Illinois Center for Advanced Studies of the Universe

# Deconstructing Relativistic Fluid Dynamics

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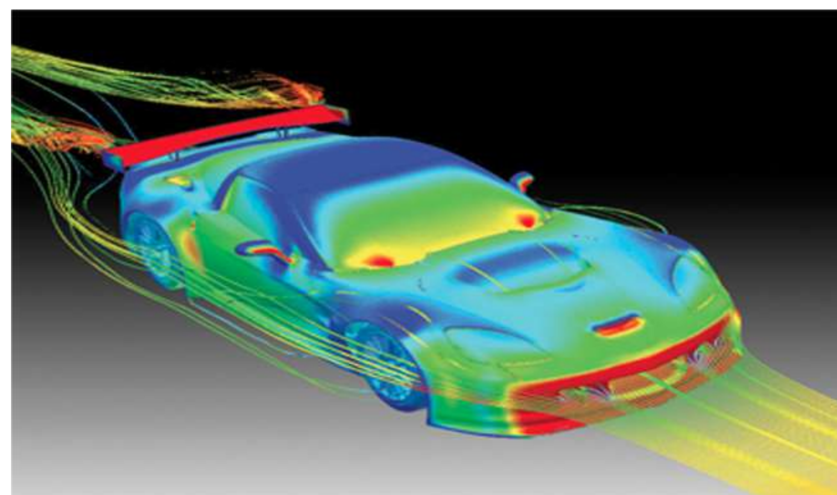
# Fluid Dynamics Is Everywhere

# The Ubiquitousness of Fluid Dynamics

Based on conservations laws + large separation of length scales

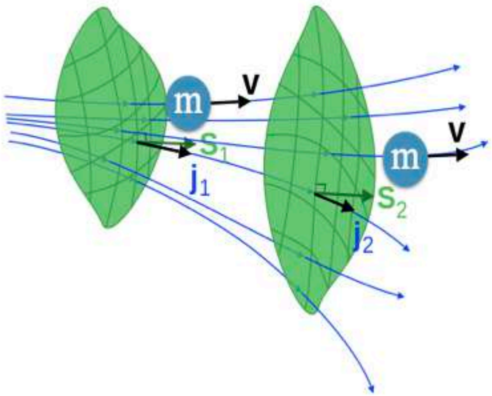
**Separation of scales** $\rightarrow$ macroscopic: $L$ microscopic: $\ell$

**Knudsen number** $$K_N \sim \frac{\ell}{L} \ll 1$$ → **FLUID**



# How does one describe fluid dynamics?

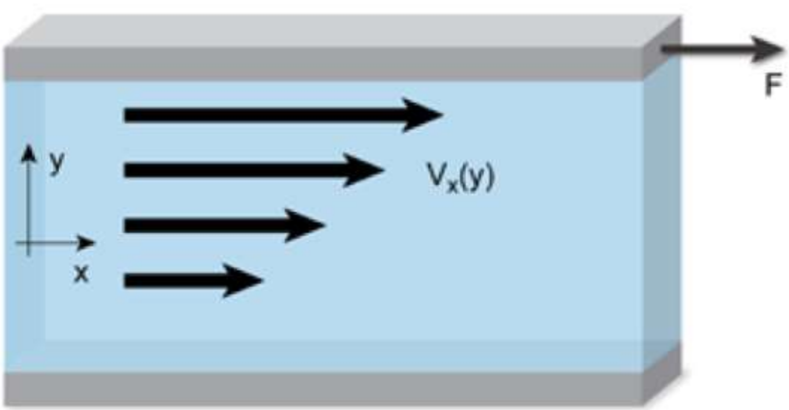
Conservation of mass + Newton's 2nd law + isotropy + incompressibility $(\rho_0)$

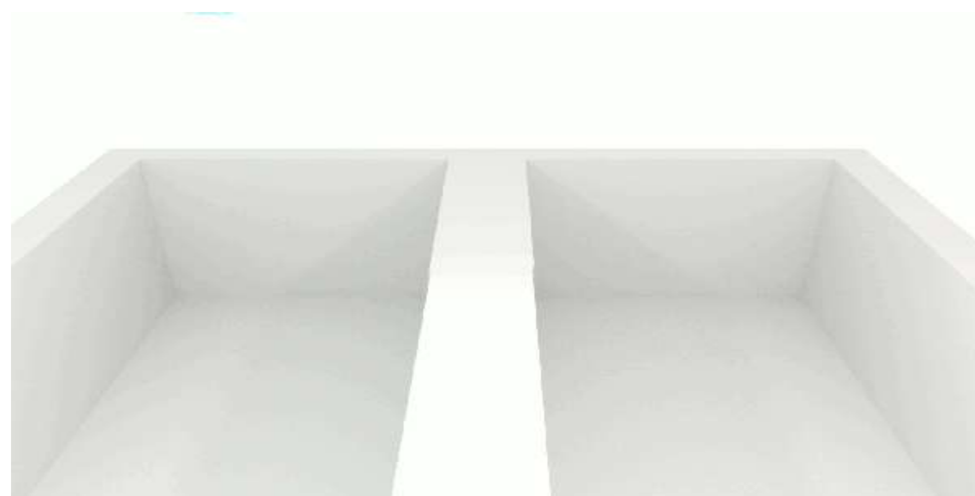



$$\partial_t \vec{V} + (\vec{V} \cdot \vec{\nabla})\vec{V} + \frac{\vec{\nabla} P}{\rho_0} = \frac{\eta}{\rho_0} \nabla^2 \vec{V} + \mathcal{O}(K_N^2)$$

Ideal fluid $\sim \mathcal{O}(K_n^0)$ $\sim \mathcal{O}(K_n)$ Higher order

$\eta \rightarrow$ shear viscosity



# Navier-Stokes equations

Navier

Stokes
~ 1845

Valid when $K_N \ll 1$

$$\partial_t \vec{V} + (\vec{V} \cdot \vec{\nabla})\vec{V} + \frac{\vec{\nabla} P}{\rho_0} = \frac{\eta}{\rho_0} \nabla^2 \vec{V}$$

- Notoriously hard **nonlinear** problem to solve in three dimensions.

- Turbulence

da Vinci, 1508-1513

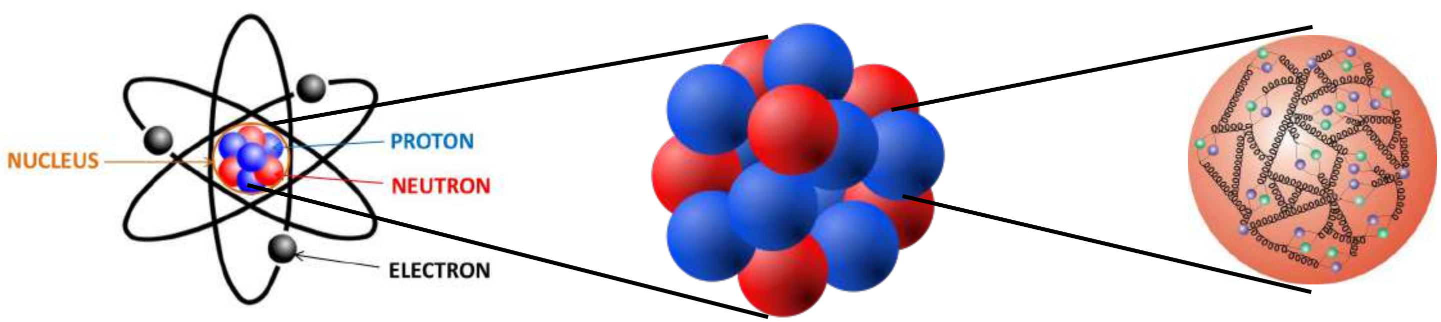
Millennium Prize Problem

*"Global existence and smoothness of solutions of the Navier-Stokes equations"*

# Frontiers of Fluid Dynamics Behavior
## "in the lab"

Redefine "macro" scales ➡ Nuclear/Particle Physics

Atom Nucleus Proton




**LIQUID??**

Heavy-Ion Collisions

$10^{-10}\,\mathrm{m}$ → $\mathrm{fm} = 10^{-15}\,\mathrm{m}$

Natural units: $\hbar = c = k_B = 1$

# Frontiers of Fluid Dynamics Behavior "in the sky"

## Fluid dynamics in strong gravitational fields

## Neutron Star Mergers

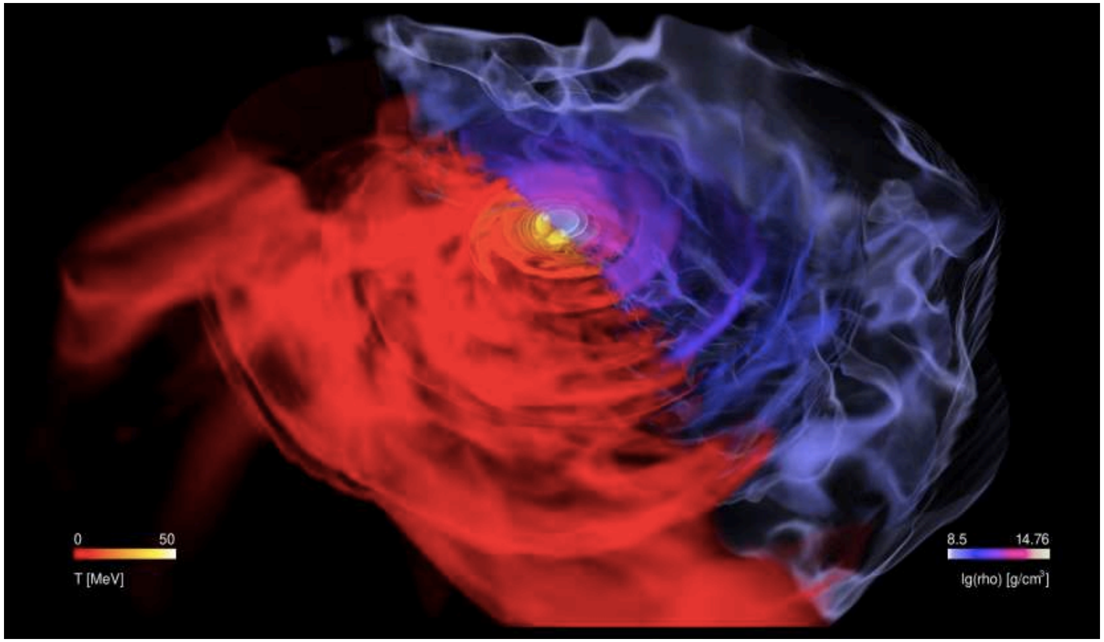
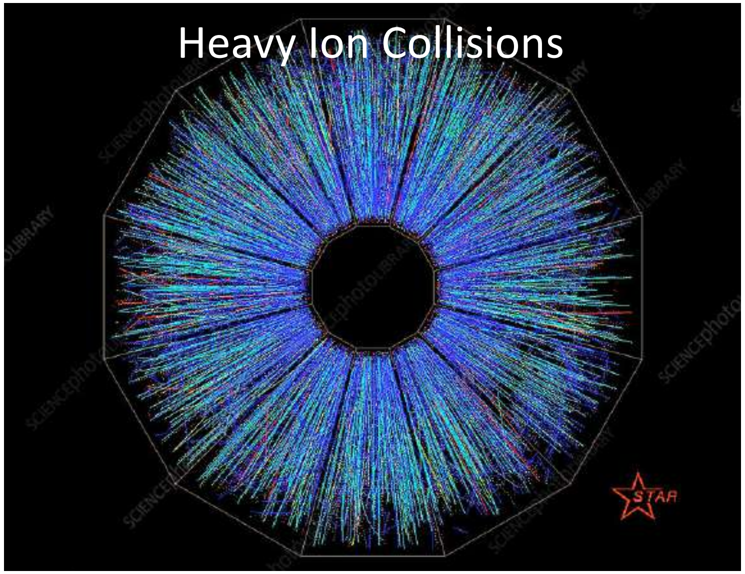


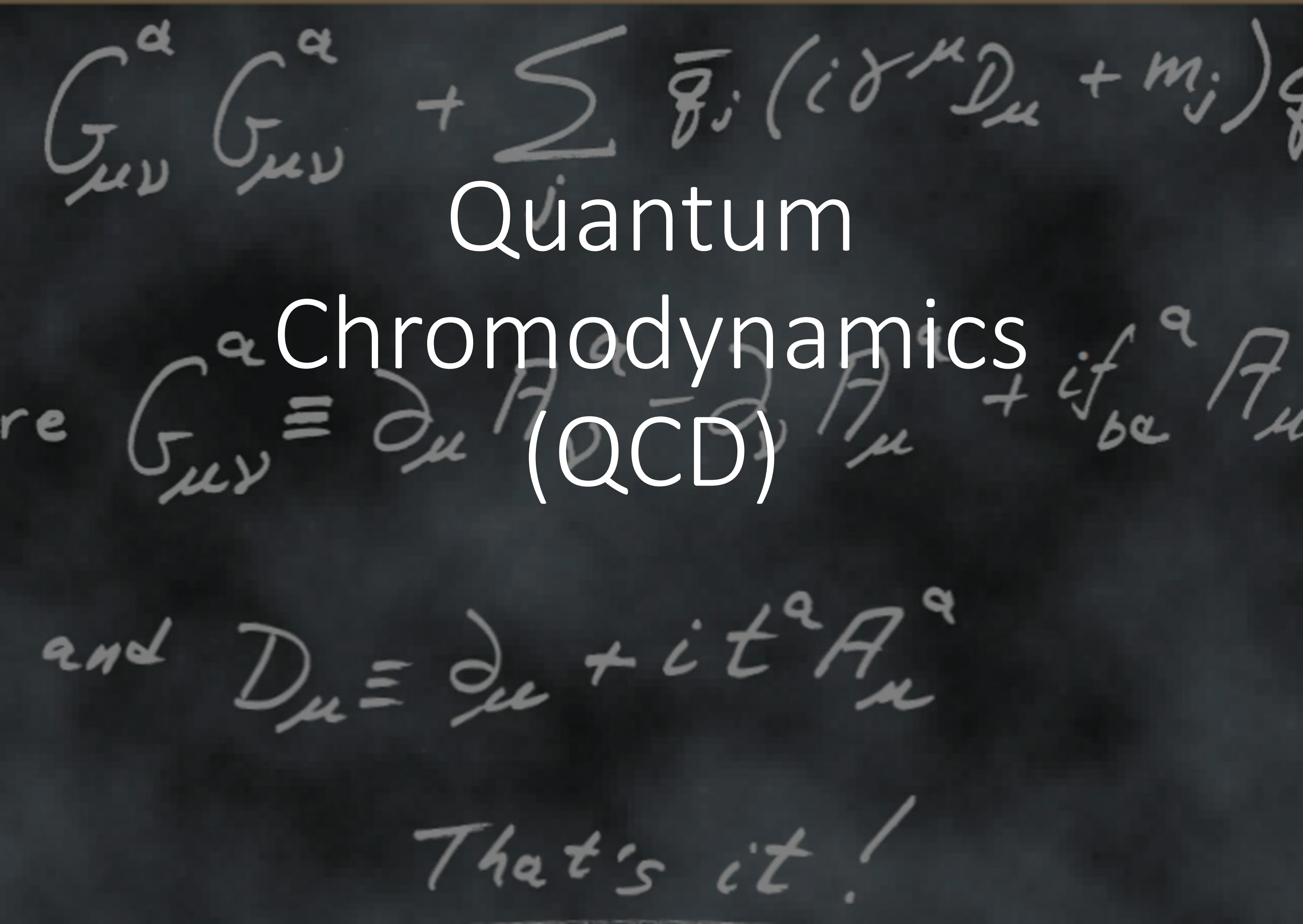

Figure by Most et al., PRL (2019)

