Gain

\-



Lose

Asymptotics in the Boltzmann equation

Usually the distribution function is expanded as series in Kn, i.e.,

$$f(x^\mu, p) = \sum_{k=0}^{\infty} (Kn)^k f_k(x^\mu, p)$$

Macroscopic quantities are simply averages , e.g.,

$$T^{\mu\nu} = \int_{\mathbf{p}} p^\mu p^\nu f(x^\mu, \mathbf{p}) \quad \Rightarrow \quad T^{\mu\nu} = \sum_{k=0}^{\infty} (Kn)^k T_k^{\mu\nu}$$

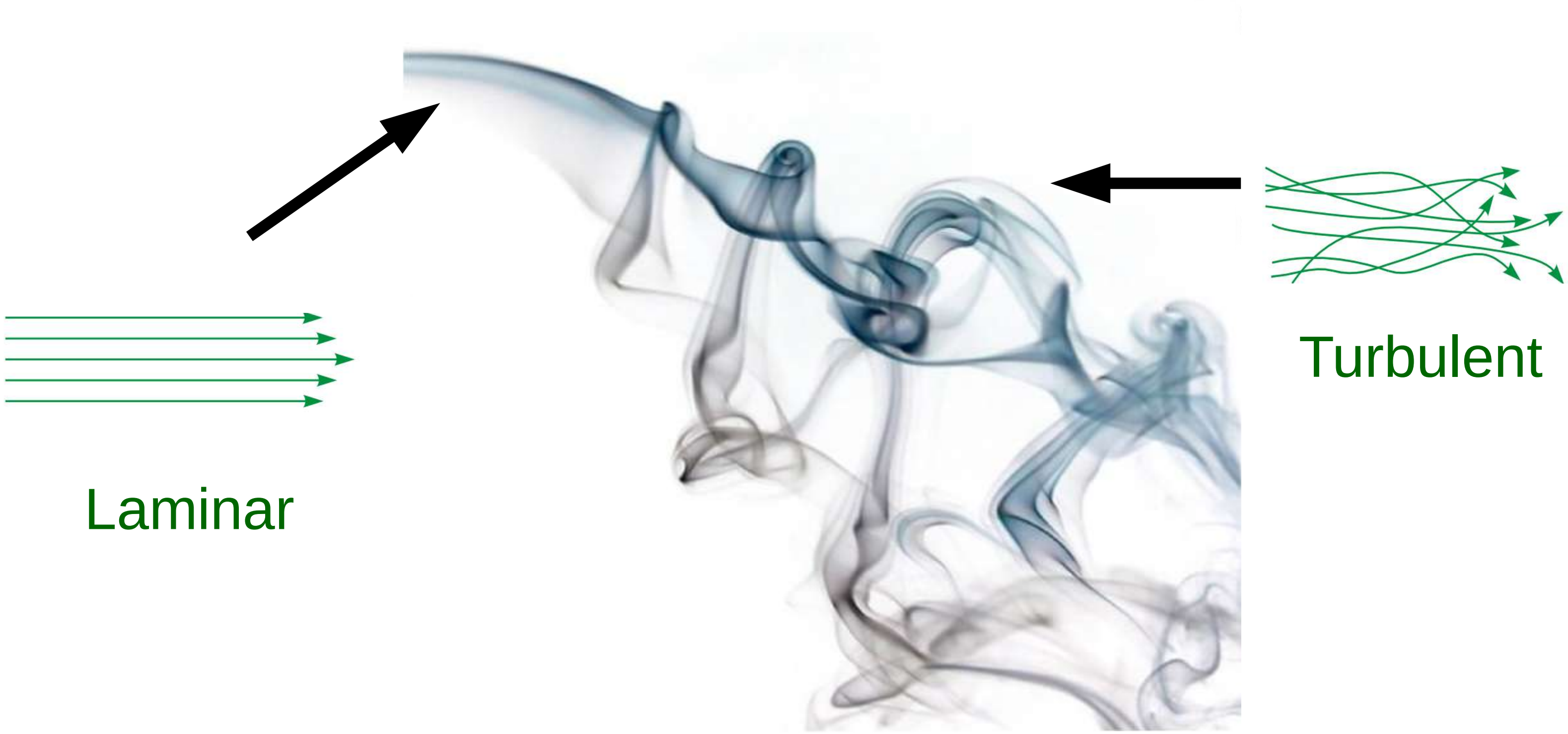
$$T_0^{\mu\nu} = (\epsilon + p(\epsilon))\, u^\mu u^\nu + p(\epsilon) g^{\mu\nu}$$ → Ideal fluid $\mathcal{O}(\mathrm{Kn}^0)$

$$T_1^{\mu\nu} = -\eta\, \sigma^{\mu\nu}$$ → $\mathcal{O}(\mathrm{Kn})$: Navier-Stokes

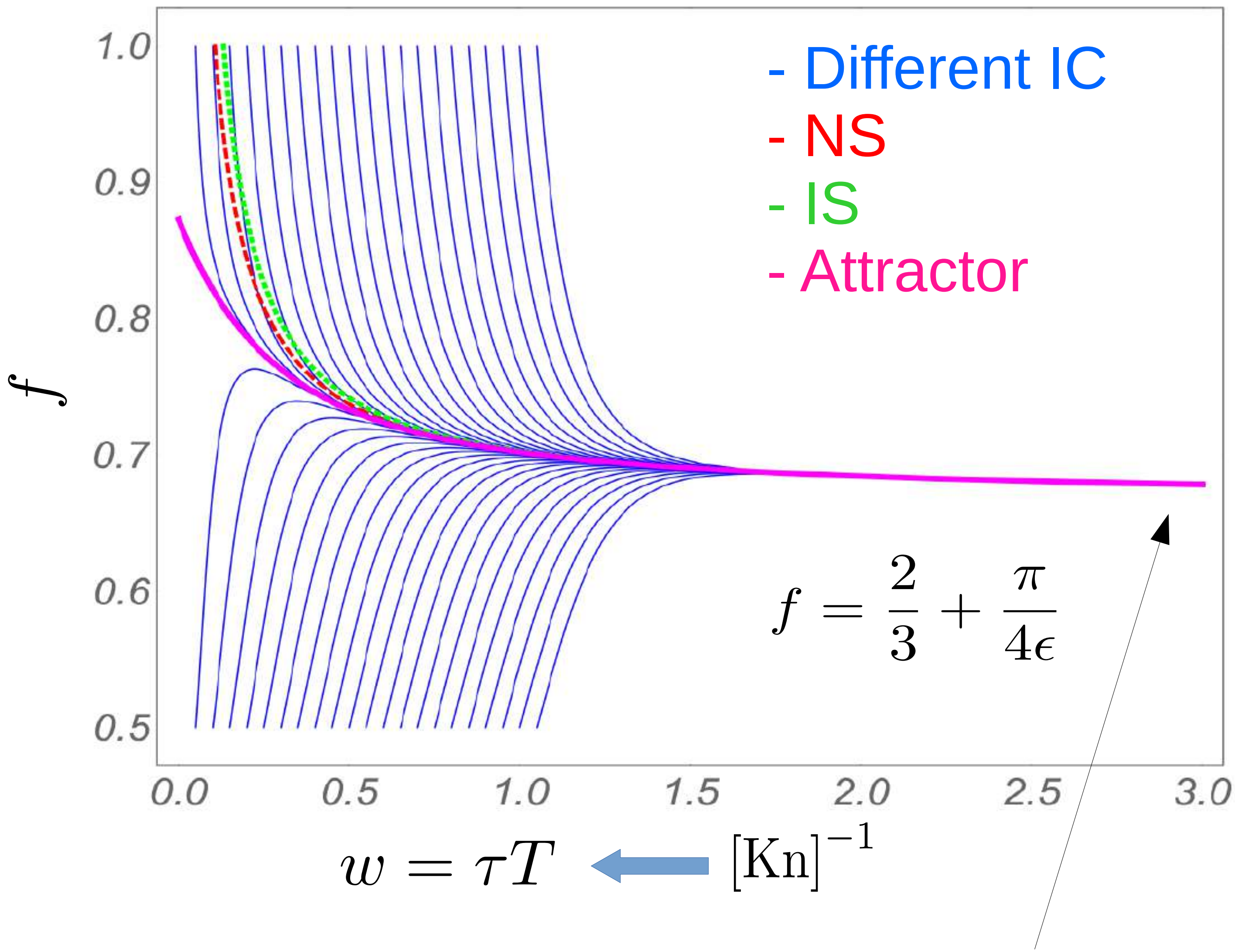
$$T_2^{\mu\nu}$$ → $\mathcal{O}(\mathrm{Kn}^2)$: IS, etc

Warning

$$Kn \sim \frac{l}{L} \sim \lambda_{mfp} \vec{\nabla} \cdot \vec{v} \sim \mathcal{O}(1)$$



Attractor in hydrodynamics



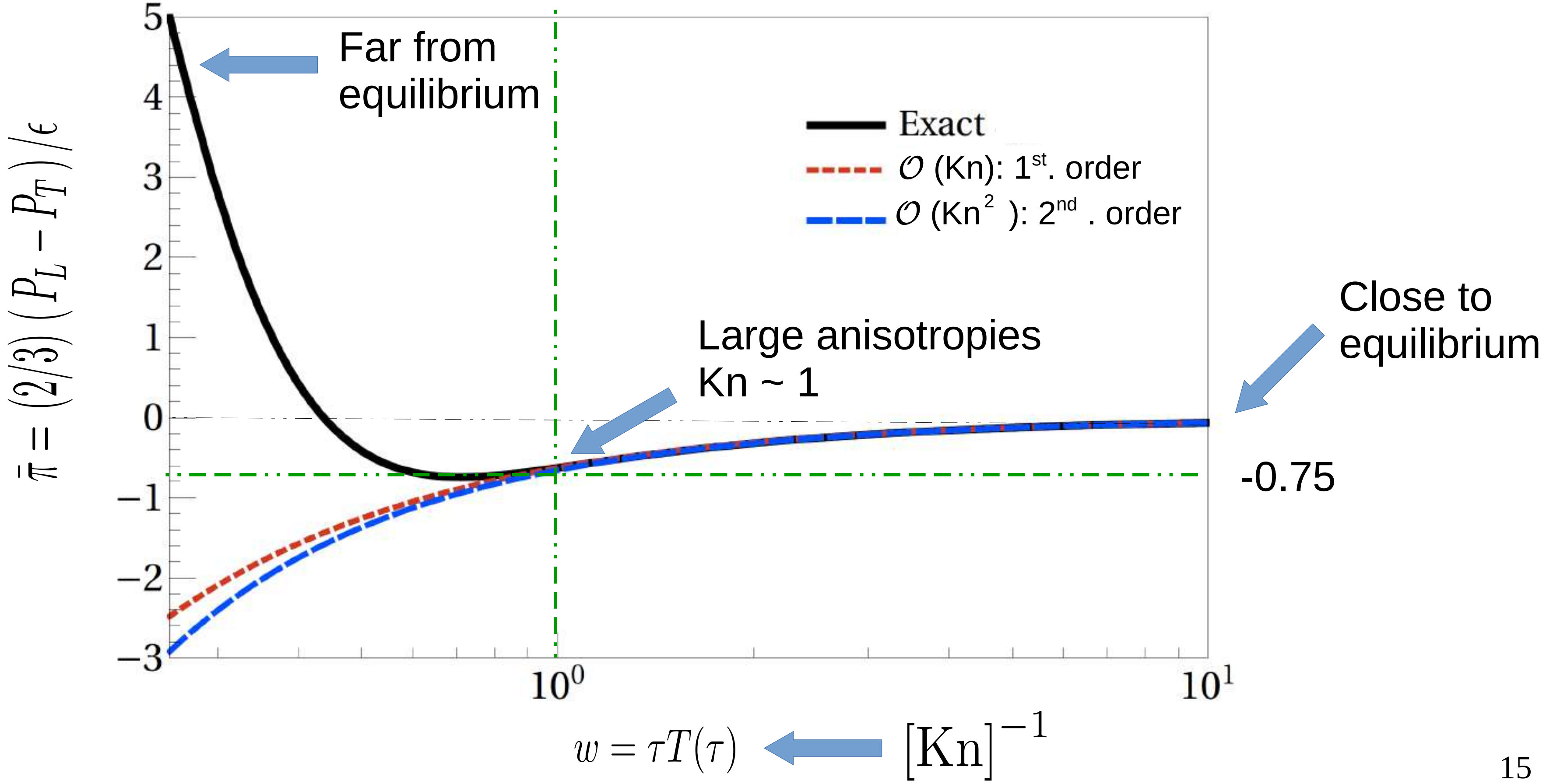
Same late time behavior independent of the IC!!!