In this talk we will consider both frontiers

Let us start with the frontier defined at very small distances …



# Quantum Chromodynamics (QCD)

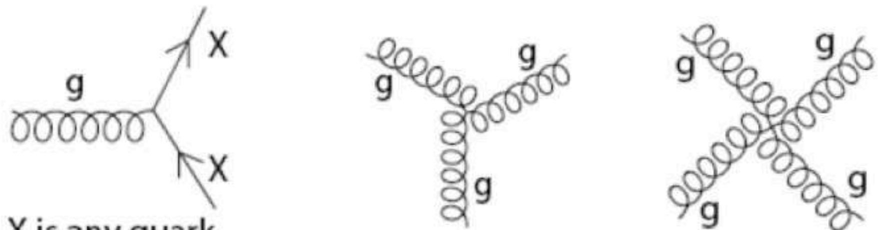


# Quantum Chromodynamics (QCD)

2004

*The fundamental theory of the strong interactions*

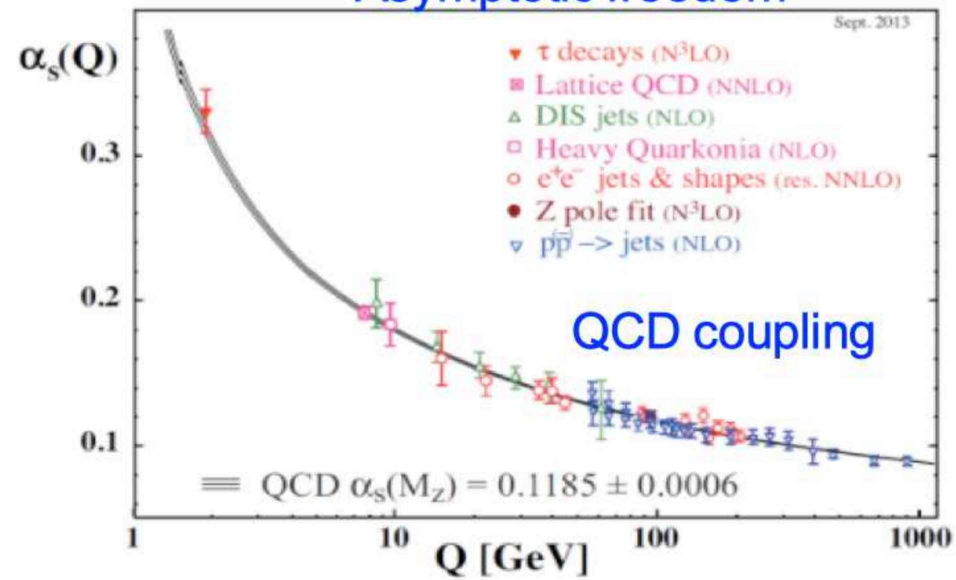
## Non-Abelian gauge theory




X is any quark.

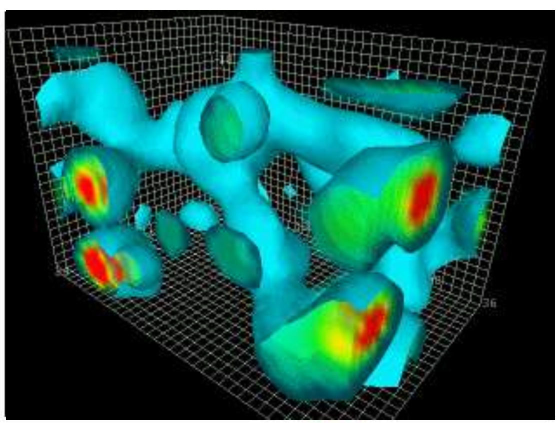
gluon self-interactions

## Asymptotic freedom




QCD coupling

## The QCD vacuum

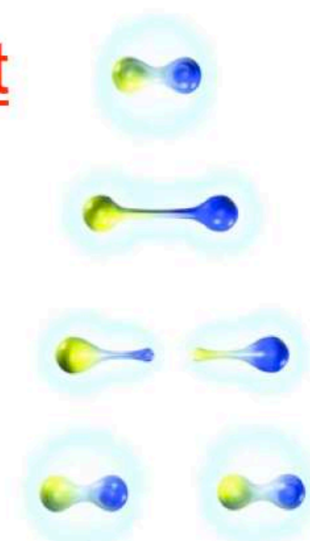


from D. B. Leinweber

## Color confinement

Strong coupling phenomenon !!!

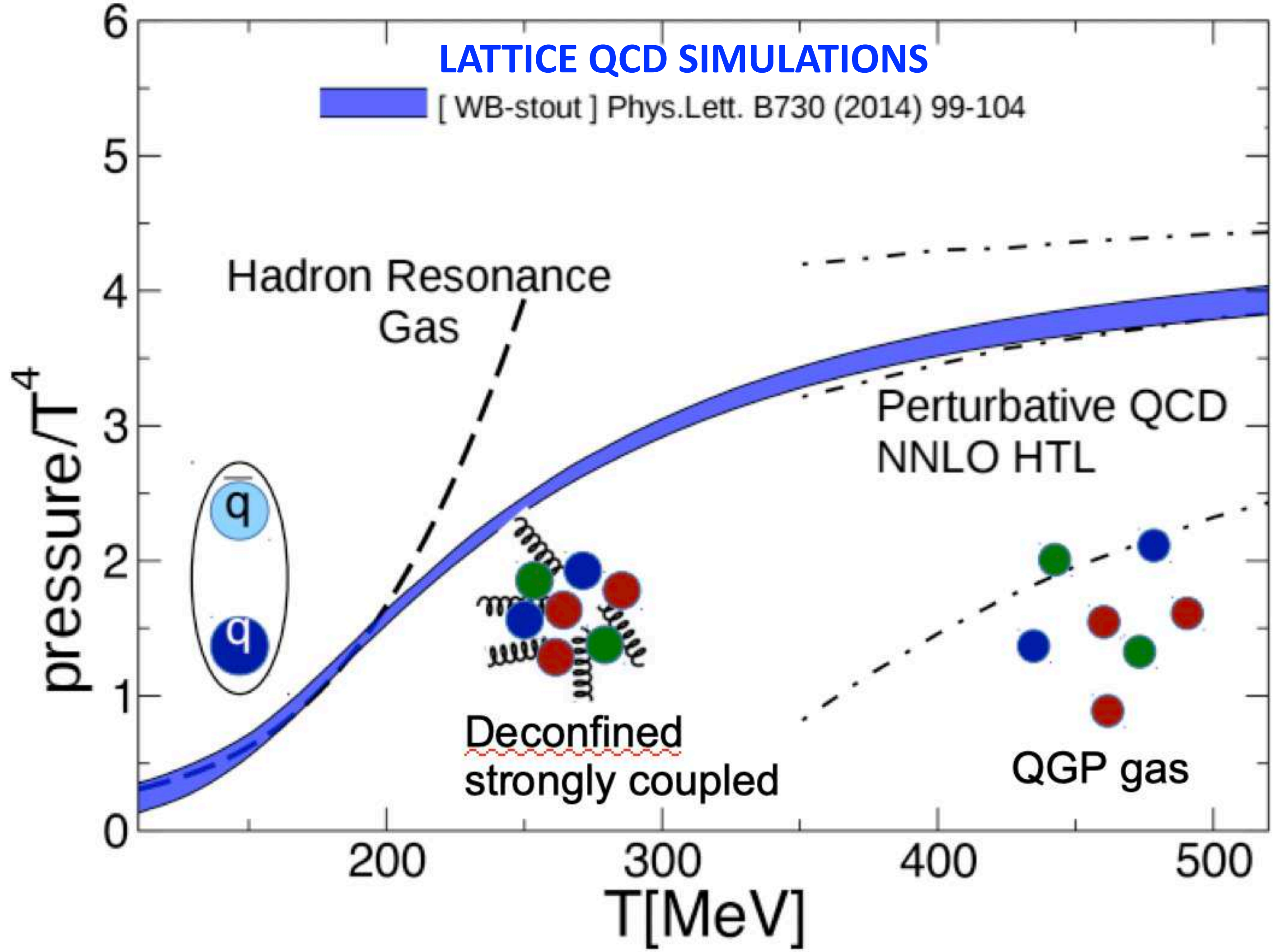
Quarks and gluons can never be truly free



Hadron (pion)

# Quark-Gluon Plasma (QGP) in equilibrium

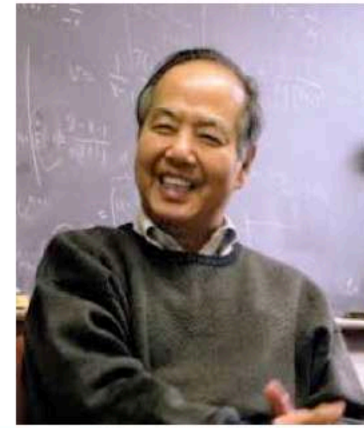
**QCD phase transition in the early universe was a crossover**



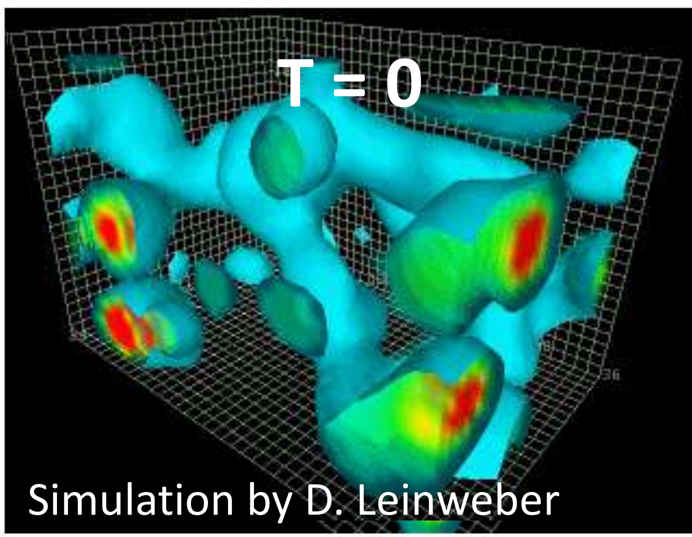
# Out-of-equilibrium properties of QCD?