Heller and Spalinski (2015)

Divergence of the late-time perturbative expansion

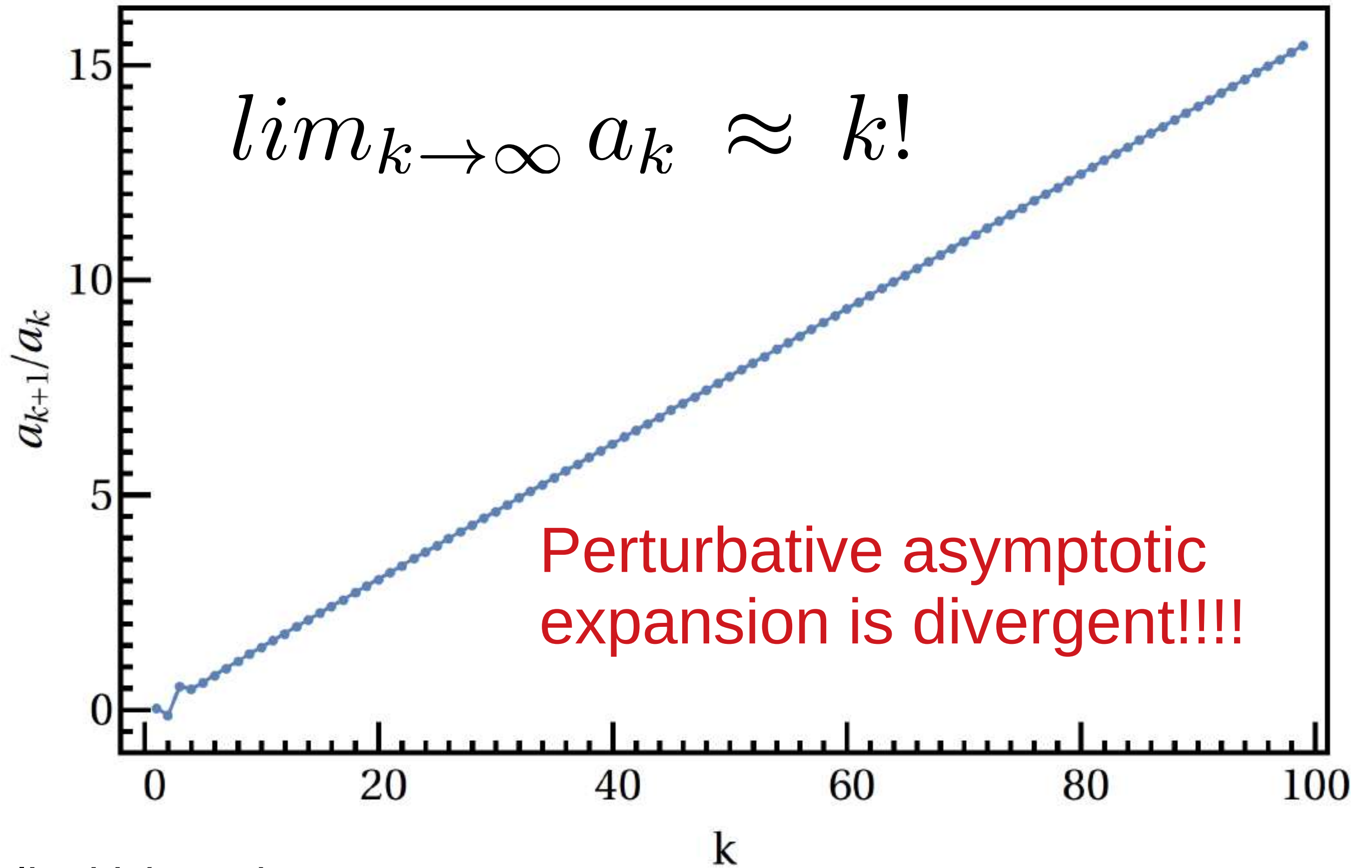
Heller & Spalinski:

$$\bar{\pi} = \sum_{k=1}^{\infty} a_k \, [\mathrm{Kn}]^k$$



Divergence of perturbative series

$$\bar{\pi} = \sum_{k=1}^{\infty} a_k \, [\mathrm{Kn}]^k$$



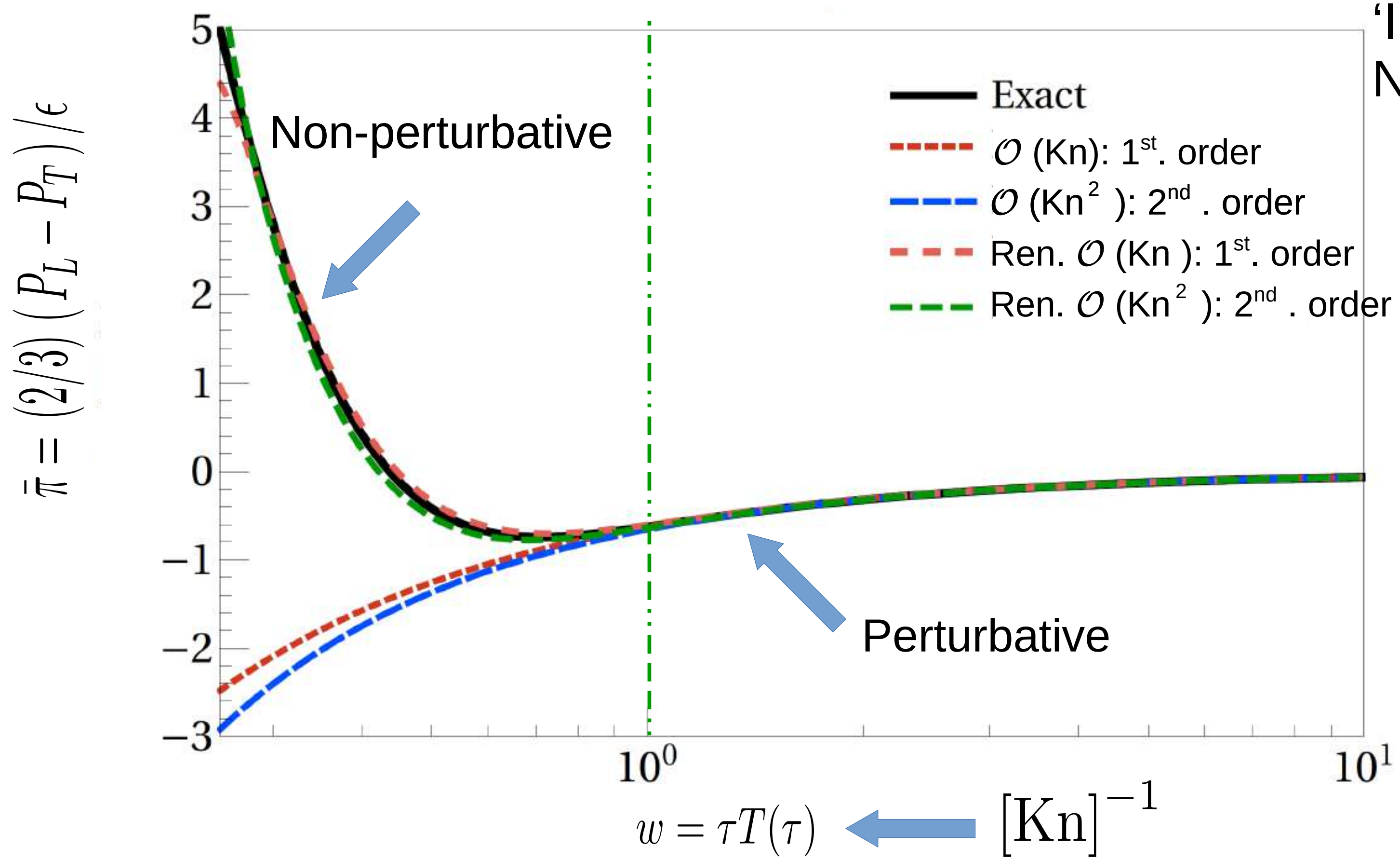
Heller and Spalinski (2015)

Resurgence and transseries

Asymptotic expansion → Transseries solutions
Costin (1998)

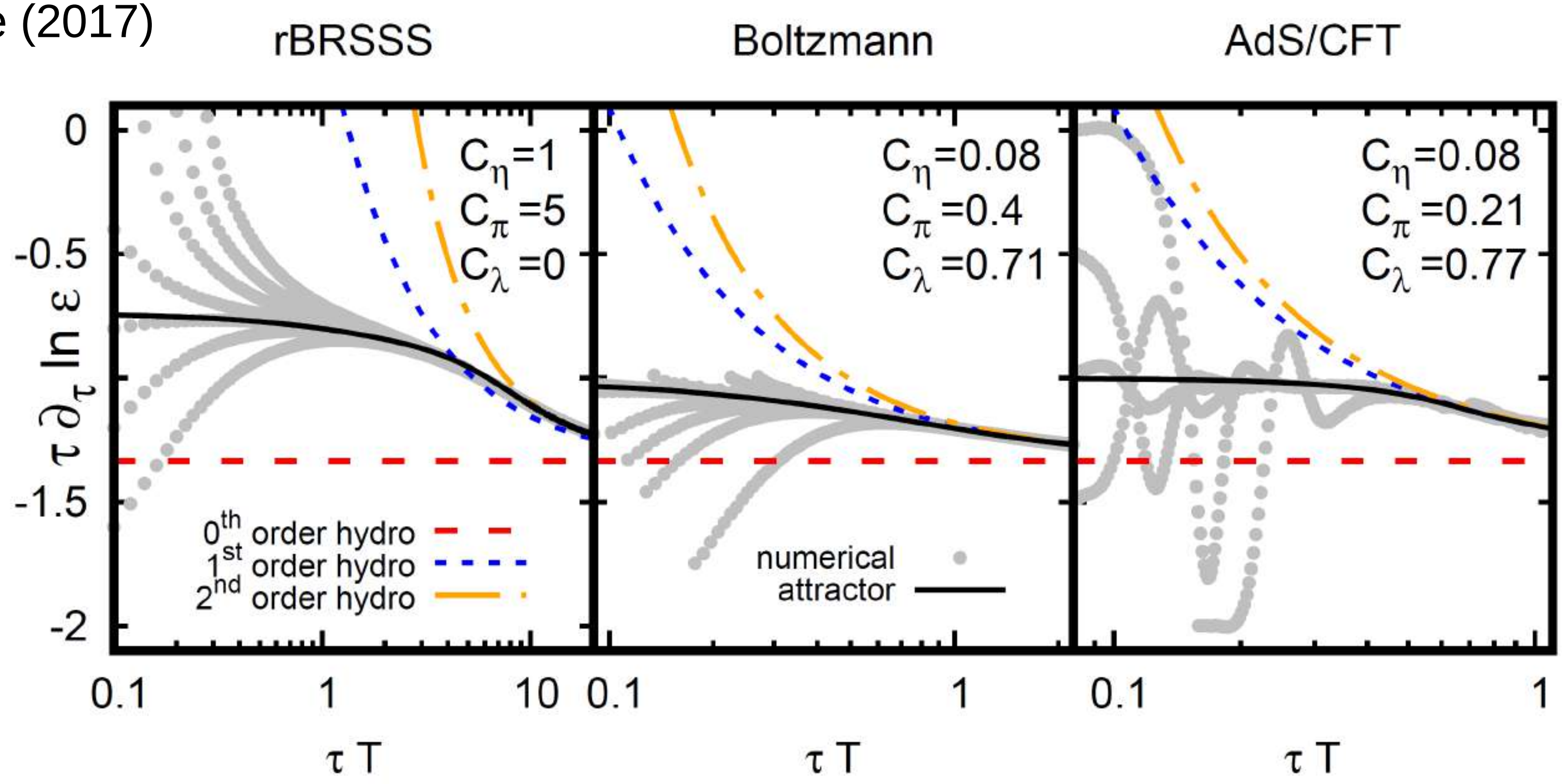
$$\bar{\pi} = \sum_{k=1}^{\infty} a_k \, [\mathrm{Kn}]^k \quad \longrightarrow \quad \bar{\pi} = \sum_{k=1}^{\infty} \left[a_k + \sum_{l=1}^{\infty} u_{k,l} \left(\sigma \, e^{-S/\mathrm{Kn}} \, [\mathrm{Kn}]^{\beta} \right)^{l} \right] [\mathrm{Kn}]^k$$

'Instanton'
Non-hydro modes



Message to take I

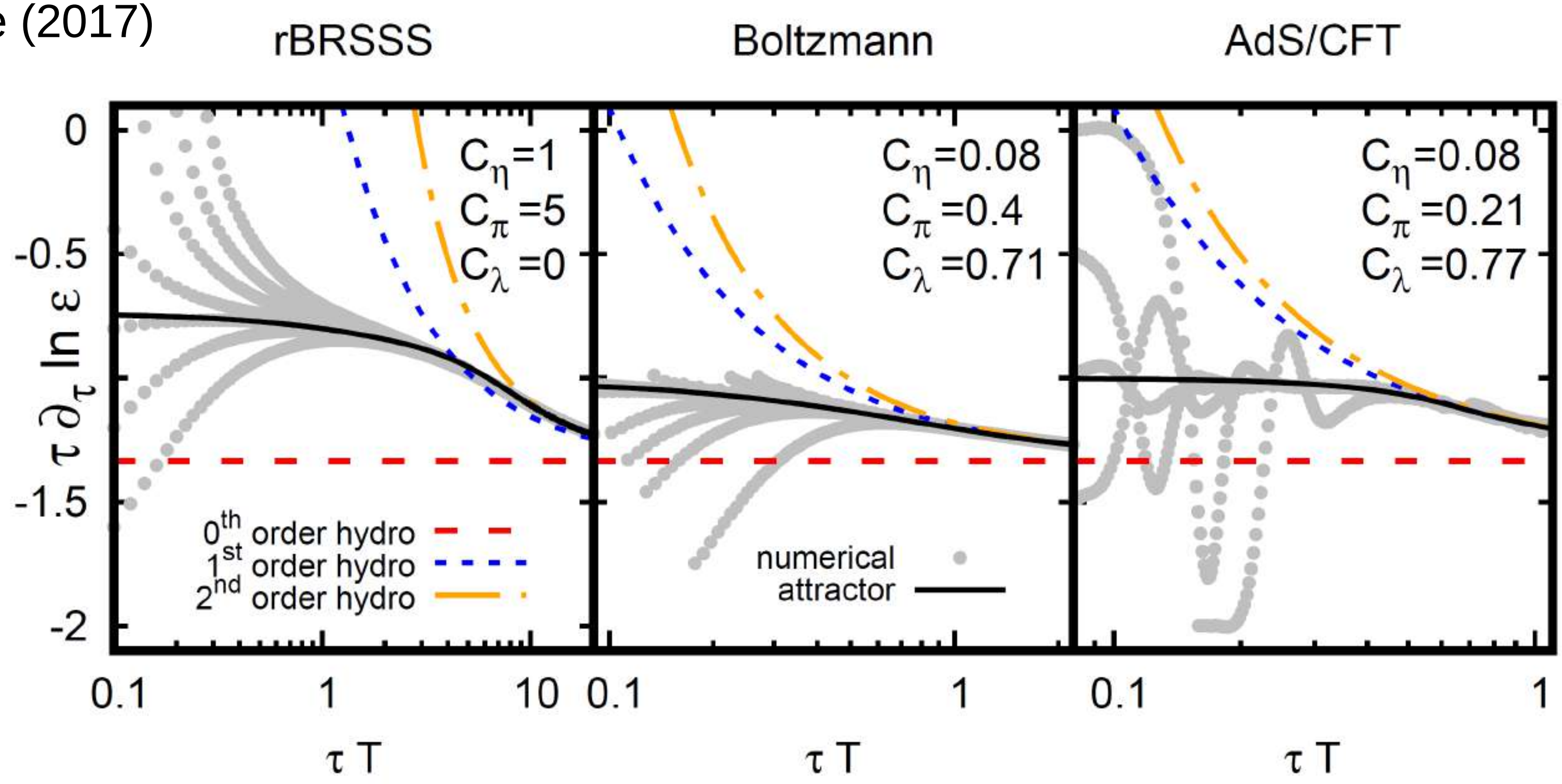
Romatschke (2017)



- *arbitrarily far-from-equilibrium initial conditions used to solve hydro equations merge towards a unique line (attractor).*
- *Independent of the coupling regime.*
- *Attractors can be determined from very few terms of the gradient expansion*
- *At the time when hydrodynamical gradient expansion merges to the attractor, the system is far-from-equilibrium, i.e. large pressure anisotropies are present in the system* $P_L \neq P_T$

Message to take I

Romatschke (2017)

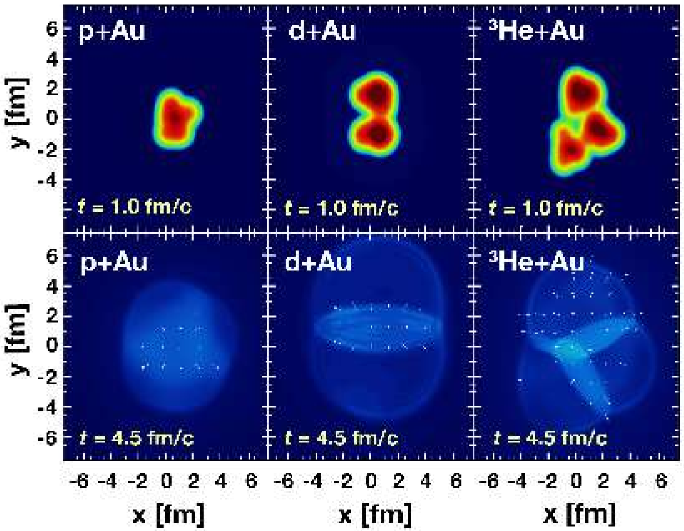


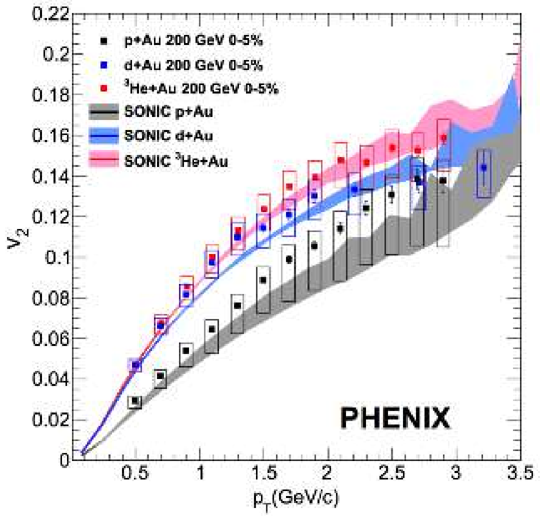
Existence of a new theory for far-from-equilibrium fluids

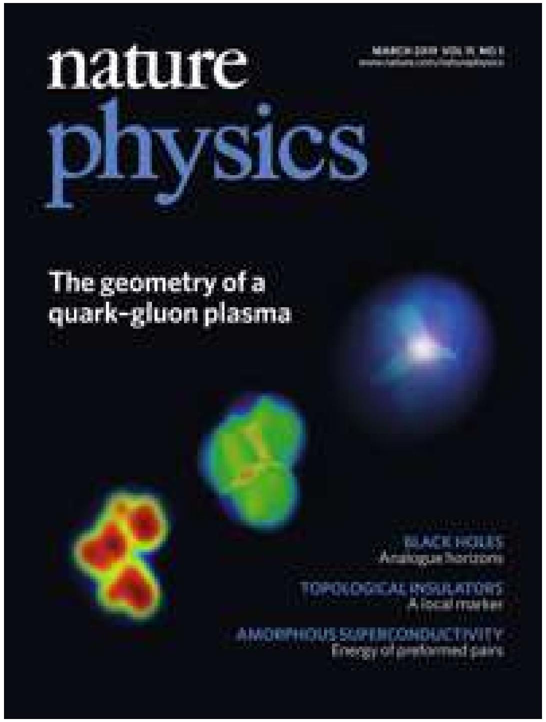
- *What are their properties?*

Do we have experimental evidence?

Nagle, Zajc (2018)







Flow-like behavior has been measured in collisions of small systems

- Hydrodynamical models seem to work in p-Au and d-Au collisions

Physical meaning:
Transient non-newtonian behavior

$$\bar{\pi} = \sum_{k=1}^{\infty} [\mathrm{Kn}]^k \underbrace{\left[a_k + \sum_{l=1}^{\infty} u_{k,l} \left(\sigma\, e^{-S/\mathrm{Kn}} [\mathrm{Kn}]^{\beta} \right)^l \right]}_{F_k(\sigma\, e^{-S/\mathrm{Kn}} [\mathrm{Kn}]^{\beta})}$$

k Each function F_k satisfies:

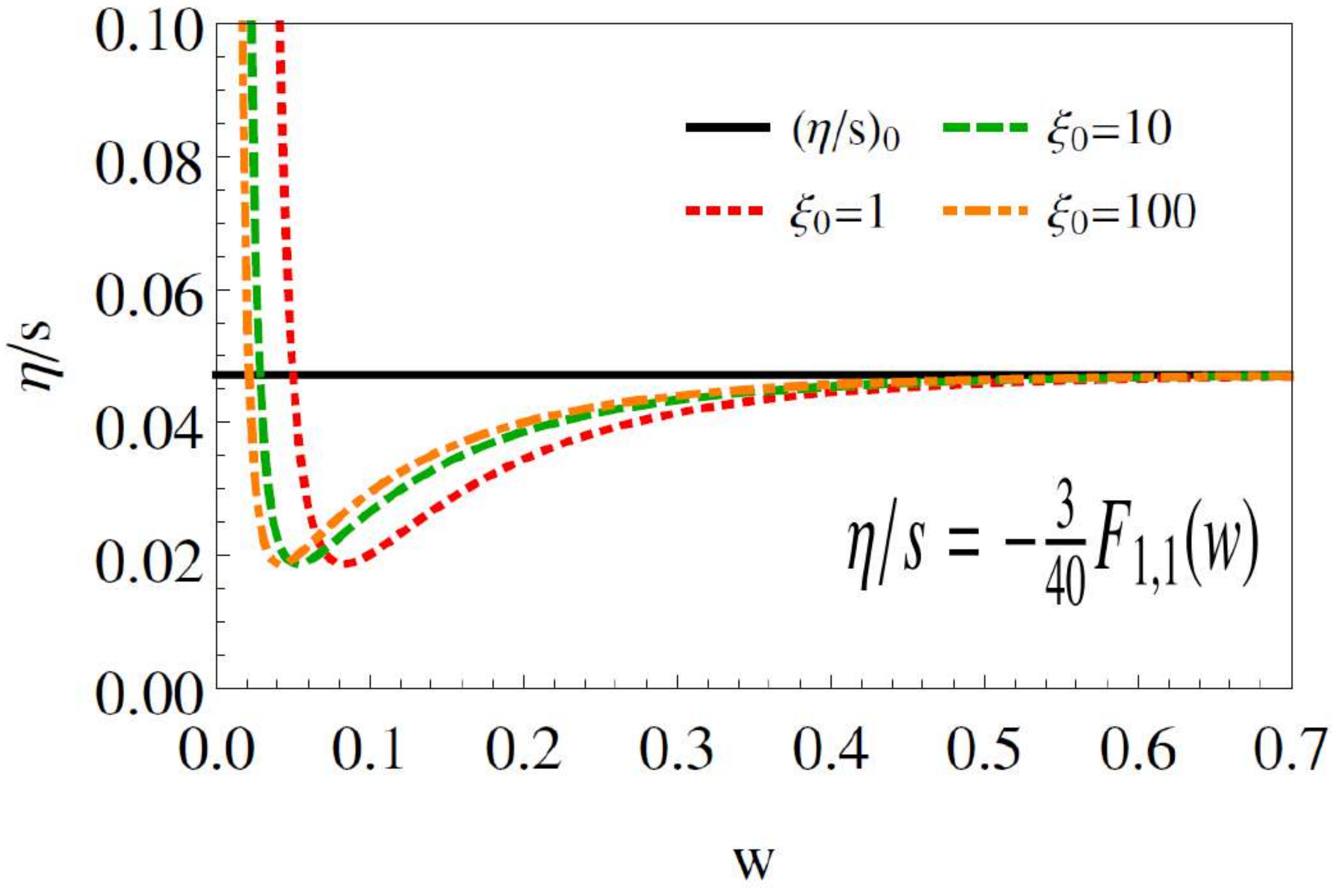
$$\lim_{Kn \to 0} F_k = a_k$$

$$\frac{F_k}{d(Kn^{-1})} = \beta_k(Kn, F_k, F_{k+1}, F_{k+2}, \ldots.)$$

Dynamical RG flow structure!!!

Behtash, Martinez, Kamata, Shi, Cruz-Camacho

Physical meaning: Transient non-newtonian behavior



Generalizes the concept of transport coefficient for far-from-equilibrium!!!

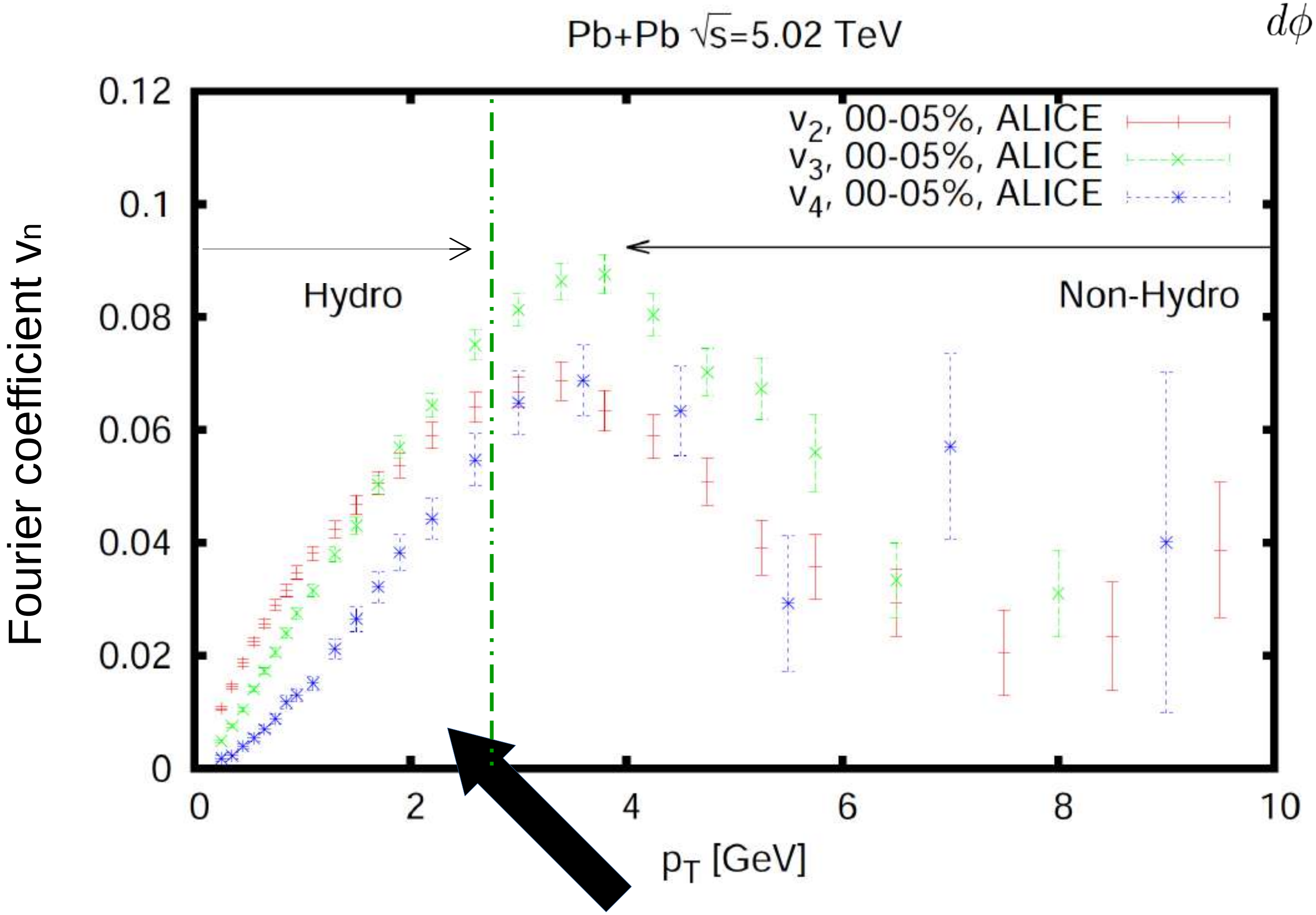
- It depends on the story of the fluid and thus, its rheology
- It presents shear thinning and shear thickening