*In order to study the question of the QCD "vacuum", we must turn to a different direction, we should investigate some "bulk" phenomena by distributing large energy over a large volume.*

T. D. Lee, Rev. Mod. Phys. 47 (1975)



QCD vacuum
Zero Temperature





# Heavy Ion Collisions in a Nutshell – The Little Bang

The way to study non-equilibrium hot QCD phenomena in the lab

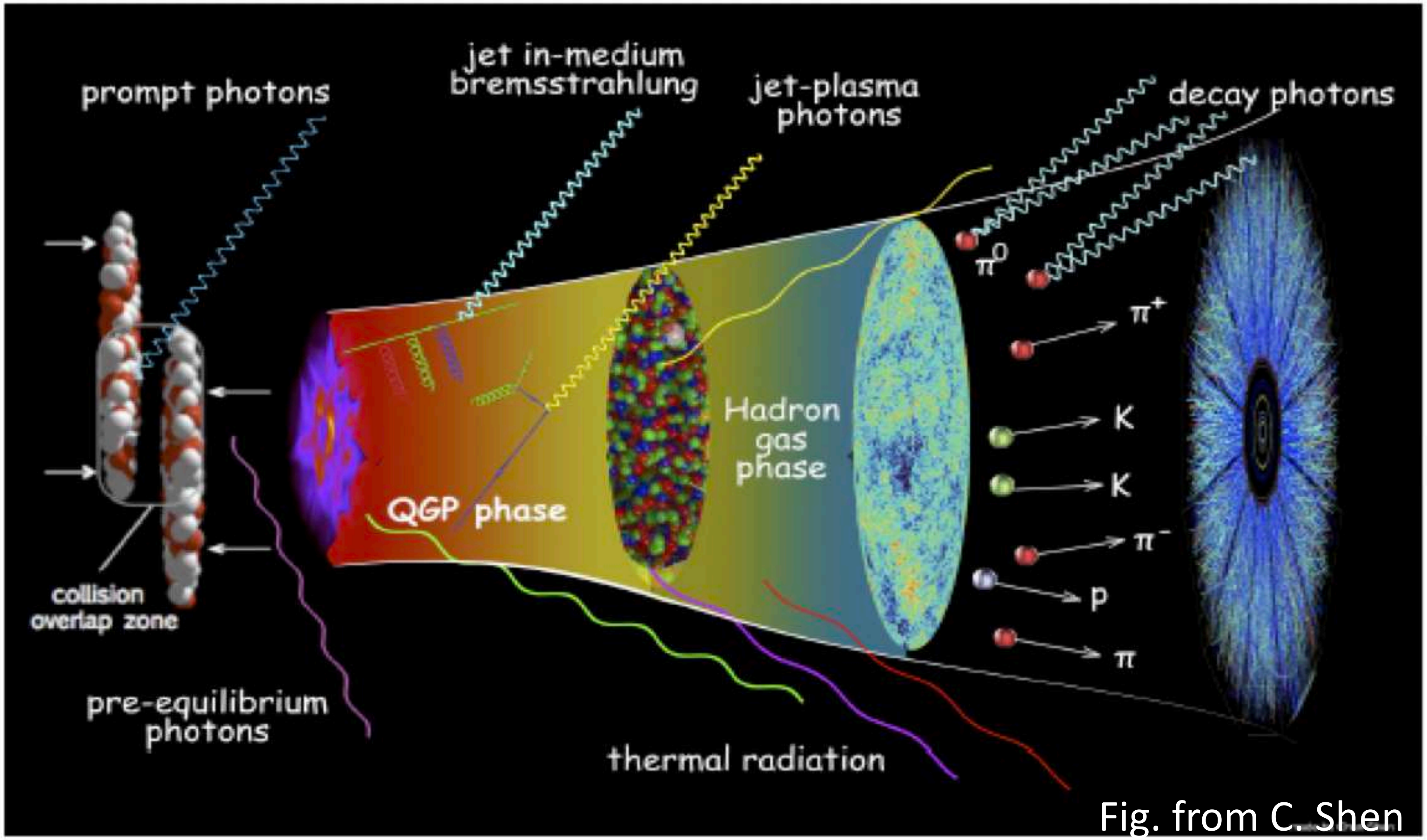



Fig. from C. Shen

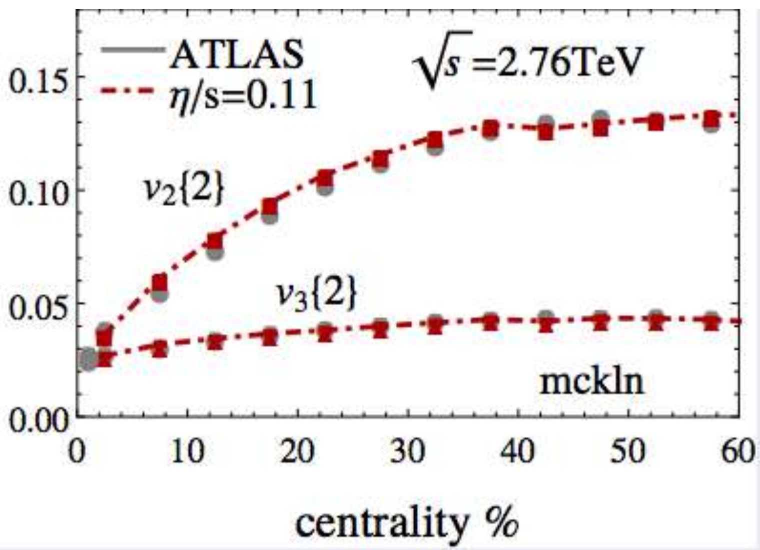
QGP = The hottest, densest, smallest, most perfect liquid

# Nearly perfect fluidity: An emergent property of QCD

## QGP behaves as a strongly coupled liquid !!!!

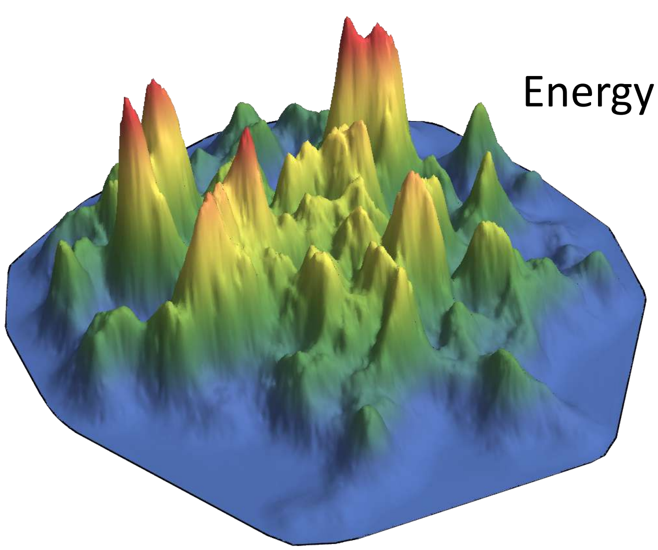
Shear viscosity to entropy density ratio

$$\eta/s \sim 0.05 - 0.2$$




Noronha-Hostler, Betz, JN, Gyulassy, PRL 2016

## Relativistic Hydrodynamics

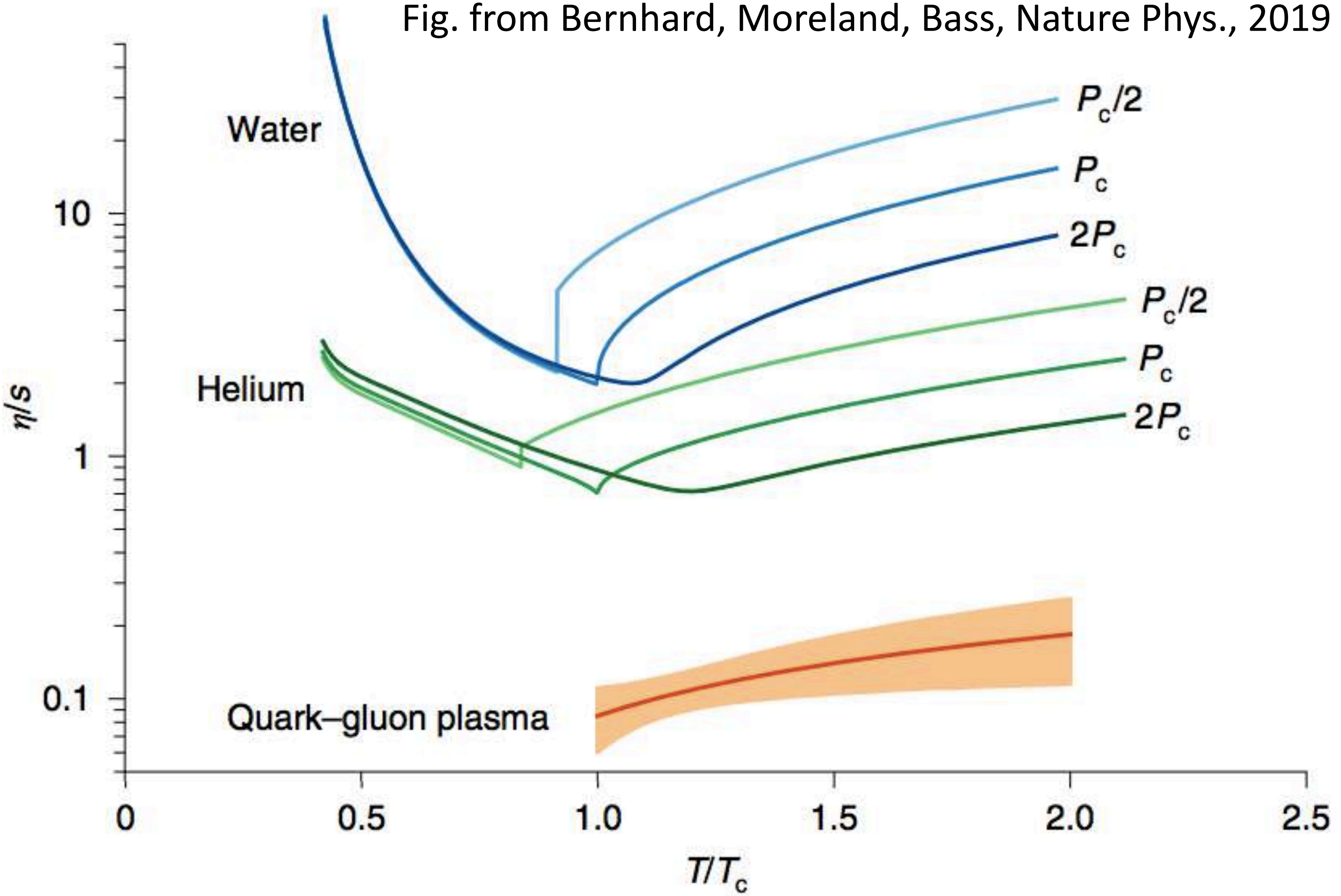


Energy density

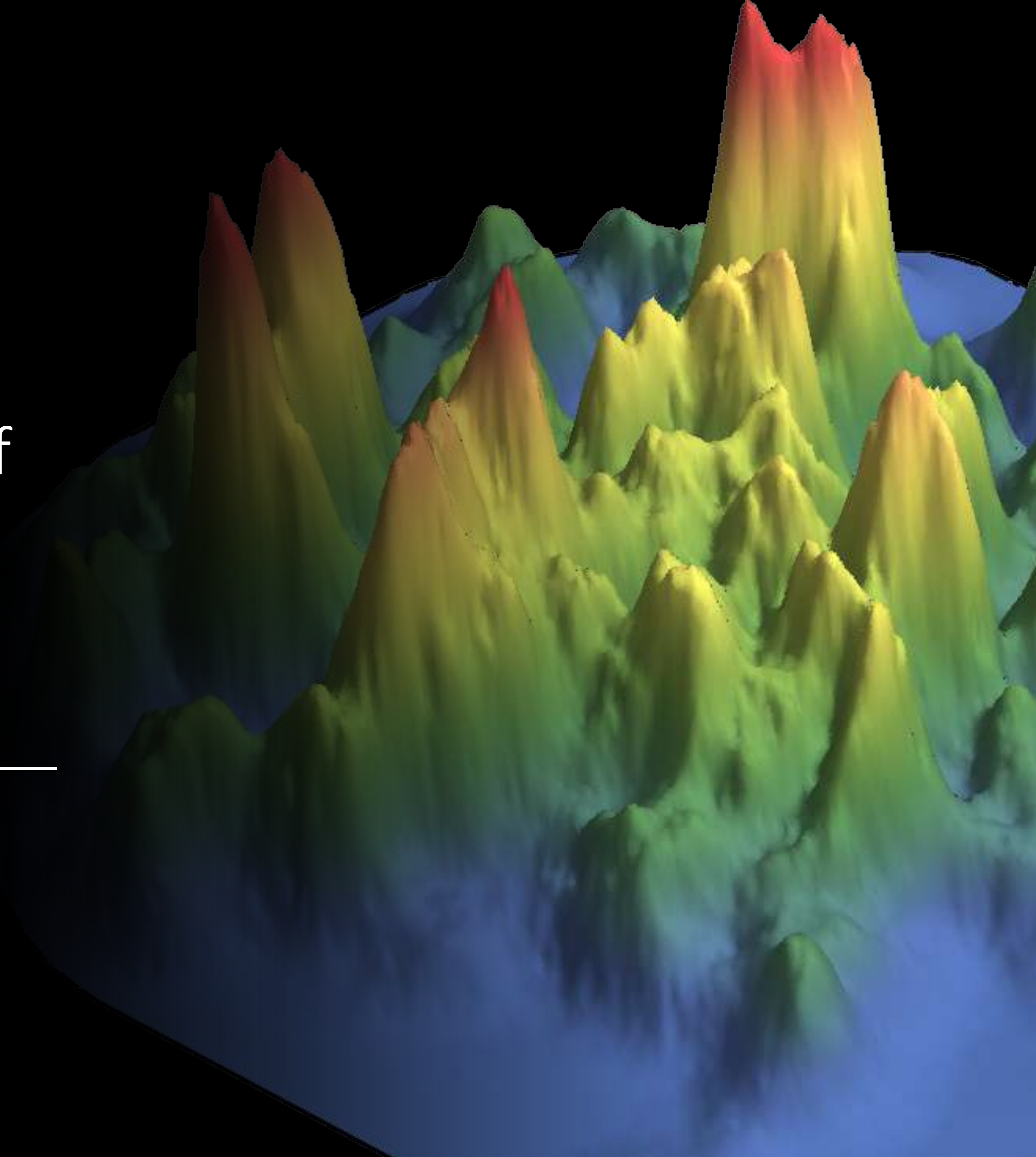
# (Nearly) Perfect fluidity: an emergent property of QCD

## QGP behaves as a strongly coupled relativistic fluid !!!

Fig. from Bernhard, Moreland, Bass, Nature Phys., 2019



# The unreasonable effectiveness of hydrodynamics in heavy-ion collisions



# How does one describe fluid dynamics in relativity?

Conservation laws
(energy and momentum)

$$\nabla_\mu T^{\mu\nu} = 0$$

energy density
energy flux

$$T^{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}$$

shear stress
pressure
momentum density
momentum flux

Energy-Momentum Tensor

$$T^{\mu\nu} = \underbrace{(\varepsilon + P(\varepsilon))u^\mu u^\nu + P(\varepsilon)g^{\mu\nu}}_{\text{ideal part}} + \Pi^{\mu\nu}$$

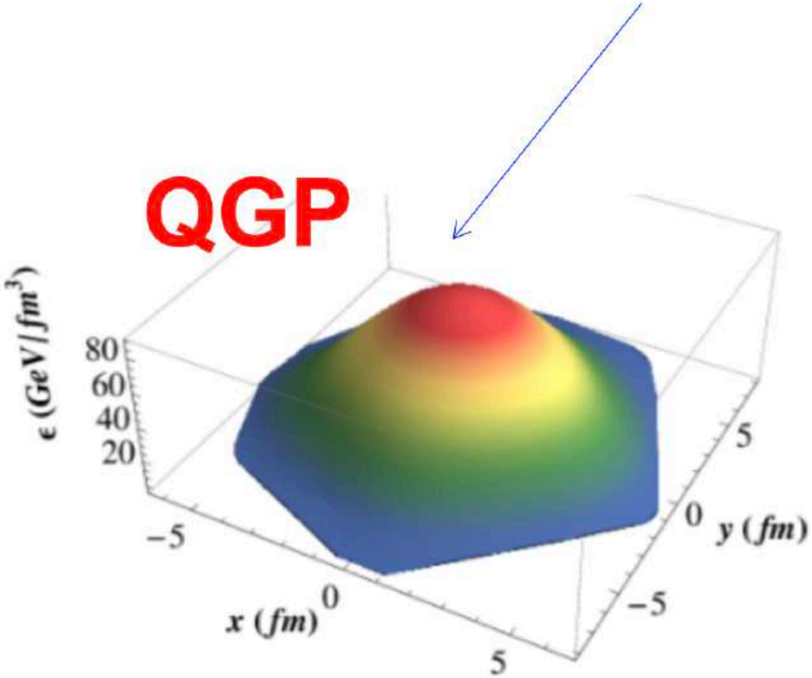
ideal part
dissipative part ($\Pi^{\mu\nu}$)

What about heavy-ion collisions?