Behtash, Martinez, Kamata, Shi, Cruz-Camacho; Yan & Blaizot

Non-hydrodynamic transport

Hydro vs. Non-hydro modes

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right)$$

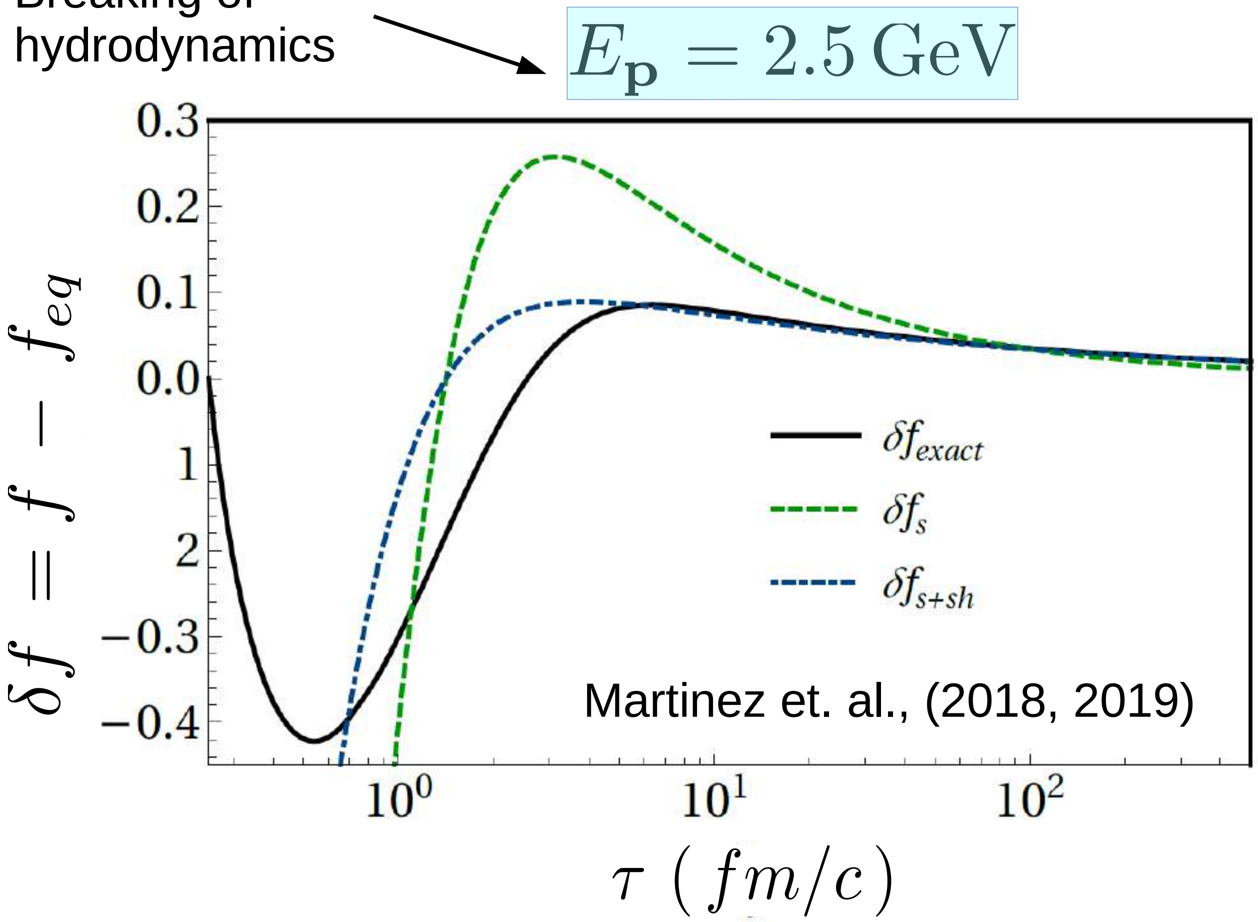


Romatschke ( )

- Hydro **breaks down** around $p_T \sim 2.5$ GeV
- **Non-hydro** modes are **dominant** at $p_T \gtrsim 2.5$ GeV

Non-hydrodynamic transport

Breaking of hydrodynamics



δf measures deviations from equilibrium of the full distribution function

Including only one mode (hydro)

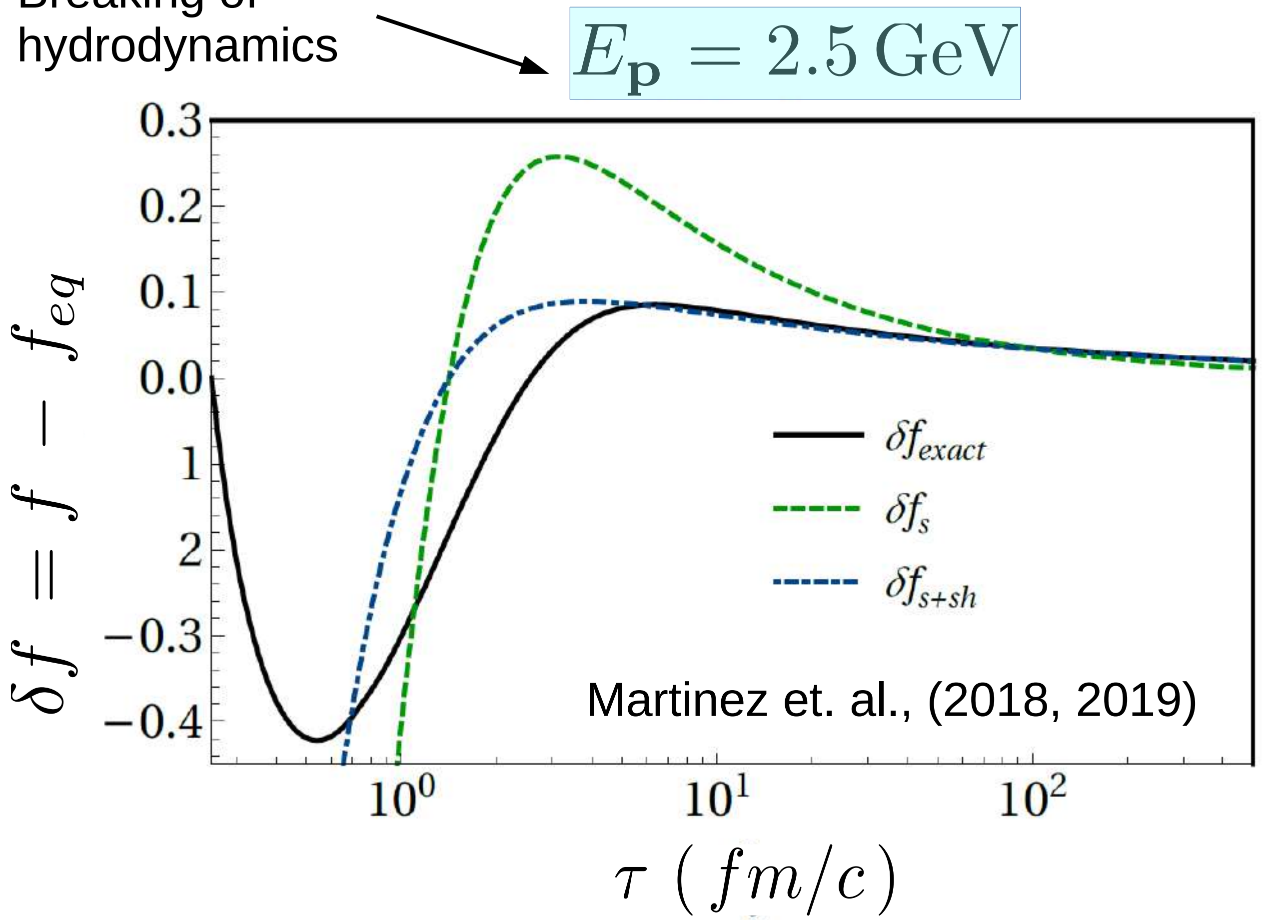
$$\delta f_s \sim a_{\bar{\pi}}\, \bar{\pi}$$

Including two modes (non-hydro)

$$\delta f_{s+sh} \sim a_{\bar{\pi}}\, \bar{\pi} + a_{\bar{c}_{sh}}\, \bar{c}_{sh}$$

Non-hydrodynamic transport

Breaking of hydrodynamics



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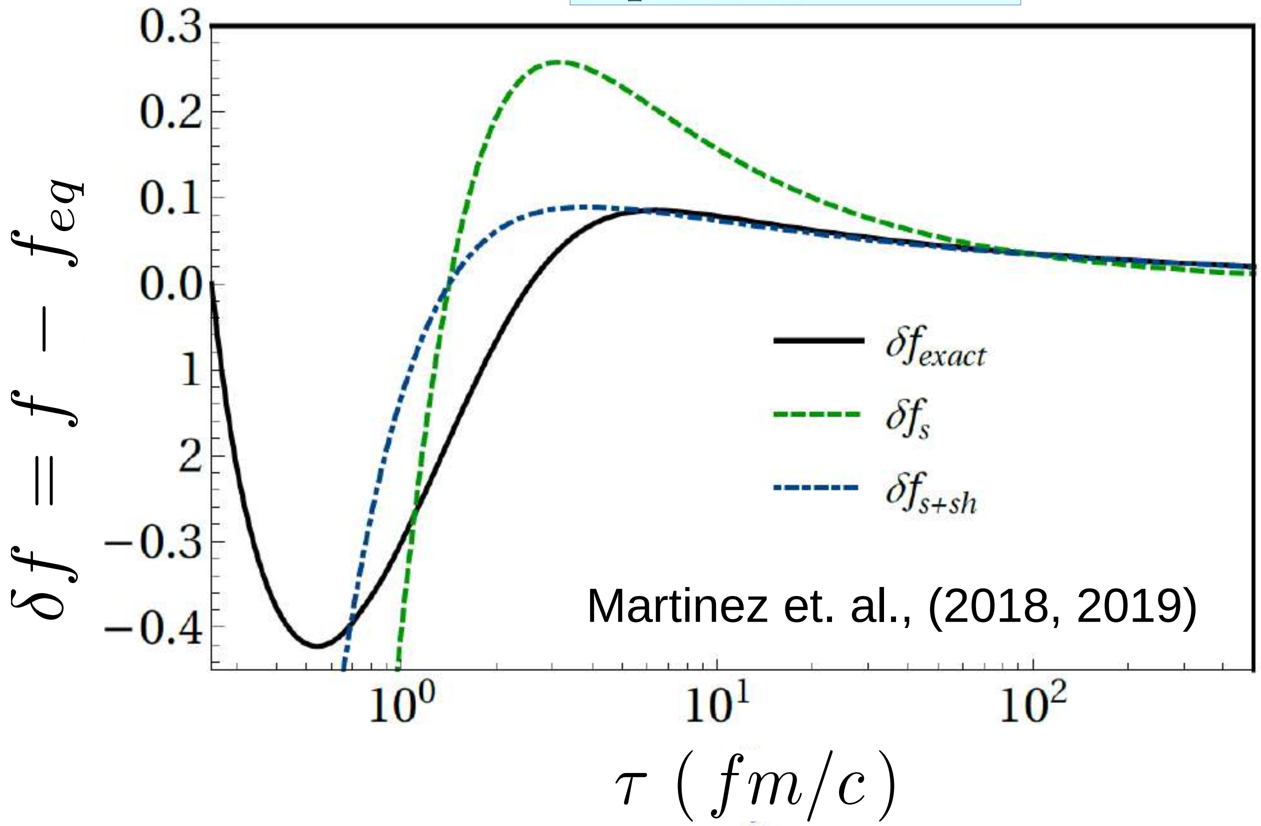
- For intermediate scales of momentum δf(t,**x**,**p**) requires the two **slowest non-hydro** modes in the **soft** and **semi-hard** momentum sectors
- **Non-hydrodynamic transport:** dynamics of non-hydro modes and hydro modes
  ⇒ **Cold atoms** : pressure anisotropies as non-hydrodynamic degrees of freedom (Bluhm & Schaefer, 2015-2017)