# Quark-Gluon Plasma and Relativistic Fluid Dynamics

At first (< 2010), it seemed that hydrodynamics was justifiable

**<u>Very smooth fluid</u> over nuclear length scales**




**near equilibrium dynamics**

macro $\partial\varepsilon/\varepsilon_0 \sim 1/L$

micro $\ell \sim 1/T \sim 1/\Lambda_{QCD}$

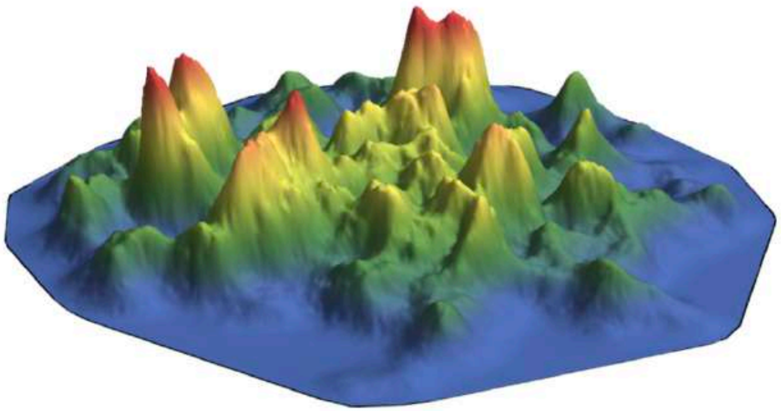
**Knudsen number**

$$K_N \sim \ell\,\partial\varepsilon < 0.1$$

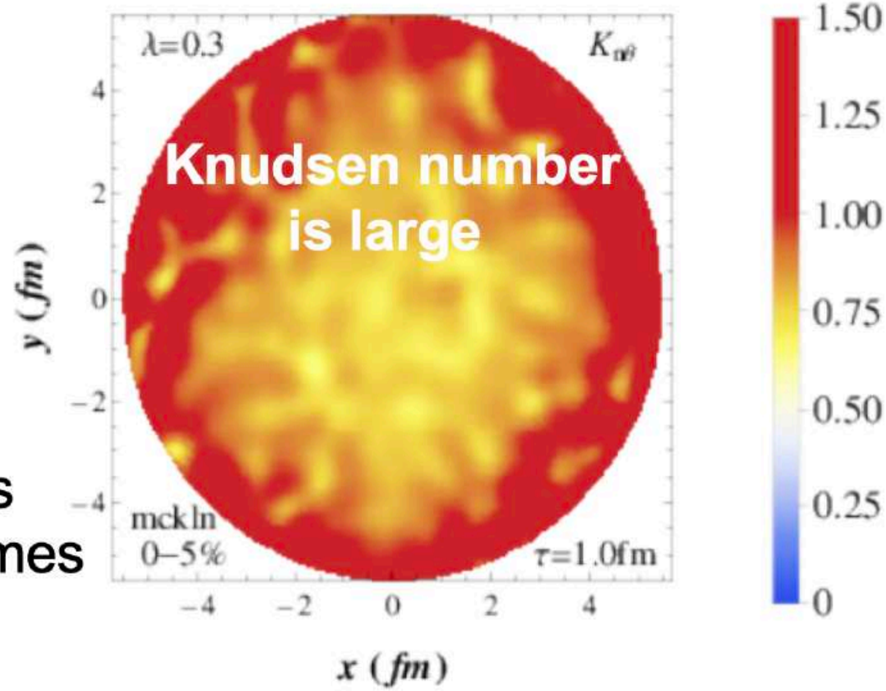
Fluid dynamics at scales of the size of a large nucleus

# Reality is much more complicated …

QGP energy density



J. Noronha-Hostler, JN, M. Gyulassy, PRC 2016

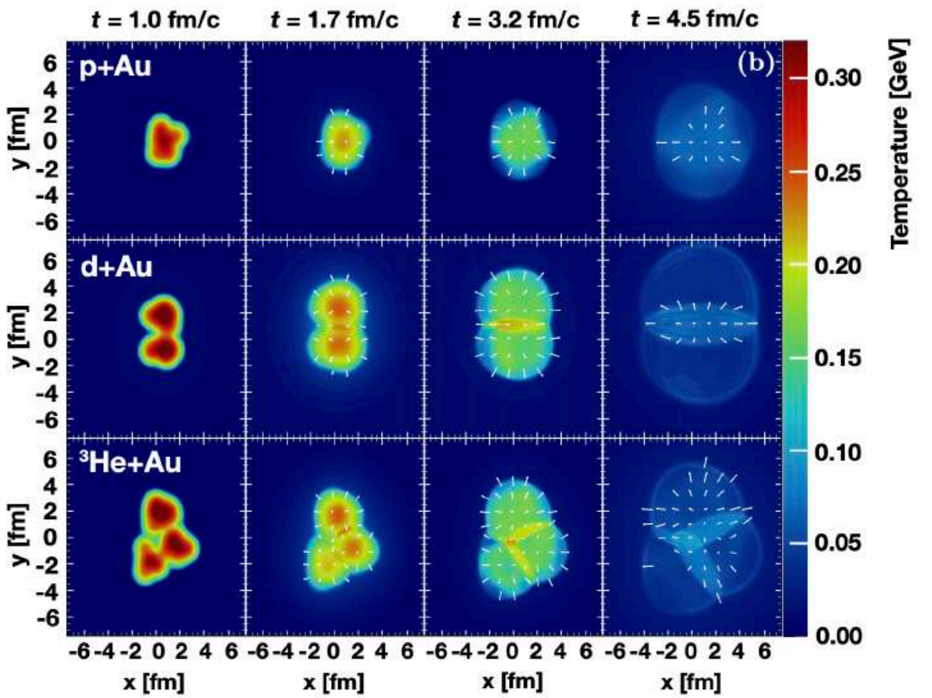



- Unavoidable quantum fluctuations
- Large spatial gradients at early times

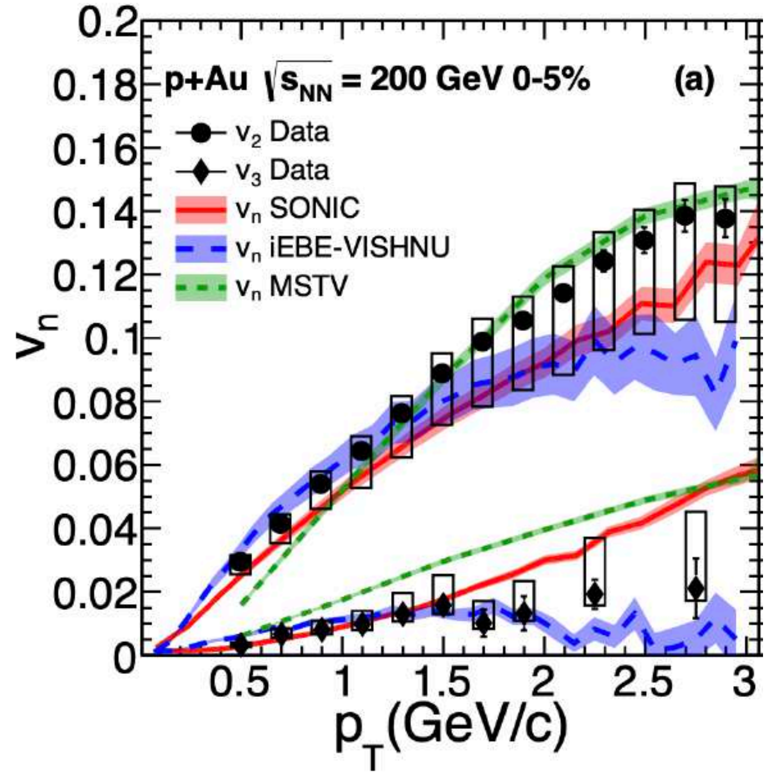
PARADOX: Knudsen number is large but “hydro” still works

This issue must be understood …

# What is the smallest droplet of QCD liquid?

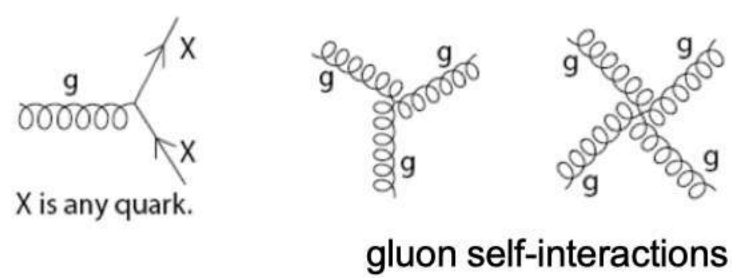



PHENIX Collab., Nature Physics (2019)




Collective behavior at scales of the size of the proton !

How does liquid behavior emerge from QCD




?

# What goes into relativistic viscous hydro simulations of heavy-ion collisions?

$\frac{\eta}{s} = 0.16$

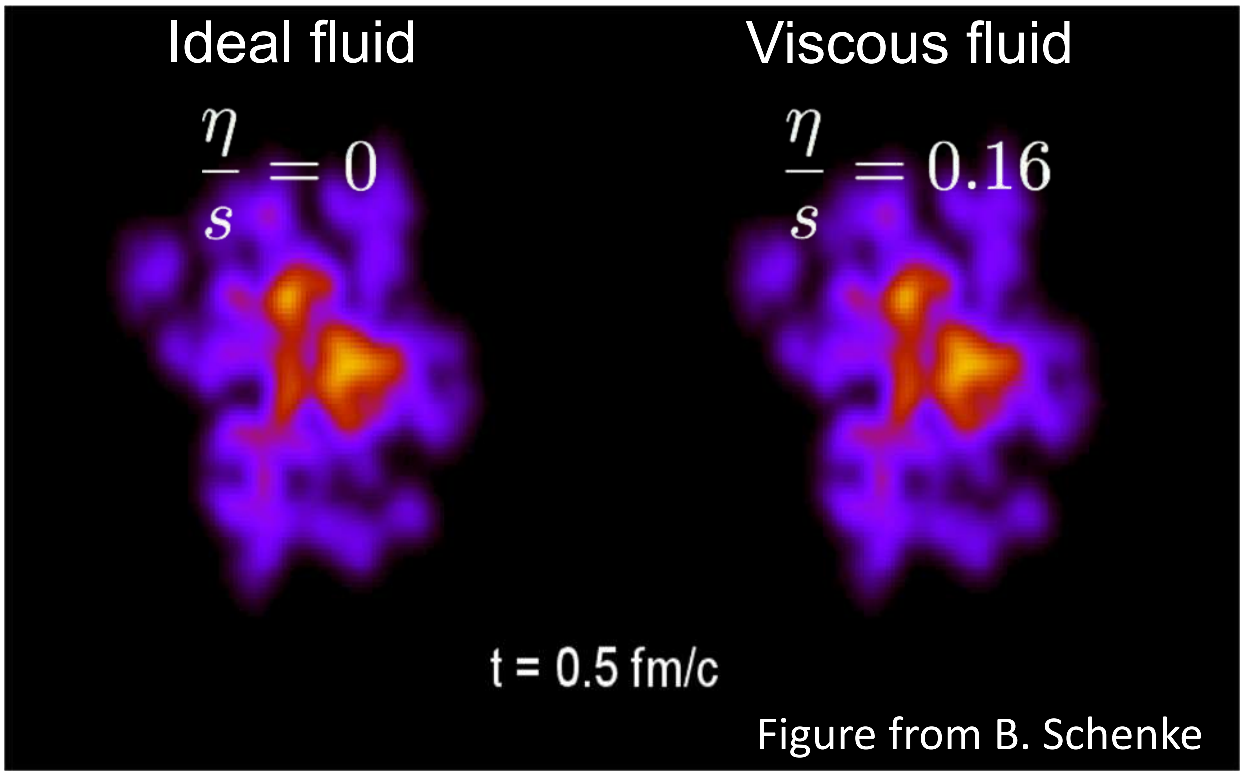
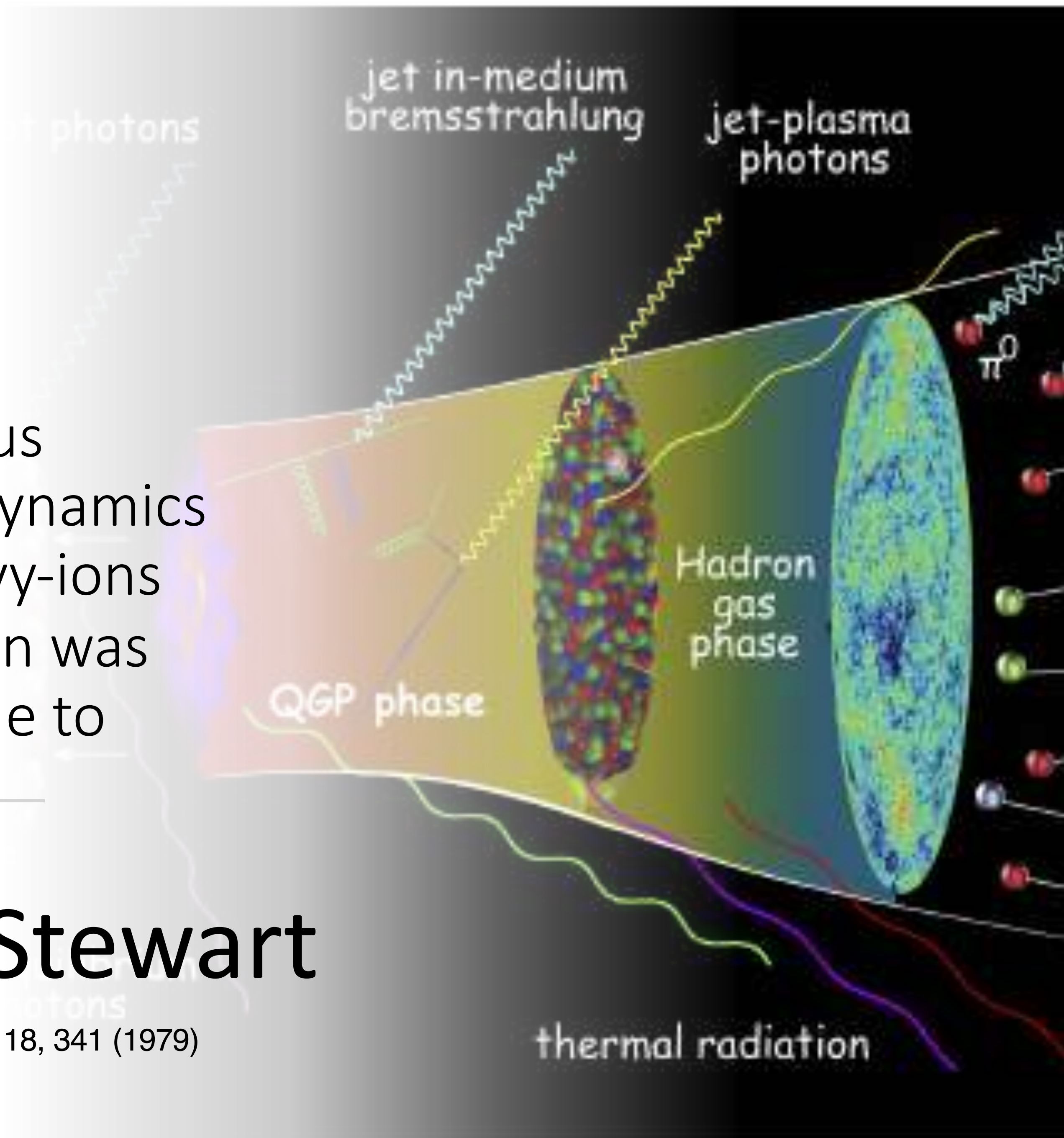



Figure from B. Schenke

Every single viscous relativistic hydrodynamics simulation in heavy-ions you have ever seen was based on ideas due to

# Israel and Stewart

Israel, Stewart, Ann. Phys. 118, 341 (1979)



# “Hydro” in HIC is not simple textbook hydro

## Israel-Stewart (IS) theory

Israel, Stewart, Ann. Phys. 118, 341 (1979)

Energy-momentum tensor

Dissipative

$$T_{\mu\nu} \rightarrow \varepsilon, u_\mu, \pi_{\mu\nu}, \Pi$$ as dynamical variables

A theory for **hydrodynamic fields** and **non-hydrodynamic** fields

**Dynamics:** $\nabla_\mu T^{\mu\nu} = 0$ (energy-momentum conservation)

$$u^\lambda \nabla_\lambda \Pi + F(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0$$ (bulk)

$$u^\lambda \nabla_\lambda \pi^{\mu\nu} + F^{\mu\nu}(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0$$ (shear)

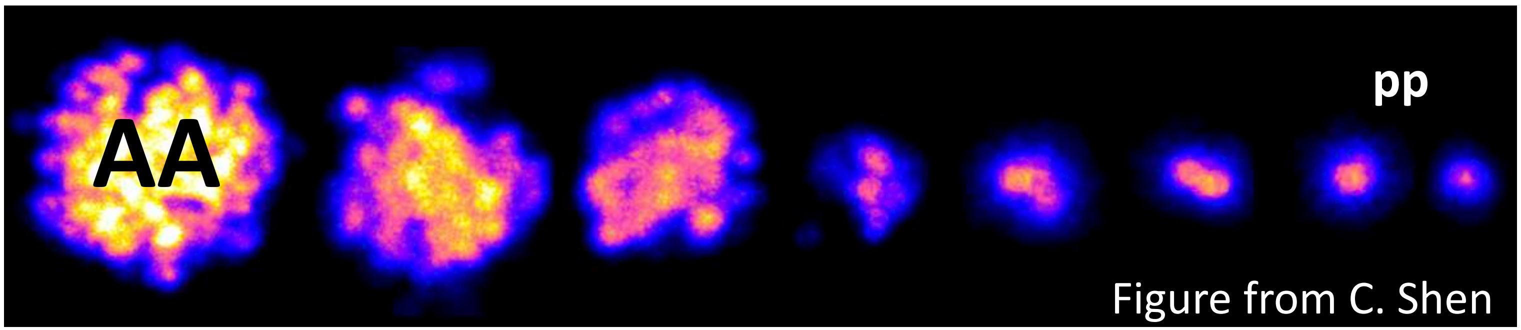
Many terms!
Many coefficients!
14 EOM

+ equation for diffusion

*“Israel-Stewart eqs. make sense even far from equilibrium ”*

Implicit assumption made in current hydro simulations

Decreasing the system size: from AA to pA and pp




Increasing the theoretical uncertainty

- When does this stop behaving hydrodynamically?
- Are there fundamental constraints that must be fulfilled?

*"Israel-Stewart eqs. were proven to be causal, under any circumstances, by Israel and Stewart (1979) themselves"*

*“Israel-Stewart eqs. were proven to be causal, under any circumstances, by Israel and Stewart (1979) themselves”*

**False**

This rating indicates that the primary elements of a claim are demonstrably false.

*“Israel-Stewart eqs. were proven to be causal, under any circumstances, by Israel and Stewart themselves”*

**False**

- Israel and Stewart (1979) proved causality only in the linearized regime around equilibrium.

**True**

- Such results say NOTHING about the theory in the far from equilibrium regime probed in heavy-ions.

$$\frac{\pi_{\mu\nu}}{\varepsilon + P}, \frac{\Pi}{\varepsilon + P} \sim \mathcal{O}(1)$$

- Known to occur in the initial state and at the edges of QGP

Niemi, Denicol, (2014), C. Shen et al. (iEBE-VISHNU), Comput. Phys. Commun. (2016); Niemi, Denicol, (2014); Bzdak, Schenke, Tribedy, Venugopalan, PRC (2013); Gallmeister, Niemi, Greiner, Rischke, PRC (2018); Schenke, Shen, Tribedy, PRC (2020); Shen, Yan, arxiv: 2010.12377

# Relativistic fluids far from equilibrium: Constraints

Proof: Bemfica, Disconzi, Hoang, JN, Radosz, arXiv:2005.11632

Causality in the nonlinear regime: shear + bulk effects

$$\tau_\Pi u^\mu \nabla_\mu \Pi + \Pi = -\zeta \nabla_\mu u^\mu - \delta_{\Pi\Pi} \Pi \nabla_\mu u^\mu - \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu},$$

$$\tau_\pi \Delta^{\mu\nu}_{\alpha\beta} u^\lambda \nabla_\lambda \pi^{\alpha\beta} + \pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} - \delta_{\pi\pi} \pi^{\mu\nu} \nabla_\alpha u^\alpha - \tau_{\pi\pi} \pi_\alpha^{\langle\mu} \sigma^{\nu\rangle\alpha} - \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu},$$

Necessary constraints for meaningful evolution

$$(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{1}{2}\tau_{\pi\pi}|\Lambda_1| \geq 0 \tag{4a}$$

$$\varepsilon + P + \Pi - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\Lambda \tag{4b}$$

$$\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_a + \Lambda_d) \geq 0 \tag{4c}$$

$$\varepsilon + P + \Pi + \Lambda_a - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \quad d \tag{4d}$$

$$\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d + \frac{1}{6\tau_\pi}[2\eta + \quad + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} + (\varepsilon + P + \Pi + \tag{4e}$$

$$\varepsilon + P + \Pi + \Lambda_d - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d - \frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] - \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} - (\varepsilon + P + \Pi + \Lambda_d)c_s^2 \geq 0, \tag{4f}$$


$$(\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 \geq 0, \tag{5a}$$

$$(2\eta + \lambda_{\pi\Pi}\Pi) - \tau_{\pi\pi}|\Lambda_1| > 0, \tag{5b}$$

$$\tau_{\pi\pi} \leq 6\delta_{\pi\pi}, \tag{5c}$$

$$\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi} \geq 0, \tag{5d}$$

$$\frac{1}{3\tau_\pi}[4\eta + 2\lambda_{\pi\Pi}\Pi + (3\delta_{\pi\pi} + \tau_{\pi\pi})\Lambda_3] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi} + |\Lambda_1| + \Lambda_3 c_s^2 + \frac{\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3 + |\Lambda_1|)^2}{\varepsilon + P + \Pi - |\Lambda_1| - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3} \leq (\varepsilon + P + \Pi)(1 - c_s^2), \tag{5e}$$

$$\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (\tau_{\pi\pi} - 6\delta_{\pi\pi})|\Lambda_1|] + \frac{\zeta + \delta_{\Pi\Pi}\Pi - \lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi} + (\varepsilon + P + \Pi - |\Lambda_1|)c_s^2 \geq 0, \tag{5f}$$

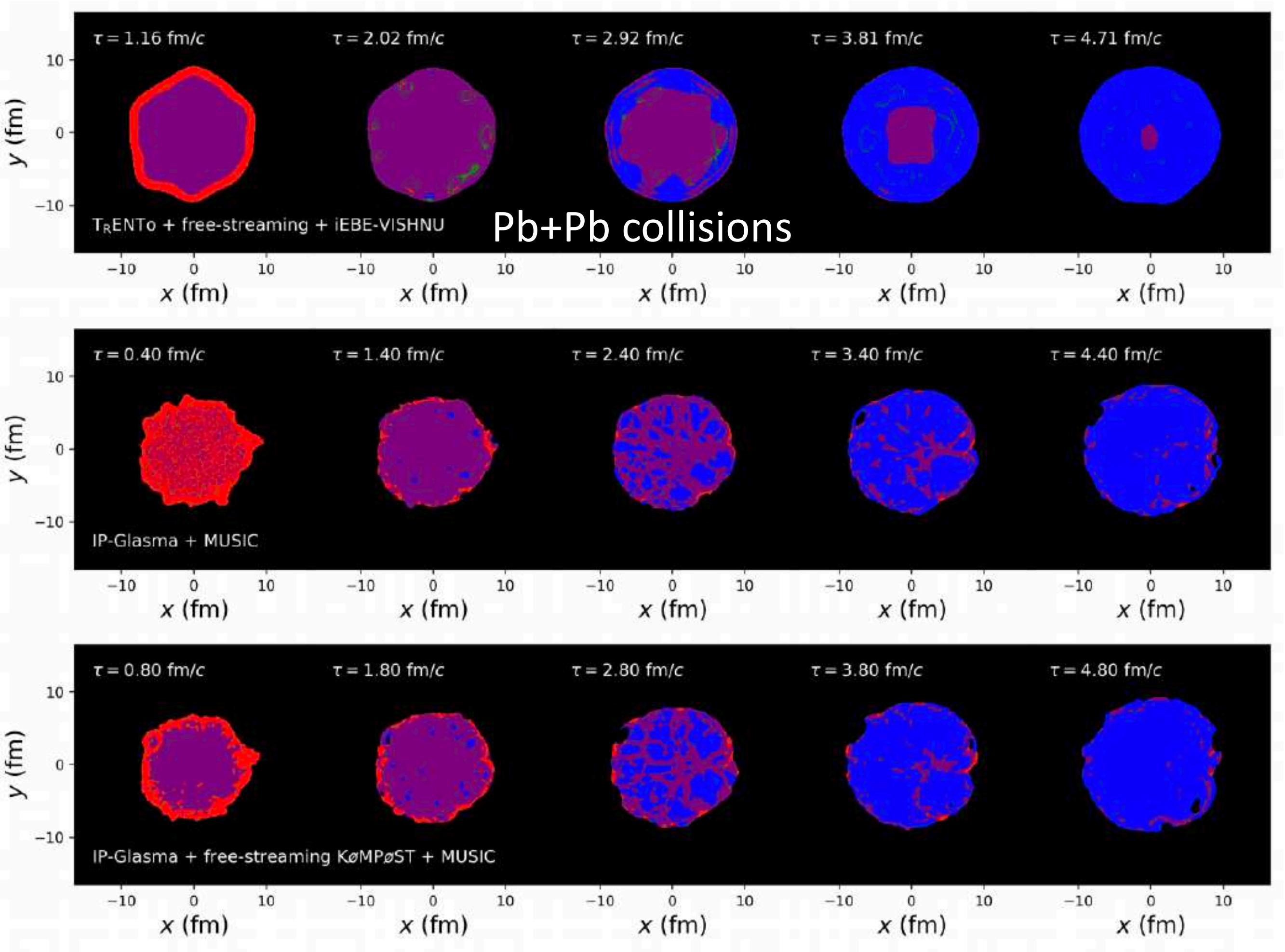
$$1 \geq \frac{\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3 + |\Lambda_1|)^2}{\left[\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right]^2} \tag{5g}$$

$$\frac{1}{3\tau_\pi}[4\eta + 2\lambda_{\pi\Pi}\Pi - (3\delta_{\pi\pi} + \tau_{\pi\pi})|\Lambda_1|] + \frac{\zeta + \delta_{\Pi\Pi}\Pi - \lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi} + (\varepsilon + P + \Pi - |\Lambda_1|)c_s^2 \geq \frac{(\varepsilon + P + \Pi + \Lambda_2)(\varepsilon + P + \Pi + \Lambda_3)}{3(\varepsilon + P + \Pi - |\Lambda_1|)}\left\{1 + \frac{2\left[\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]}{\varepsilon + P + \Pi - |\Lambda_1|}\right\}, \tag{5h}$$

# Causality Violation in State-of-the-Art Simulations

C. Plumberg, D. Almaalol, T. Dore, JN, J. Noronha-Hostler
arXiv: [2103.15889](2103.15889)

See also C. Cheng and C. Shen
arXiv:2103.09848




Diagnostics:

**Red: Acausal**

**Purple: Unknown**

**Blue: Causal**

Scary

# Causality Violation in State-of-the-Art Simulations

C. Plumberg, D. Almaalol, T. Dore, JN, J. Noronha-Hostler
arXiv: 2103.15889

~30% of initial cells acausal in state-of-the-art simulations

This issue must be fixed!