Non-hydrodynamic transport

Breaking of hydrodynamics → $E_{\mathbf{p}} = 2.5\,\mathrm{GeV}$



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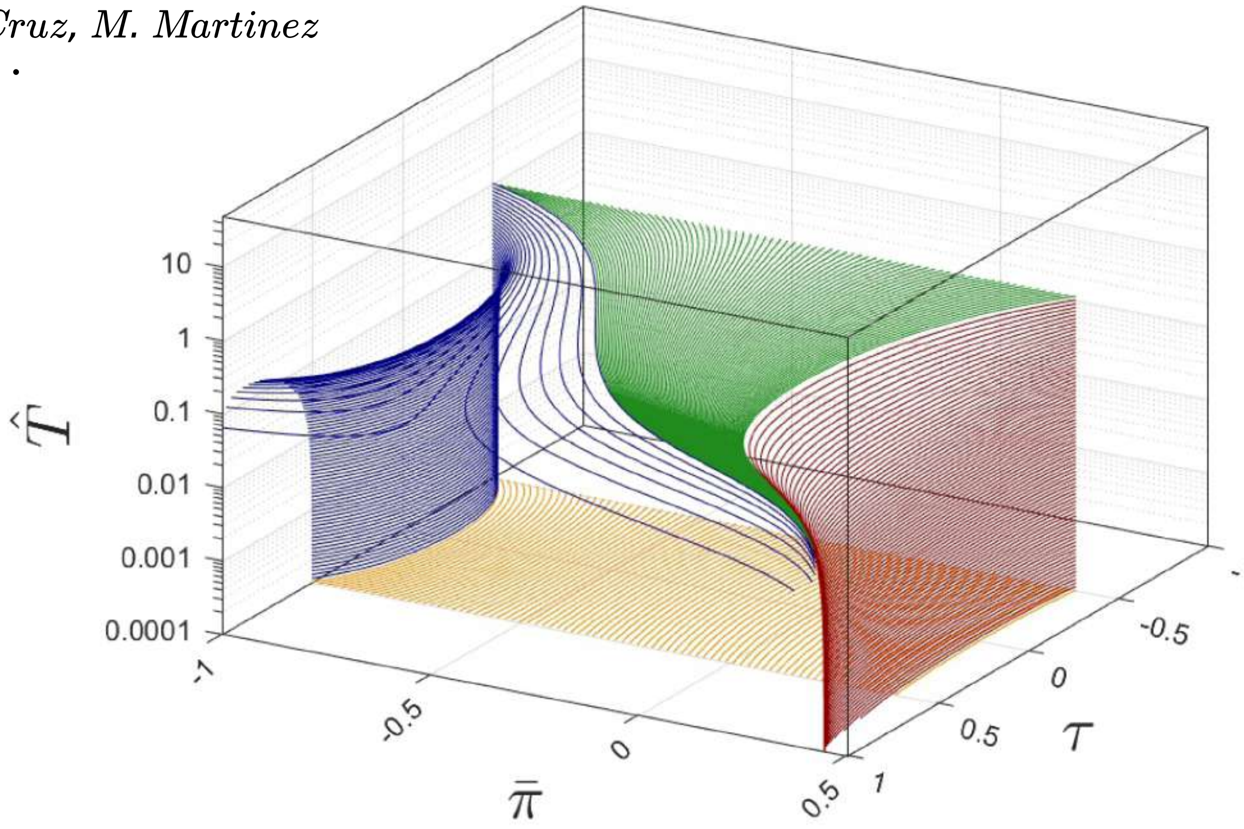
- For intermediate scales of momentum δf(t,**x**,**p**) requires the two **slowest non-hydro** modes in the **soft** and **semi-hard** momentum sectors
- **Non-hydrodynamic transport:** dynamics of non-hydro modes and hydro modes

  The asymptotic late time attractor of the distribution function depends not only on the shear but also on other slowest non-hydro modes!!!

Attractors in higher dimensions: Gubser flow for IS theory

A. Behtash, CN Cruz, M. Martinez
arXiv: .
PRD in press



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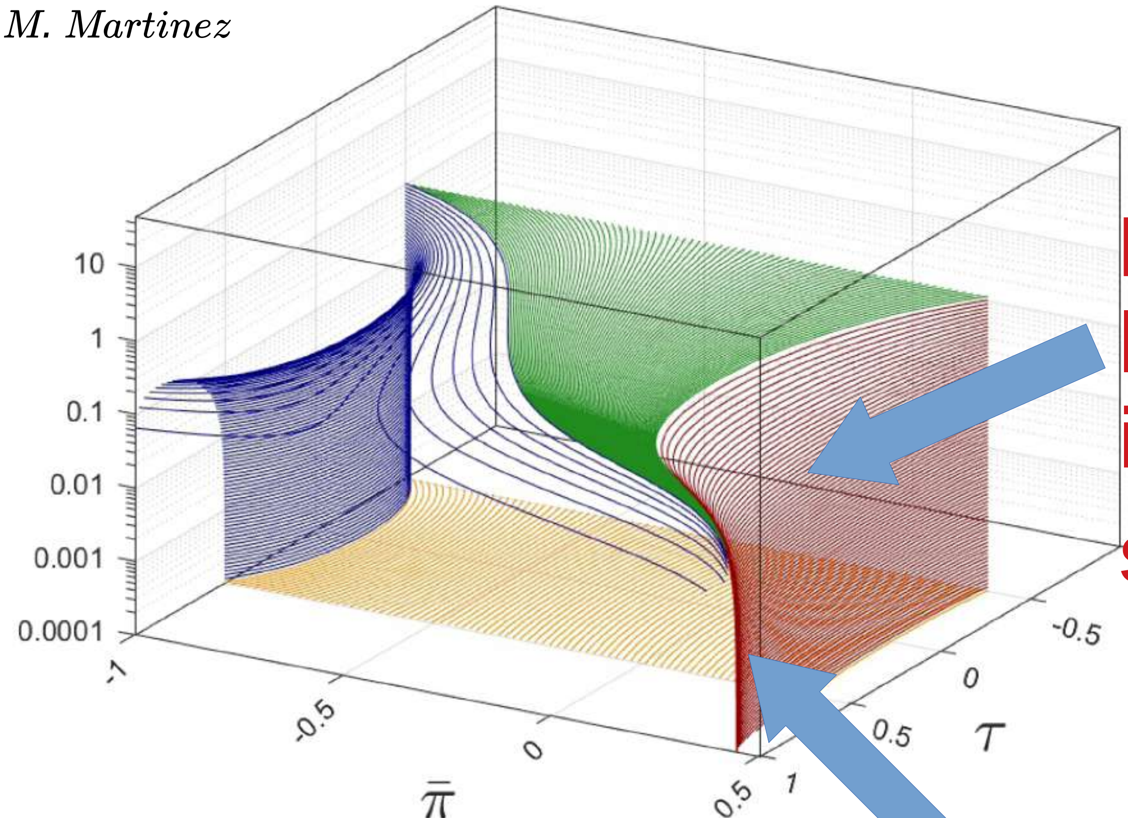

No universal line during intermediate stages

Late time asymptotic attractor

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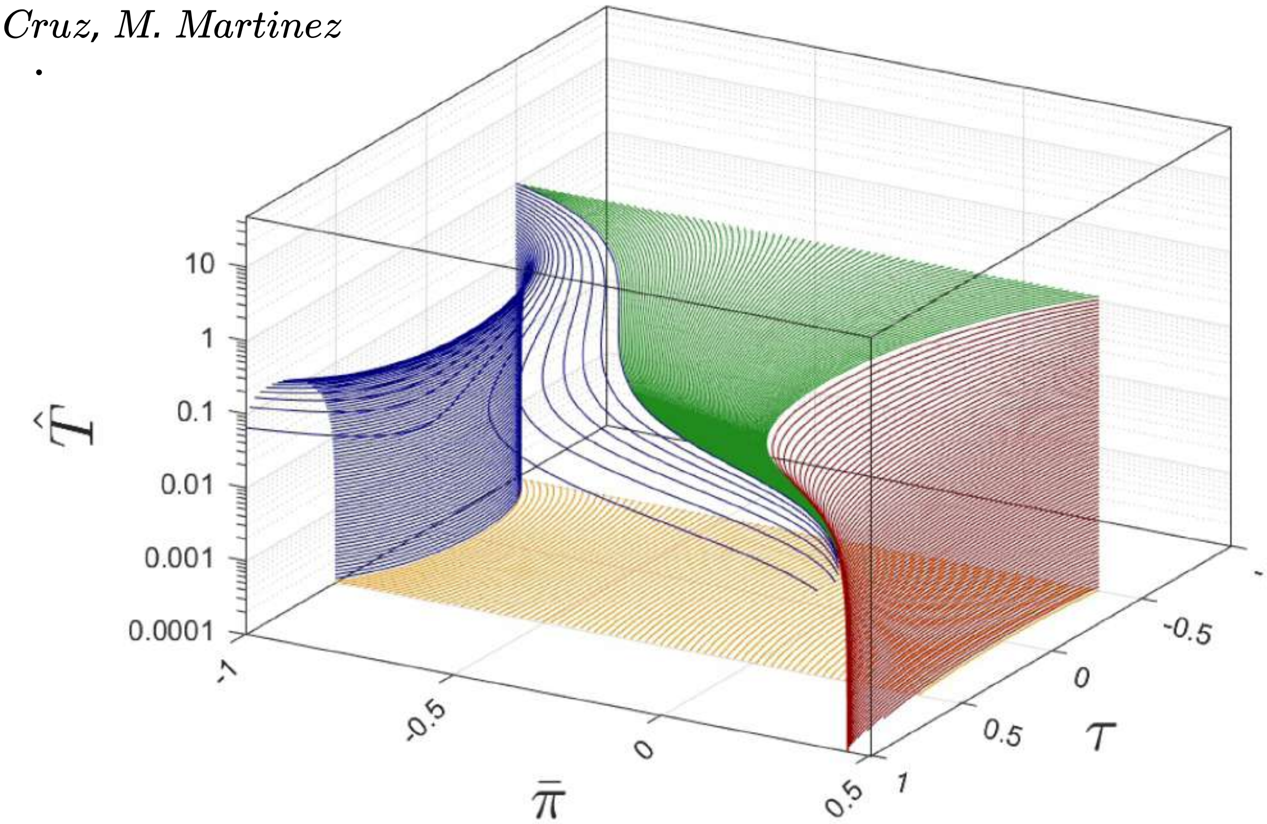

Attractor is a 1-d non planar manifold

- **In Bjorken you see a unique line cause the attractor is a 1d planar curve**

Attractors in higher dimensions: Gubser flow for IS theory

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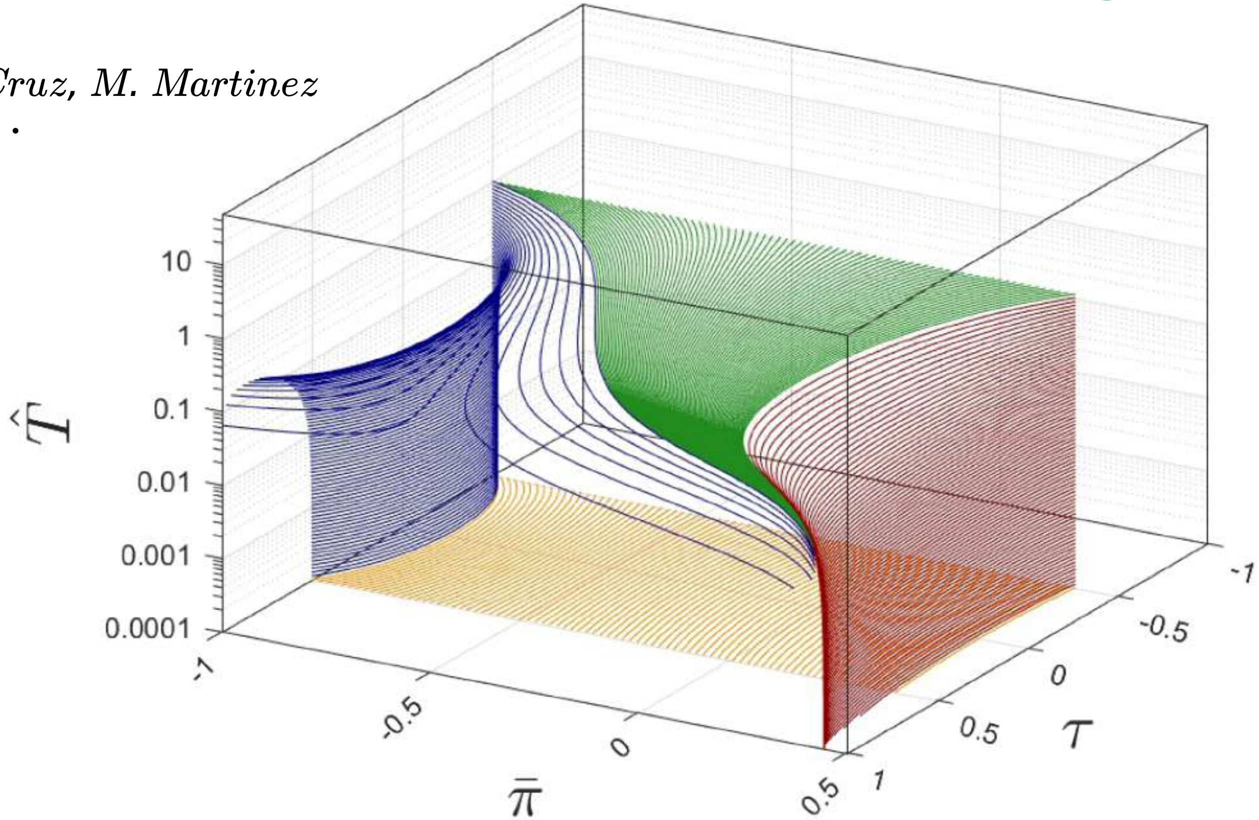