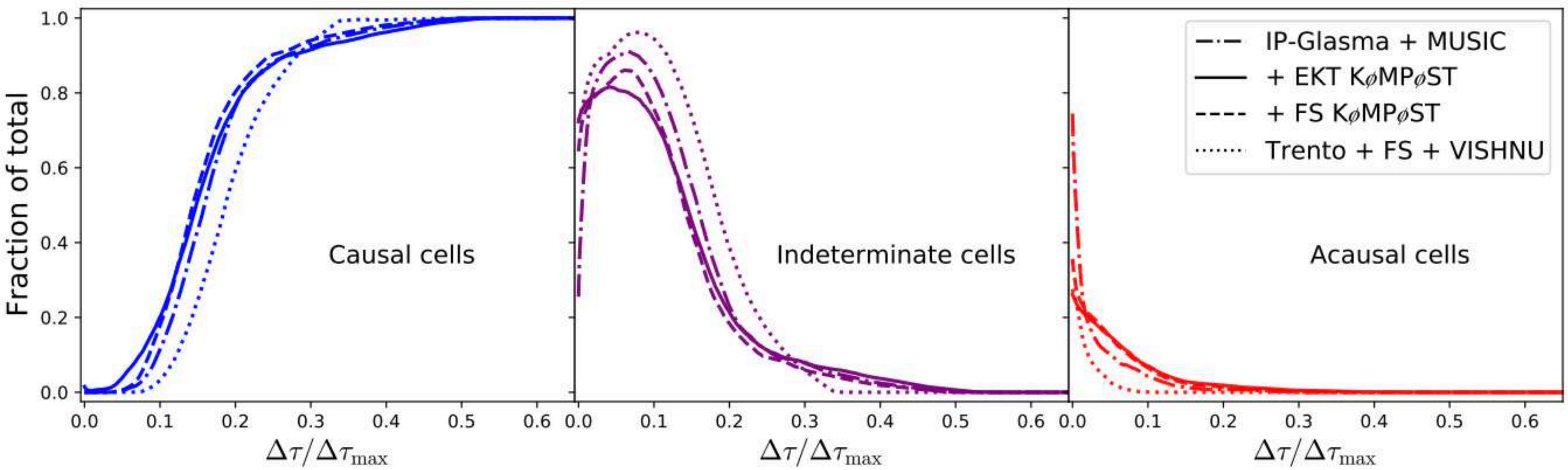
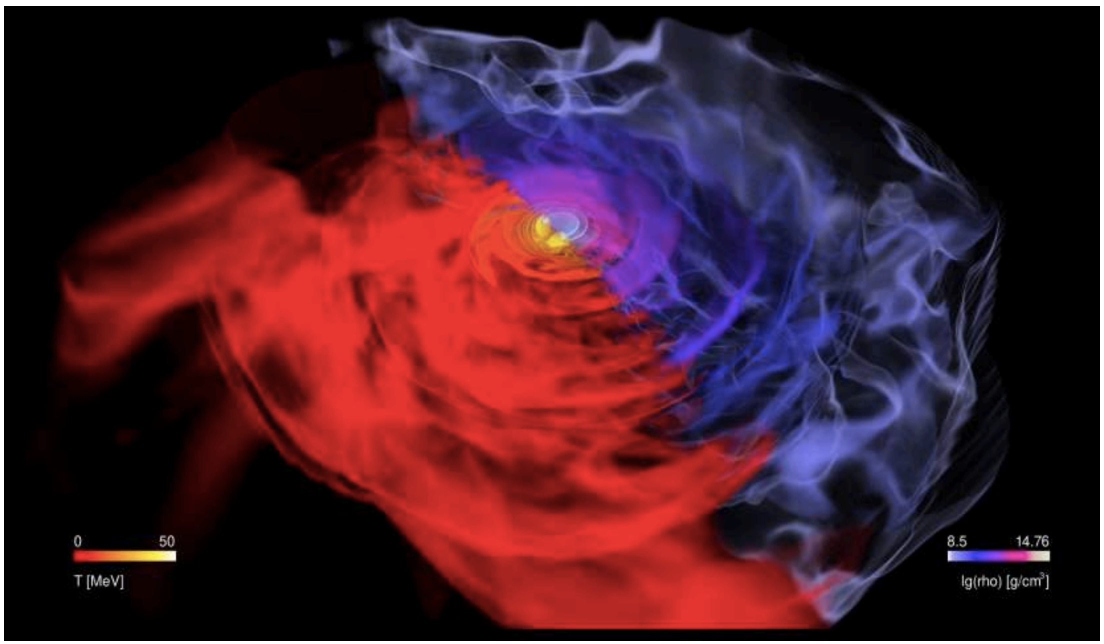



FIG. 2. Fractions of the total number of hydrodynamic cells ($e \geq e_{FO}$) in the causal (left), indeterminate (center), or acausal (right) categories, plotted as functions of the rescaled time evolution in each framework.

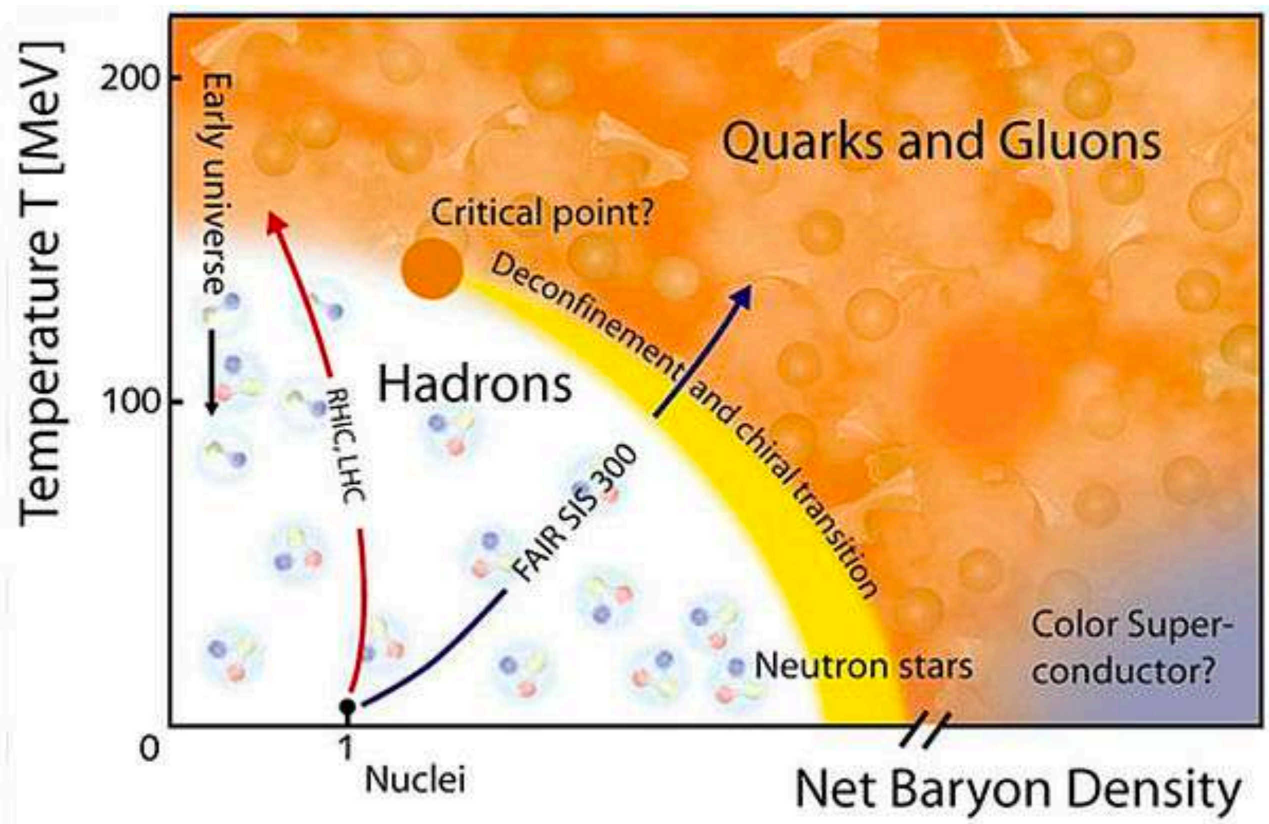
# Let us now consider the other frontier

## Neutron Star Mergers



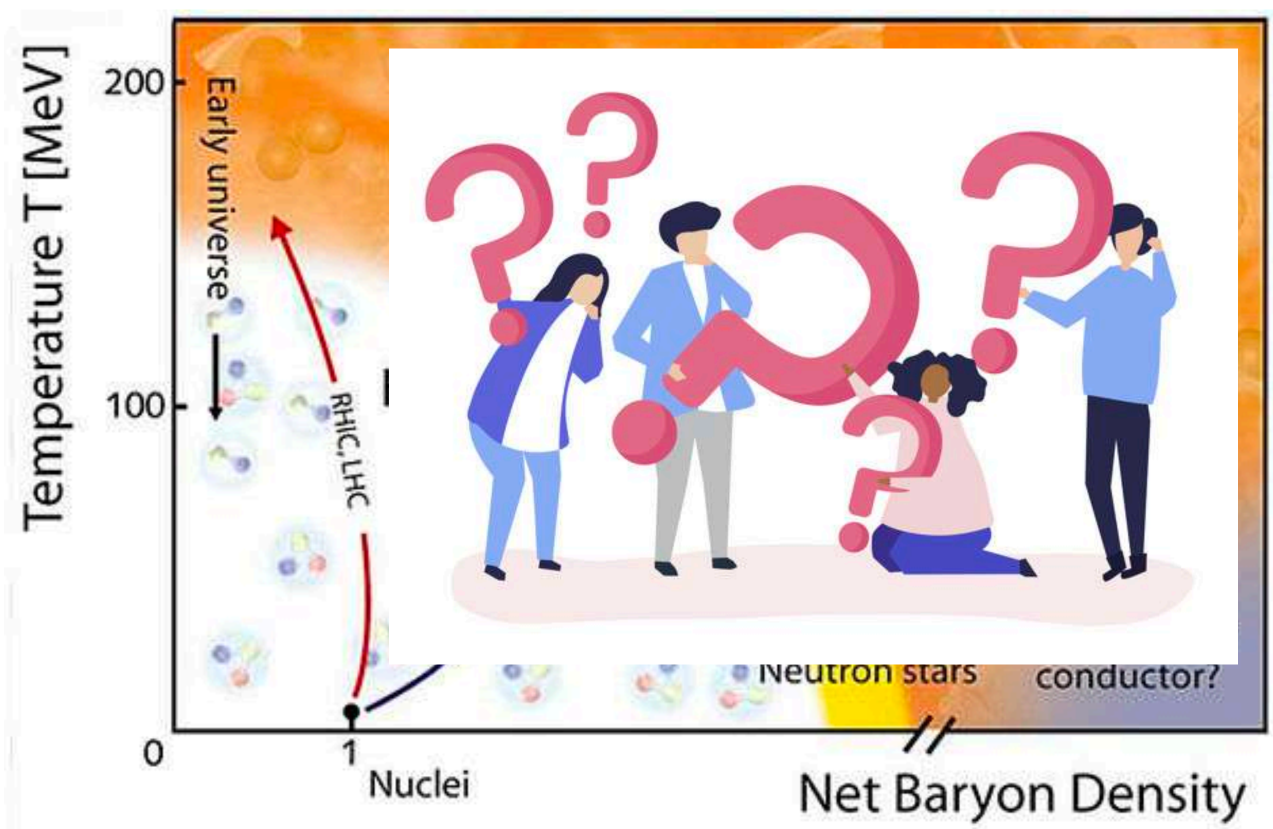
# Properties of matter at extreme baryon densities (neutron stars) remain unknown even in equilibrium

## QCD Phase Diagram



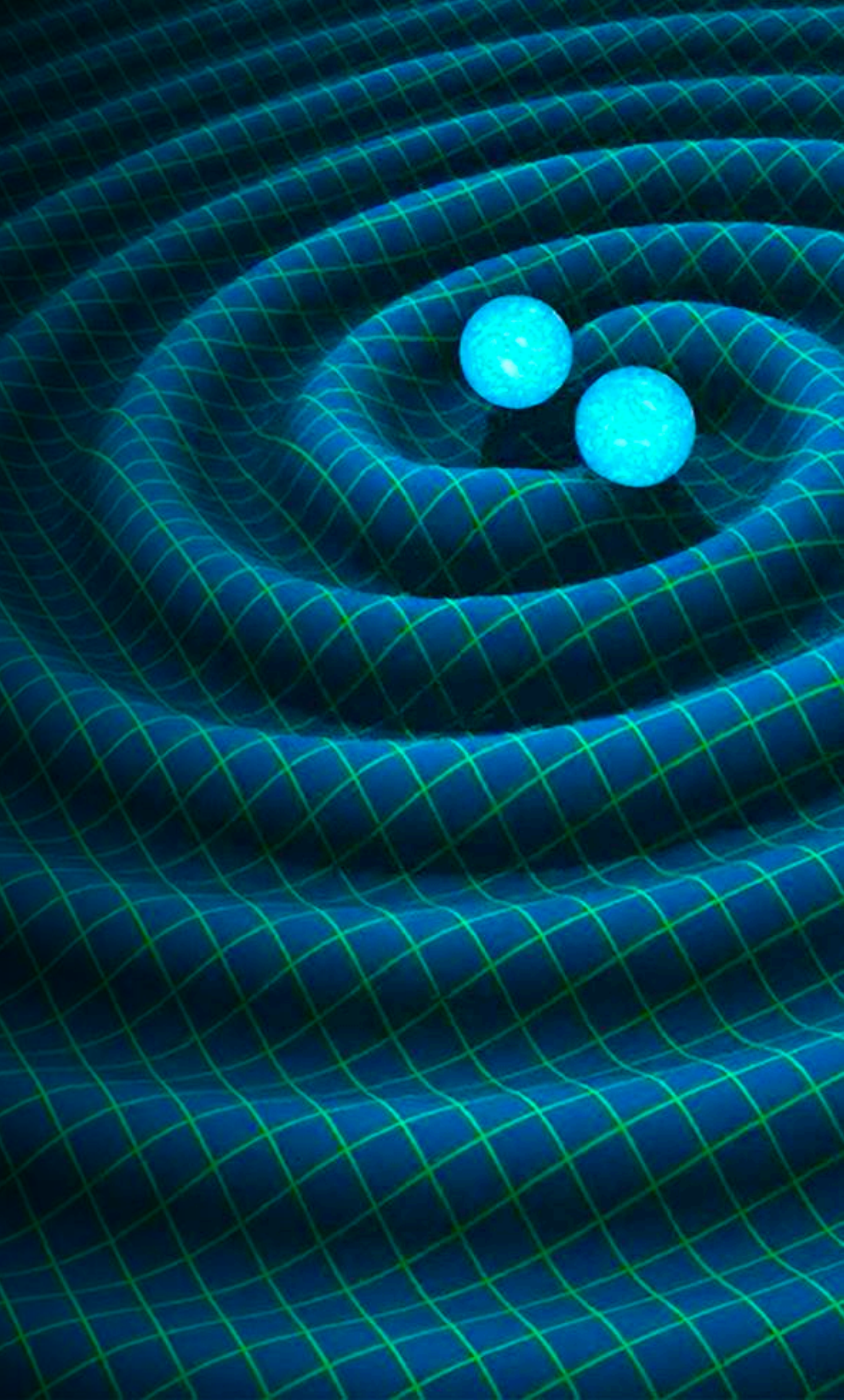
# Properties of matter at extreme baryon densities (neutron stars) remain unknown even in equilibrium

## QCD Phase Diagram



# Gravitational Waves from Neutron Star Mergers

- Equation of state (not this talk)
- Probing viscous ultradense matter when gravity waves



# How does a lump of baryon rich QCD matter flow under strong gravitational fields?

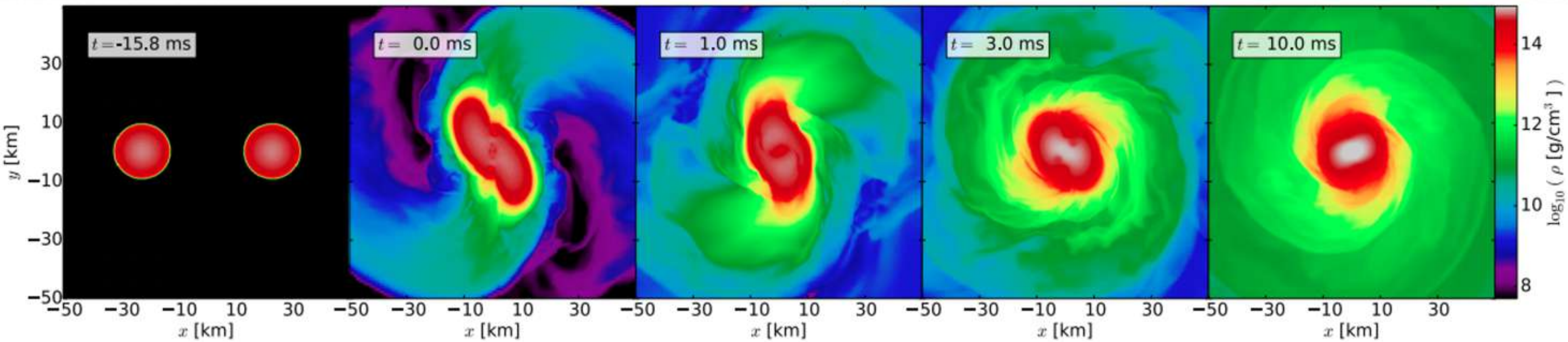



Fig. by L. Rezzolla

New signatures for deconfinement/phase transitions?

e.g. Most et al., PRL (2019)

Viscous fluid dynamics + strong gravitational fields?

Viscous effects in neutron star mergers?

Duez et al PRD (2004), Shibata et al. PRD (2017), Alford et al. PRL (2018)

# Viscous effects in binary neutron-star mergers?

Previous work by: Duez et al PRD (2004), Shibata, Kiuchi, PRD (2017)

Alford, Bovard, Hanauske, Rezzolla, Schwenzer, PRL (2018):

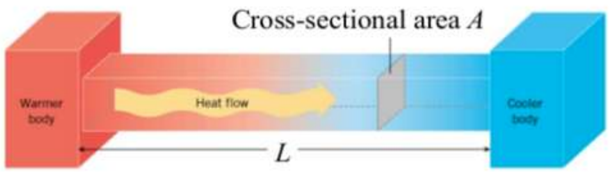
## Post-merger phase

Shear dissipation:


Relevant for trapped neutrinos if T > 10 MeV and gradients at small scales ~ 0.01 km (e.g, turbulence).

Thermal transport:

"heat conductivity"

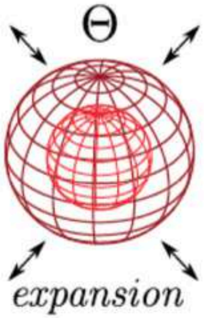



Relevant for trapped electron neutrinos if T > 10 MeV and gradients ~ 0.1 km

# Viscous effects in binary neutron-star mergers?

Alford, Bovard, Hanauske, Rezzolla, Schwenzer, PRL (2018)

## Bulk viscous damping in neutron star mergers



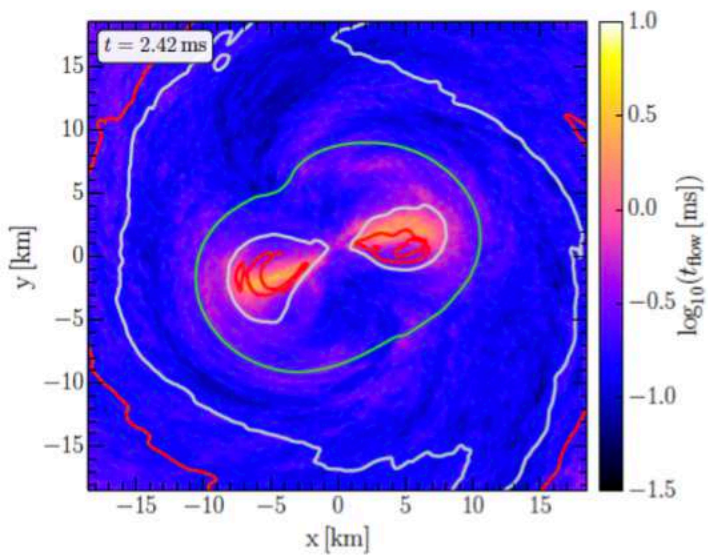




Figure 3: The flow timescale $t_{\rm flow}$ obtained from a numerical-relativity simulation of two 1.35 $M_\odot$ neutron stars [40]. The red (4 MeV) and gray (7 MeV) contours show the boundaries of the temperature range in which the bulk viscosity roughly takes its maximum value, while the green contour shows the inner region where the rest-mass density exceeds nuclear saturation density.

Alford and Harris, PRC (2019)

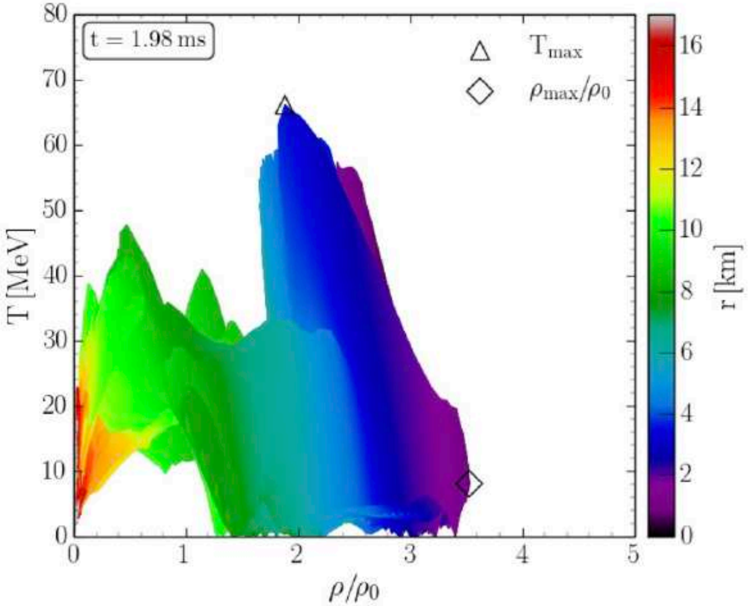
- Low densities (EOS)
- Neutrino transparency (low T)
- High frequencies (f > 1 kHz)

See also:

Alford, Harutyunyan, Sedrakian,
PRD (2019) ; Alford, Haber, 2009.05181

Significant variations in:

**Temperature**
**Density**
**Fluid velocity**




Rezzolla group, Frankfurt

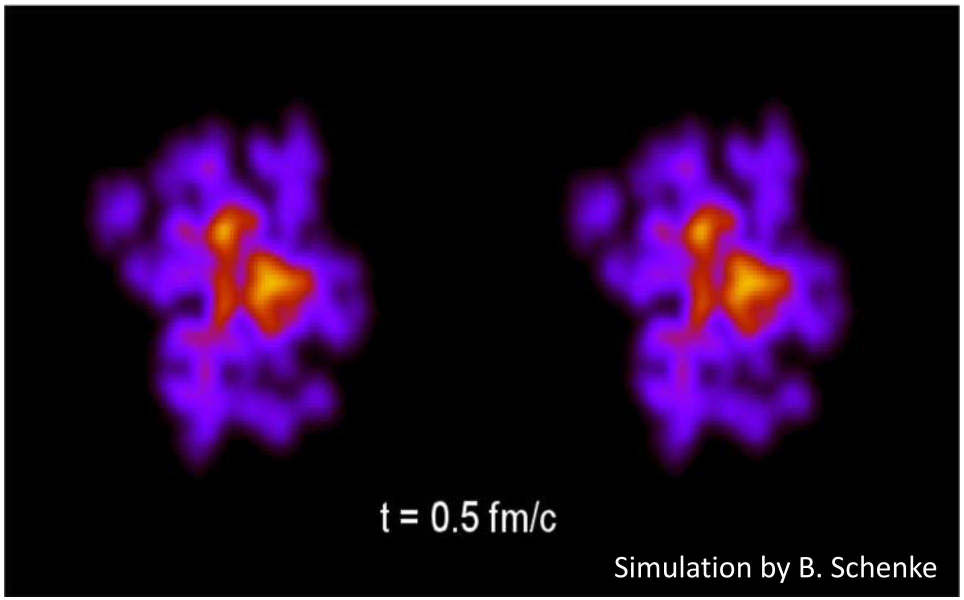
Alford et al. PRL (2018)

- Bulk viscosity effects should be investigated in simulations.

- Other viscous contributions deserve further investigation (e.g., thermal conductivity, shear viscosity).