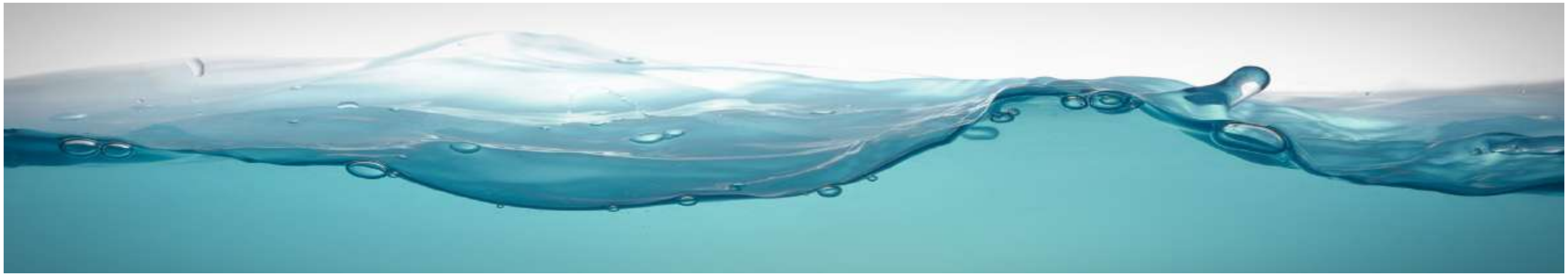
- **Asymptotic behavior of temperature is not determined by the Knudsen number**
- **Breaking of asymptotic gradient expansion (see also Denicol & Noronha)**


THE FUTURE


Research directions and opportunities

- **Emergence of liquid-like behavior in systems at extreme conditions**
  Neutron star mergers, cosmology, chiral effects in nuclear and condensed matter systems
- **Early time behavior of attractors**
  Behtash et. al., Wiedemann et. al., Heinz et. al.
- **Entropy production & experiments**
  Giacalone et. al.
- **Higher dimensional attractors via machine learning**
  Heller et. al.
- **Understanding scaling behavior**
  Mazeliauskas and Berges, Venugopalan et. al., Gelis & others

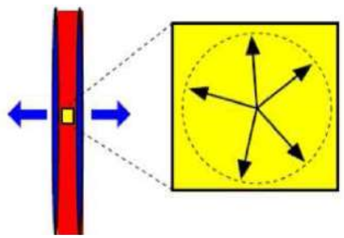
Conclusions



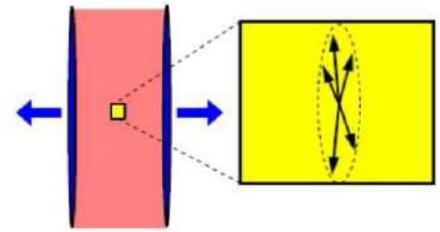
- Hydrodynamics is a beautiful 200 year old theory which remains as one of the most active research subjects in physics, chemistry, biology, etc.
- The emergence of liquid-like behavior has been observed in a large variety of systems subject to extreme conditions
- We need new ideas to formulate an universal Fluid dynamics for equilibrium and non-equilibrium
- Need to test these ideas with experiments

Backup slides

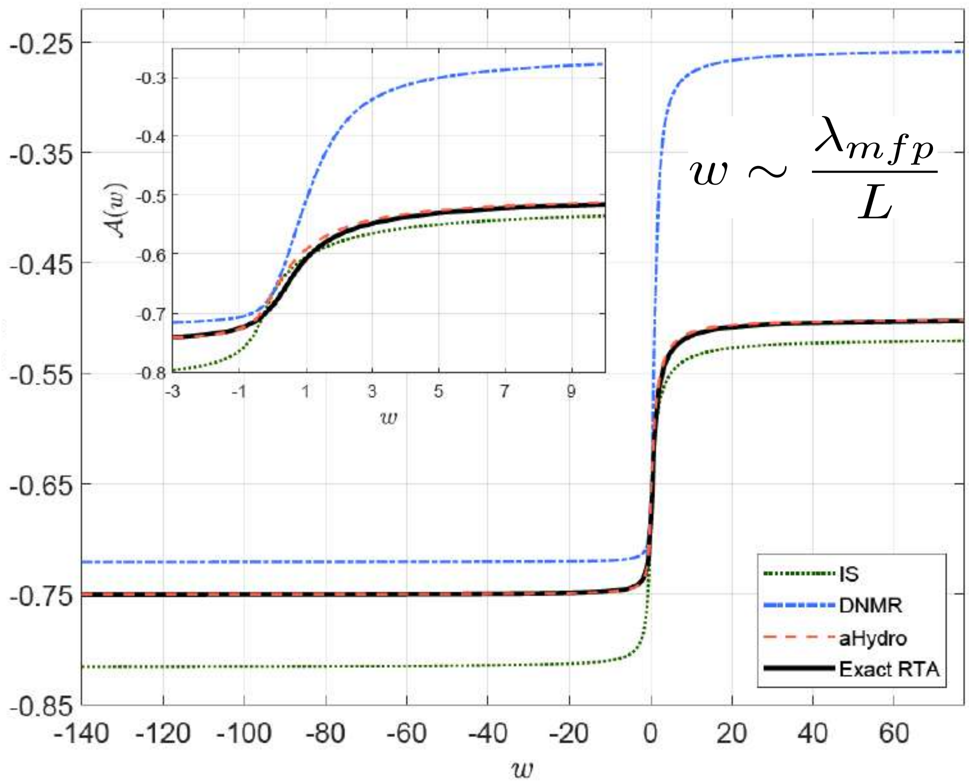
Comparing Gubser flow attractors



vs.



$$\mathcal{A}(w) = \frac{1}{3}\left(\bar{\pi} - 2\right)$$



$$w \sim \frac{\lambda_{mfp}}{L}$$

A. Behtash, CN Cruz, M. Martinez
arXiv: .
PRD in press

- **Anisotropic hydrodynamics matches the exact attractor to higher numerical accuracy !!!**
- **Anisotropic hydro is an effective theory which resumes the largest anisotropies of the system in the leading order term**

Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

$SO(3)_q$ → Special Conformal transformations + rotation along the beam line

$SO(1,1)$ → Boost invariance

Z_2 → Reflections along the beam line

Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

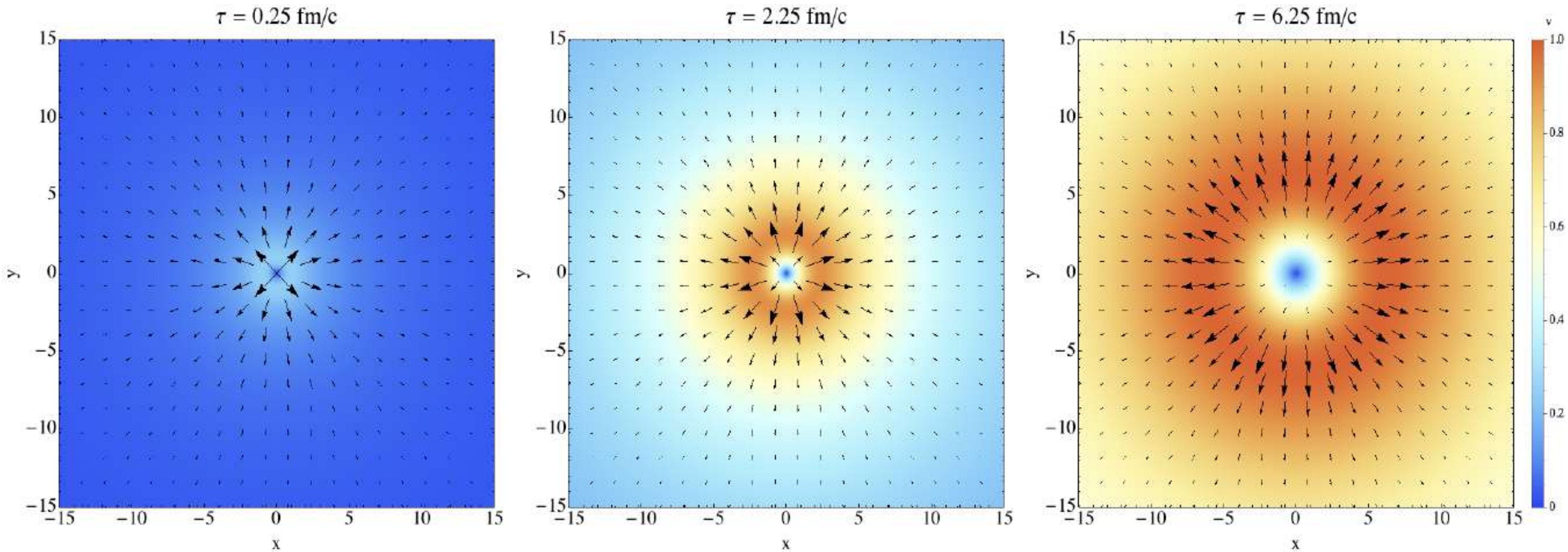
$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

In polar Milne Coordinates (τ, r, ϕ, η)

$$u^\mu = (\cosh\kappa(\tau,r), \sinh\kappa(\tau,r), 0, 0)$$

$$\kappa(\tau,r) = \tanh^{-1}\left(\frac{2q^2\tau r}{1+(qr)^2+(q\tau)^2}\right)$$

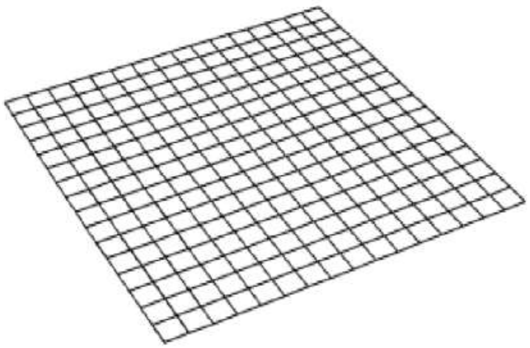
$\boldsymbol{q}$ is a scale parameter



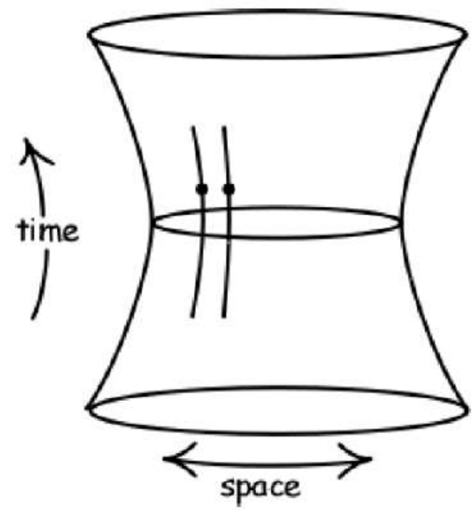
Gubser flow

$$g_{\mu\nu}(x) \rightarrow e^{-2\Omega(x)} g_{\mu\nu}(x)$$

Flat Minkowski space



dS$_3$ x R



$$\sinh\rho = -\frac{1-\tilde{\tau}^2+\tilde{r}^2}{2\tilde{\tau}}, \qquad \tan\theta = \frac{2\tilde{r}}{1+\tilde{\tau}^2-\tilde{r}^2}.$$

Complicated dynamics

3d de Sitter space

$$x^\mu = (\tau, r, \phi, \eta) \quad \Rightarrow \quad \hat{x}^\mu = (\rho, \theta, \phi, \eta)$$

$$ds^2 = -d\tau^2 + dr^2 + r^2 d\phi^2 + d\eta^2 \quad \Rightarrow \quad d\hat{s}^2 = -d\rho^2 + \cosh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right) + d\eta^2$$

$$u^\mu = (u^\tau(\tau,r), u^r(\tau,r), 0, 0) \quad \Rightarrow \quad \hat{u}^\mu = (1,0,0,0)$$

$$\epsilon(\tau, r) \quad \Rightarrow \quad \hat{\epsilon}(\rho)$$

Exact Gubser solution

- *In dS $\otimes$R the dependence of the distribution function is restricted by the symmetries of the Gubser flow*

$$f(\hat{x}^\mu, \hat{p}_i) = f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right)$$

$$\hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2\theta}$$ → *Total momentum in the* (*θ*,*ϕ*) *plane*

$$\hat{p}_\eta$$ → *Momentum along the η direction*

- *The RTA Boltzmann equation gets reduced to*

$$\frac{\partial}{\partial\rho} f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) = -\frac{\hat{T}(\rho)}{c}\left(f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) - f_{eq}\left(\hat{p}^\rho/\hat{T}(\rho)\right)\right)$$

$$c = 5\frac{\eta}{\mathcal{S}}$$

- *The exact solution to this equation is*

$$f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) = D(\rho, \rho_0)\, f_0\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\, \hat{T}(\rho')\, D(\rho, \rho')\, f_{eq}\left(\hat{p}^\rho/\hat{T}(\rho)\right)$$

Boltzmann equation

The macroscopic quantities of the system are simply averages weighted by the solution for the distribution function

$$\varepsilon(x) = \int \frac{d^3p}{\sqrt{-g}p^0}(p \cdot u)^2 f(x^\mu, p_i)\,,$$

$$\mathcal{P}(x) = \frac{1}{3}\int \frac{d^3p}{\sqrt{-g}p^0}\Delta_{\mu\nu}p^\nu p^\mu f(x^\mu, p_i)\,,$$

$$\pi^{\mu\nu}(x) = \int \frac{d^3p}{\sqrt{-g}p^0}p^{\langle\mu}p^{\nu\rangle} f(x^\mu, p_i)\,.$$

Solving exactly the Boltzmann eqn. is extremely hard so one needs some method to construct approximate solutions

Fluid models for the Gubser flow

E-M conservation law

$$\frac{\partial_\rho \hat{T}}{\hat{T}} + \frac{2}{3}\tanh\rho = \frac{\bar{\pi}}{3}\tanh\rho$$

DNMR theory

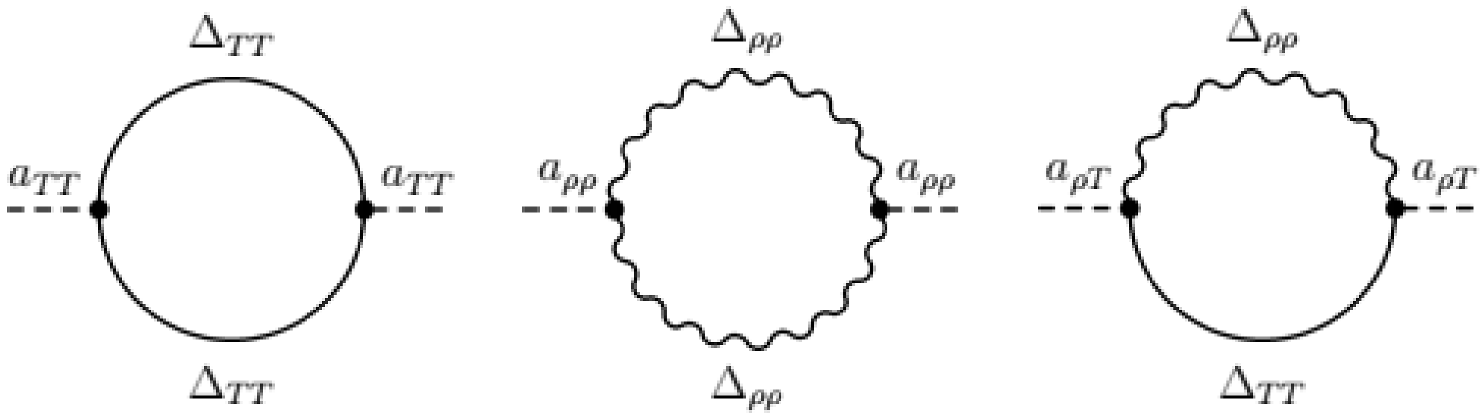
$$\hat{\tau}_{\hat{\pi}}\left(\partial_\rho \bar{\pi} + \frac{4}{3}(\bar{\pi})^2\,\tanh\rho\right) + \bar{\pi} = \frac{4}{3}\frac{\eta}{s\hat{T}}\,\tanh\rho + \frac{10}{7}\hat{\tau}_{\hat{\pi}}\bar{\pi}\,\tanh\rho$$

IS theory

Anisotropic hydrodynamics

$$\partial_\rho \bar{\pi} + \frac{\bar{\pi}}{\hat{\tau}_r} = \frac{4}{3}\tanh\rho\left(\frac{5}{16} + \bar{\pi} - \bar{\pi}^2 - \frac{9}{16}\mathcal{F}(\bar{\pi})\right)$$

Statistical field theory method



In the Gaussian approximation (white random noise)

$$G_S^{\mathcal{O}\mathcal{O}}(\omega,0) = \int \frac{d\omega'}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[2a_{\rho\rho}^2 \Delta_S^{\rho\rho}(\omega',\mathbf{k}) \Delta_S^{\rho\rho}(\omega-\omega',\mathbf{k}) + a_{\rho T}^2 \Delta_S^{\rho\rho}(\omega',\mathbf{k}) \Delta_S^{TT}(\omega-\omega',\mathbf{k}) + 2a_{TT}^2 \Delta_S^{TT}(\omega',\mathbf{k}) \Delta_S^{TT}(\omega-\omega',\mathbf{k}) \Big] .$$

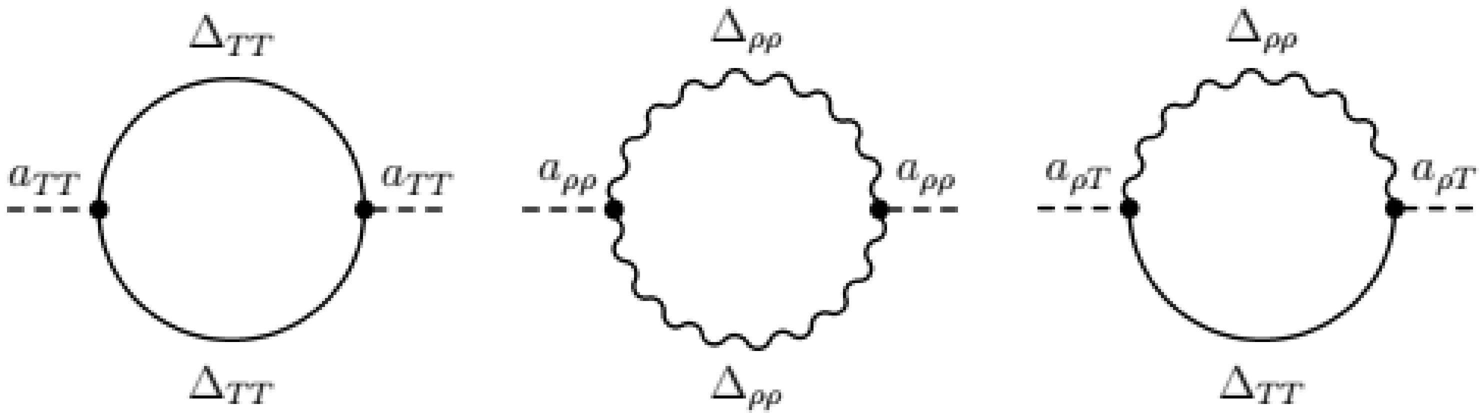
$$\Delta_S^{TT}(\omega,\mathbf{k}) = \frac{2T^2}{c_P} \frac{D_T \mathbf{k}^2}{\omega^2 + (D_T \mathbf{k}^2)^2}$$

Dominated by the diffusive heat wave

$$\Delta_S^{\rho\rho}(\omega,\mathbf{k}) = 2\rho T \left\{ \frac{\Gamma k^4}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} + \frac{\Delta c_P}{c_s^2} \frac{D_T \mathbf{k}^2}{\omega^2 + (D_T k^2)^2} - \frac{\Delta c_P}{c_s^2} \frac{(\omega^2 - c_s^2 k^2) D_T \mathbf{k}^2}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} \right\},$$

Mix of sound and diffusive modes

Statistical field theory method



After a long algebra plus pole analysis of propagators

$$G_R^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{0}) = -A_T L(\omega, \Lambda, 2D_T) - A_\Gamma L(\omega, \Lambda, \Gamma)$$

$$L(\omega, \Lambda, D_i) = \frac{1}{2\pi^2}\left\{\frac{\Lambda^3}{3} + \frac{i\omega\Lambda}{D_i} - \frac{\pi}{2\sqrt{2}}(1+i)\left(\frac{\omega}{D_i}\right)^{3/2} + \ldots\right\}.$$

Resurgence and transseries

A new time-dependent resummation scheme is needed

Asymptotic expansion

$$\bar{\pi} = \sum_{k=1}^{\infty} a_k \, [\mathrm{Kn}]^k$$

Resurgence
Costin (1998)

Transseries solution

$$\bar{\pi} = \sum_{k=1}^{\infty} \left[a_k + \sum_{l=1}^{\infty} u_{k,l} \left(\sigma \, e^{-S/\mathrm{Kn}} [\mathrm{Kn}]^{\beta} \right)^l \right] [\mathrm{Kn}]^k$$

Instantons

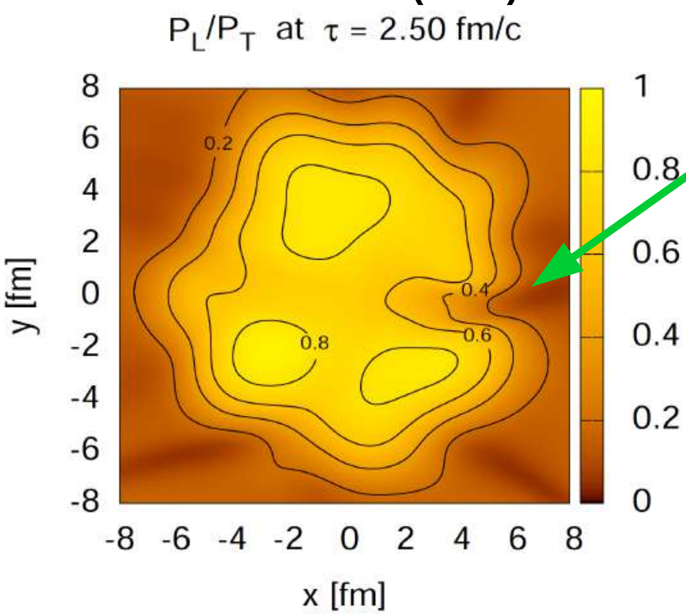
Transseries:

At a given order of the perturbative expansion, transseries resumes the non-perturbative contributions of small perturbations around the asymptotic late time fixed point

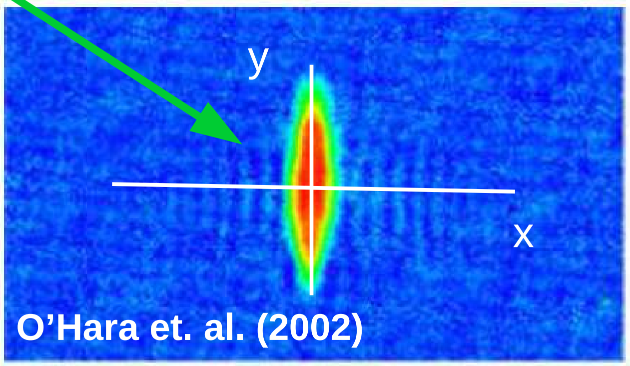
Size of the hydrodynamical gradients

Heavy Ion Collision

Martinez et. al. (2012)



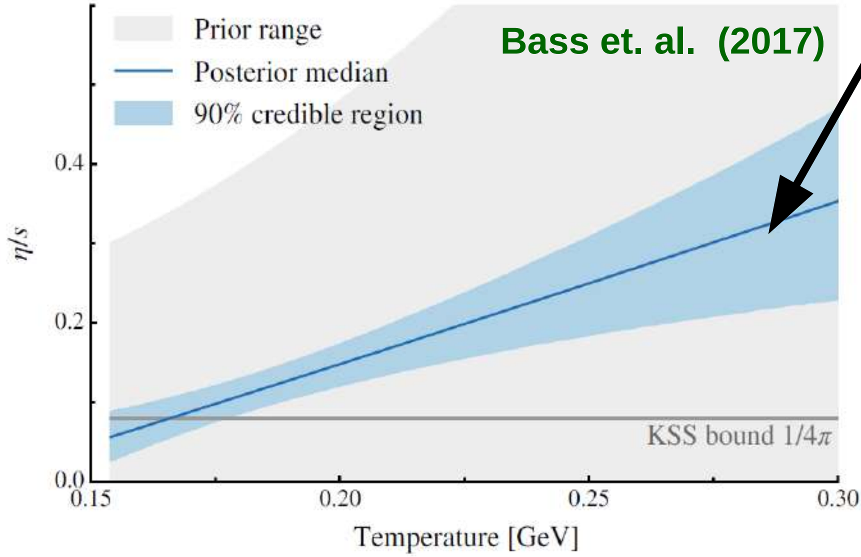
Cold Atoms

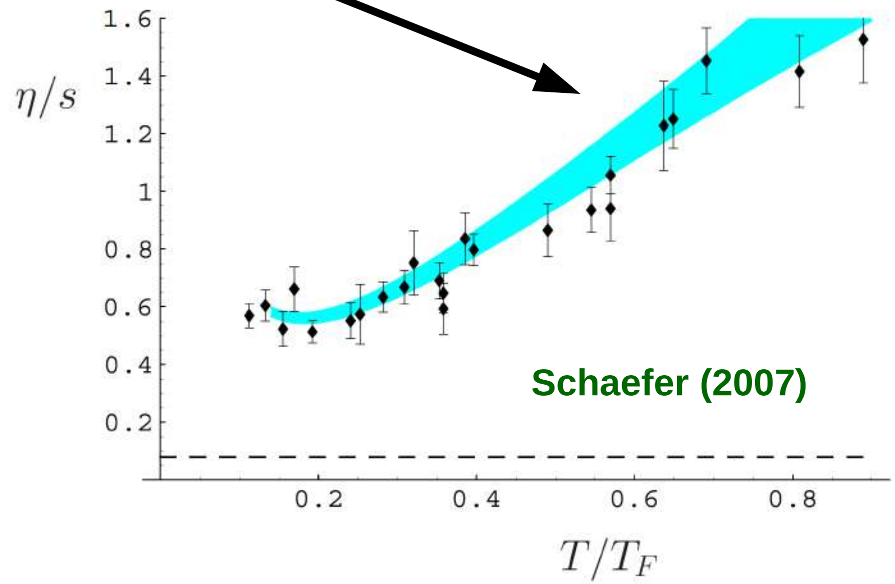


Gradients are not small

$$Kn \sim 1$$

LARGE UNCERTAINTY





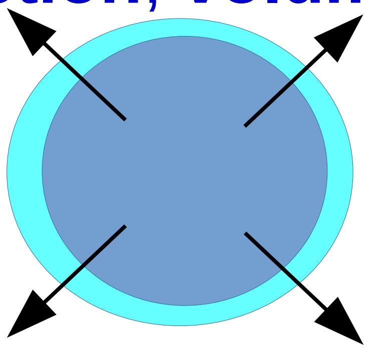
Universality of hydrodynamics

- Fluid dynamical equations of motion are **universal**

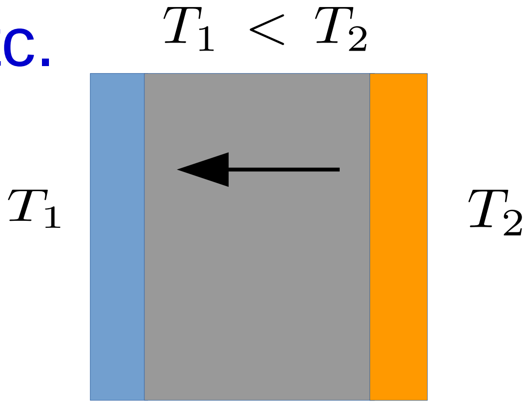
  ⇛ **In general** fluid dynamics is **not** a particular limit of a weakly (e.g. kinetic theory) or strongly coupled (e.g. AdS/CFT) theory

- **Transport coefficients** (e.g. shear viscosity) and other **thermodynamical properties** depend on microscopic details of the system

- Hydrodynamical approach also describes **heat conduction**, **volume expansion,** etc.

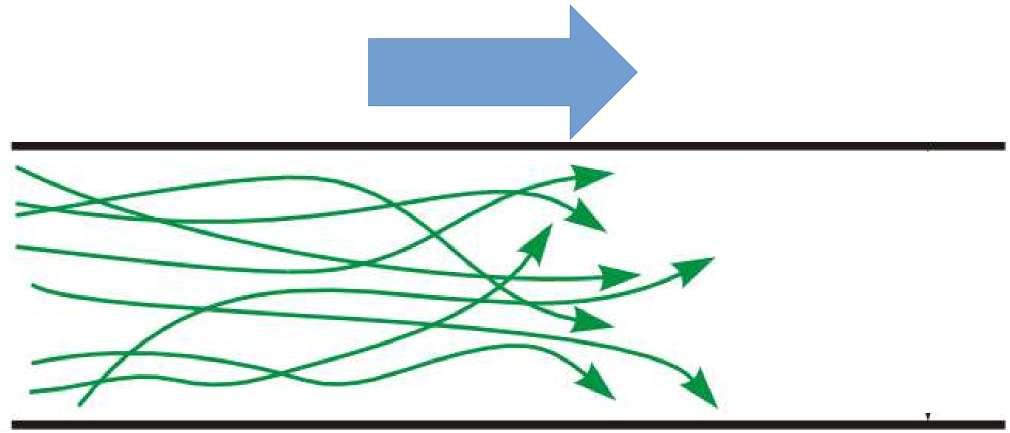


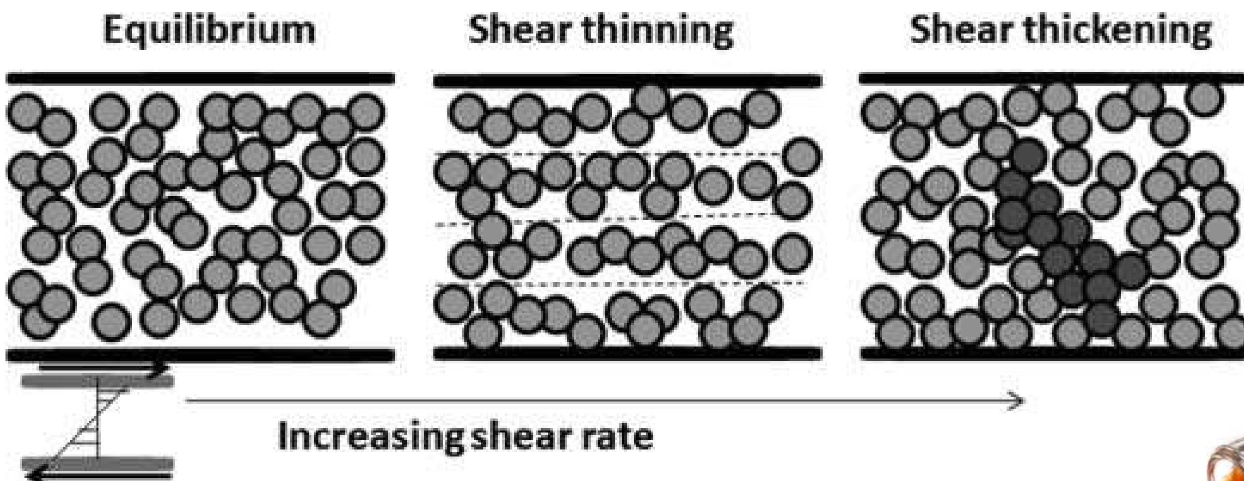
$$\Pi = -\zeta \, \vec{\nabla} \cdot \vec{v}$$



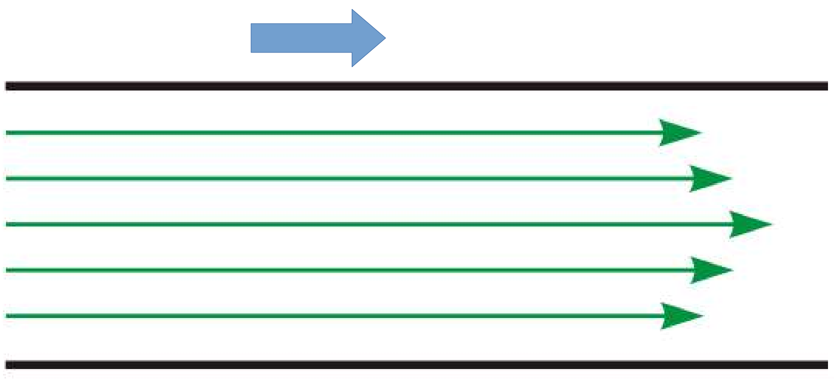
$$\vec{q} = -\kappa \, \vec{\nabla} T$$

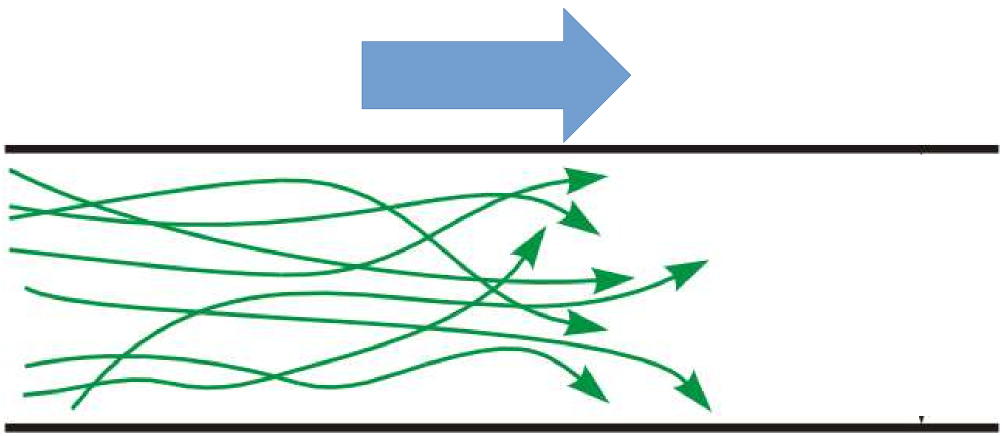
Non-newtonian fluids and rheology





Non-newtonian fluids and rheology





$$\pi_{yx} \sim \eta\, \partial_y v_x$$

$$\pi_{yx} \sim \eta(\partial_y v_x)\, \partial_y v_x$$

Shear viscosity

- Becomes a **function** of the **gradient of the flow velocity**
- can **increase** (shear thickening) or **decrease** (shear thinning) **depending** on the **size** of the **gradient of the flow velocity**

Non-newtonian fluids and rheology

$$\pi_{yx} \sim \eta \, \partial_y v_x \qquad\qquad \pi_{yx} \sim \eta(\partial_y v_x) \, \partial_y v_x$$

Does the QGP behave like a non-newtonian fluid?

Our idea

- Develop a new truncation scheme which captures some of the main features of far-from-equilibrium fluids (e.g. non-hydrodynamical modes) while being simple enough to perform concrete calculations

$$\tau_\pi \, D_\tau \pi^{\mu\nu} + \pi^{\mu\nu} = \eta \sigma^{\mu\nu}$$

$$\tau_\pi(\sigma^{\mu\nu}) \, D_\tau \pi^{\mu\nu} + \pi^{\mu\nu} = \eta(\sigma^{\mu\nu}) \sigma^{\mu\nu}$$

Keep track of the deformation history of the fluid
$\Rrightarrow$ Study its rheological properties

Effective η/s as a non-hydrodynamical series

At $\mathcal{O}(w^{-1})$ the dominant term of the trans-series is

$$c_1 = \frac{\sum_l U^{-1}_{1l} \tilde{u}^{(0)}_{l,1}}{w}$$

On the other hand, Chapman-Enskog expansion gives the asymptotic behavior of c_1

$$c_1 = -\frac{40}{3}\frac{1}{w}\left(\frac{\eta}{s}\right)_0$$

$$\Rightarrow \left(\frac{\eta}{s}\right)_0 = -\frac{3}{40}\sum_l U^{-1}_{1l}\tilde{u}^{(0)}_{l,1}$$

- **Effective η/s is the asymptotic limit of a trans-series**
- **We can study its rheology by following the 'history' of the corresponding trans-series**

A. Behtash et. al, Forthcoming

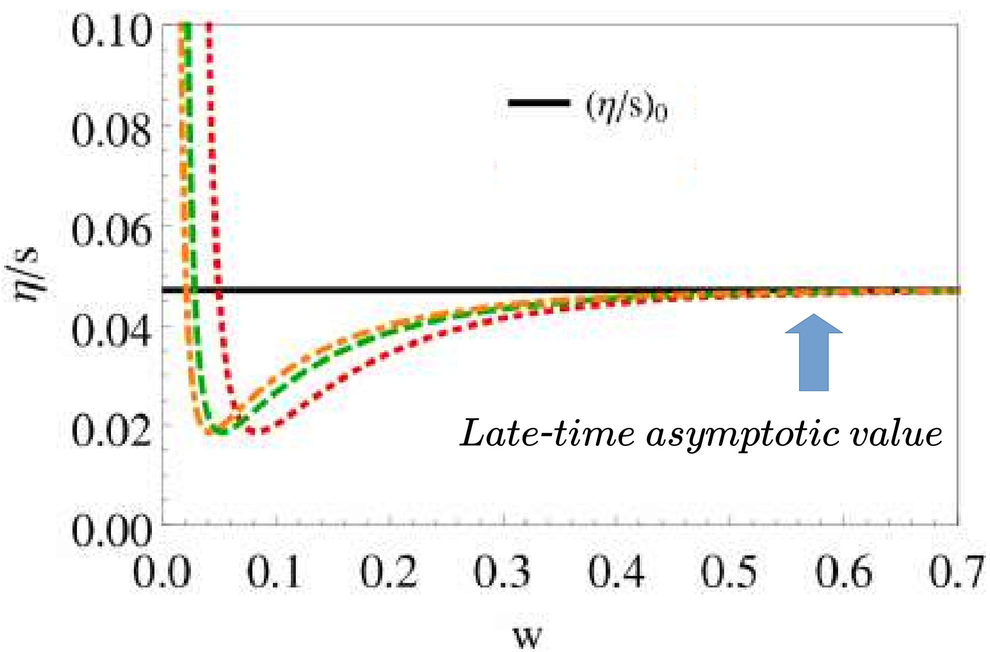
Effective η/s as a non-hydrodynamic series

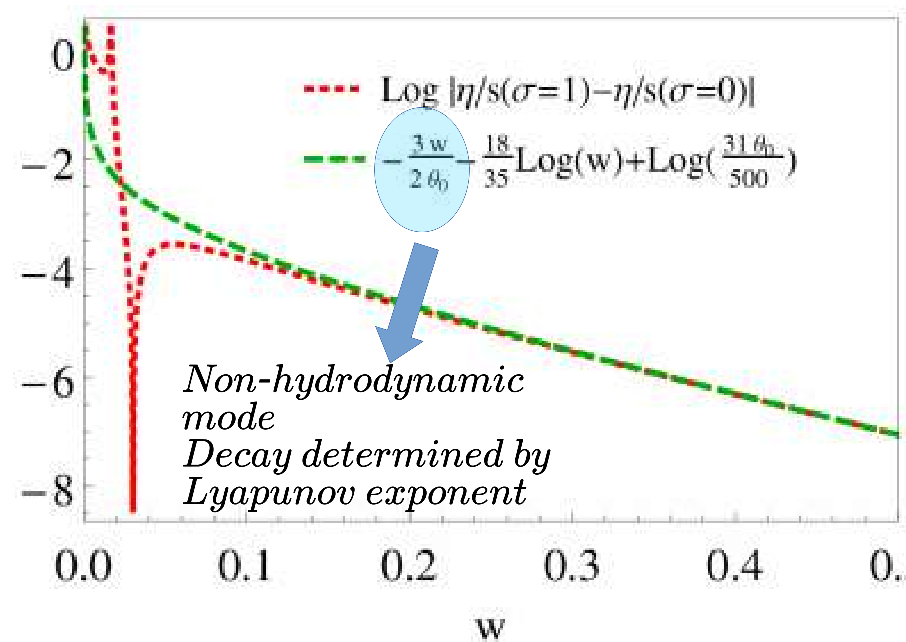
Thus effective η/s is

$$\left(\frac{\eta}{s}\right)_R = -\frac{3}{40}\sum_l U^{-1}_{1l}\tilde{C}_{l,1}(\sigma e^{-Sw}w^{\tilde{b}})$$

Its RG flow evolution is one of the differential recursive relation of the corresponding trans-series

$$\frac{d}{dw}\left(\frac{\eta}{s}\right)_R = -\frac{3}{40}\sum_l U^{-1}_{1l}\frac{d}{dw}\tilde{C}_{l,1}(\sigma e^{-Sw}w^{\tilde{b}})$$





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