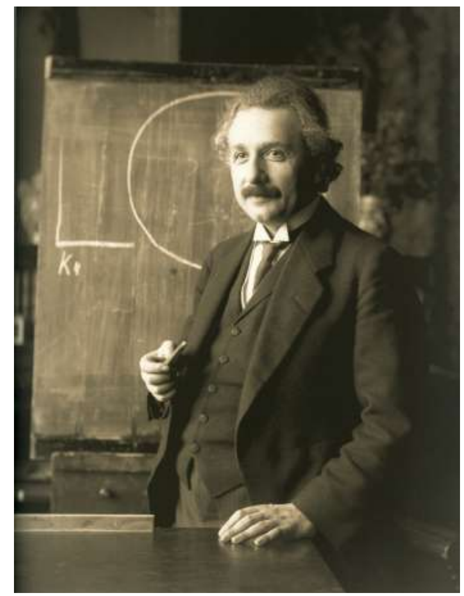
# So, what will happen when we combine

Ideal fluid viscous fluid



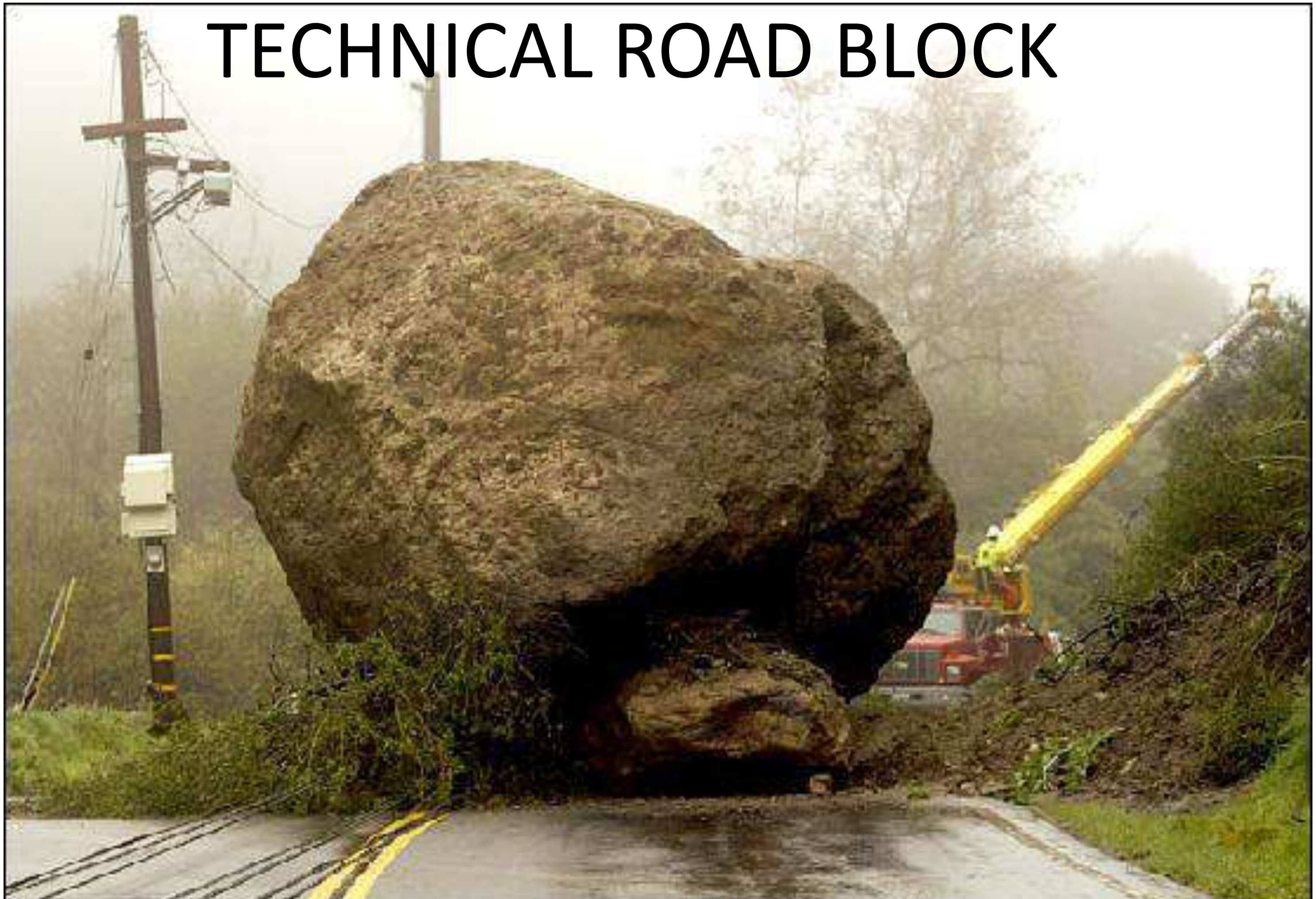

+

General Relativity



?

# Viscous Fluids in General Relativity



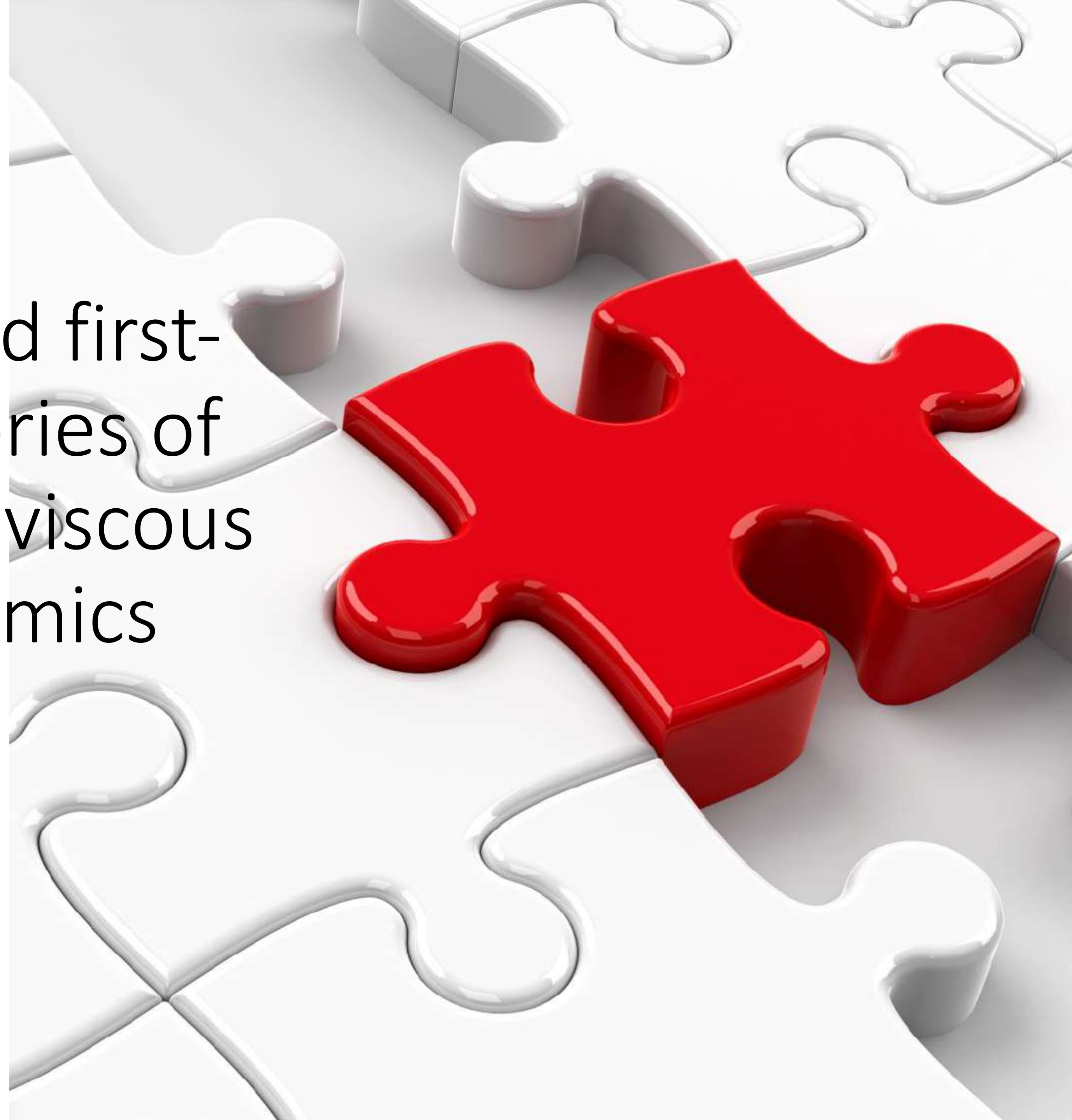
# Why is this so challenging?

- Einstein + fluid equations are highly nonlinear.

- Previous approaches to viscous fluids are either acausal/unstable or not known to have a well-posed initial-value problem in relativity.

- The situation changed, dramatically, in the last couple of years, as I will now show you.

# Generalized first-order theories of relativistic viscous hydrodynamics



# Near Equilibrium Behavior:

# The Derivative Expansion

# Generalized First-Order Relativistic Hydrodynamics

Originally proposed by Bemfica, Disconzi, JN, PRD (2018): Conformal regime
Followed by Kovtun, JHEP (2019); Bemfica, Disconzi, JN, PRD (2019): Non-conformal
Hoult, Kovtun, JHEP (2020); Bemfica, Disconzi, JN, arxiv: 2009.11388: Finite density

In equilibrium: $T^{\mu\nu}$ $J^{\mu}$ → $T, \mu, u^{\lambda}$

Baryon current

Well-defined Mapping

Temperature $T$

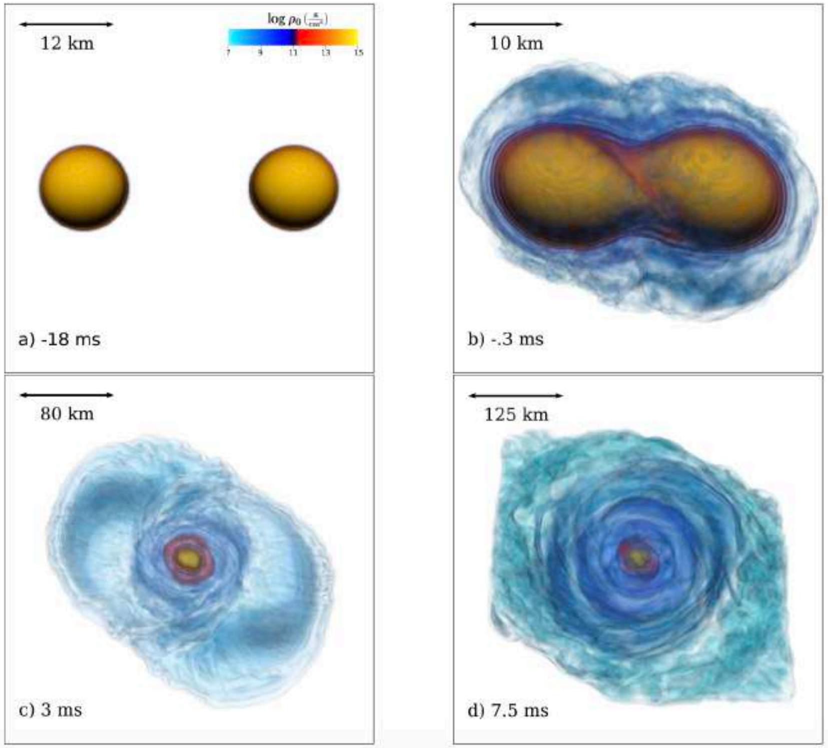
Chemical potential $\mu$

Flow velocity $u^{\lambda}$

# Generalized First-Order Relativistic Hydrodynamics

Originally proposed by Bemfica, Disconzi, JN, PRD (2018): Conformal regime
Followed by Kovtun, JHEP (2019); Bemfica, Disconzi, JN, PRD (2019): Non-conformal
Hoult, Kovtun, JHEP (2020); Bemfica, Disconzi, JN, arxiv: 2009.11388: Finite density

Out of equilibrium $T^{\mu\nu}$ $J^{\mu}$ are well defined




*Ex: Figure from Vincent et al., PRD* 101 (2020)

But hydrodynamic variables

$$T, \mu, u^{\lambda}$$

can be defined in many ways

Different hydrodynamic frames

Eckart, Landau and Lifshitz + ...

## Most general decomposition out of equilibrium

$$T_{\mu\nu} = \mathcal{E} u_\mu u_\mu + \mathcal{P} \Delta_{\mu\nu} + \pi_{\mu\nu} + \mathcal{Q}_\mu u_\nu + \mathcal{Q}_\nu u_\mu$$

Conserved baryon current: $J_\mu = \mathcal{N} u_\mu + \mathcal{J}_\mu$

Dissipation = more terms: $\mathcal{E}, \mathcal{N}, \mathcal{P}, \pi_{\mu\nu}, \mathcal{Q}_\mu, \mathcal{J}_\mu$

How does one fix such terms? Further assumptions needed.

- Standard approach: Derivative expansion
- Compatibility with the 2nd law of thermodynamics.
- Recovers non-relativistic physics of fluids when v << 1.

# Effective Field Theory Approach

- Write the most general expansion at 1st order in derivatives compatible with symmetries.

- Most general hydrodynamic frame.

$$\mathcal{E} = \varepsilon + \varepsilon_1 \frac{u^\alpha \nabla_\alpha T}{T} + \varepsilon_2 \nabla_\alpha u^\alpha + \varepsilon_3 u^\alpha \nabla_\alpha (\mu/T),$$
$$\mathcal{P} = P + \pi_1 \frac{u^\alpha \nabla_\alpha T}{T} + \pi_2 \nabla_\alpha u^\alpha + \pi_3 u^\alpha \nabla_\alpha (\mu/T),$$
$$\mathcal{N} = n + \nu_1 \frac{u^\alpha \nabla_\alpha T}{T} + \nu_2 \nabla_\alpha u^\alpha + \nu_3 u^\alpha \nabla_\alpha (\mu/T),$$
$$\mathcal{Q}^\mu = \theta_1 \frac{\Delta^{\mu\nu} \nabla_\nu T}{T} + \theta_2 u^\alpha \nabla_\alpha u^\mu + \theta_3 \Delta^{\mu\nu} \nabla_\nu (\mu/T),$$
$$\mathcal{J}^\mu = \gamma_1 \frac{\Delta^{\mu\nu} \nabla_\nu T}{T} + \gamma_2 u^\alpha \nabla_\alpha u^\mu + \gamma_3 \Delta^{\mu\nu} \nabla_\nu (\mu/T)$$

$$\pi_{\mu\nu} = -2\eta\sigma_{\mu\nu}$$

Usual transport coefficients

$$\eta, \zeta, \kappa$$

shear, bulk, heat cond.

New coefficients parametrize the freedom in choosing the hydro fields in this approach.

**What choices are physical?**

- In the regime of validity of the first-order theory, any choice of hydrodynamic frame satisfies the 2nd law of thermo if $\eta, \zeta, \kappa \geq 0$

P. Kovtun, JHEP (2019)

Entropy current at 1st order

$$S^\mu = (s + \mathcal{A} - \mu\mathcal{N})u^\mu + \frac{\mathcal{Q}^\mu - \mu\mathcal{J}^\mu}{T}$$

Entropy production (on shell)

$$\nabla_\mu S^\mu = positive + \mathcal{O}(\partial^3)$$

- New parameters can be constrained by causality and stability.

Proof: See original theorems in Bemfica, Disconzi, JN PRD (2018), PRD (2019), arxiv: 2009.11388


- This leads to a set of good definitions for $T, \mu, u^\lambda$

# How can these new developments be used to understand the hot and viscous ultradense matter in neutron star mergers?

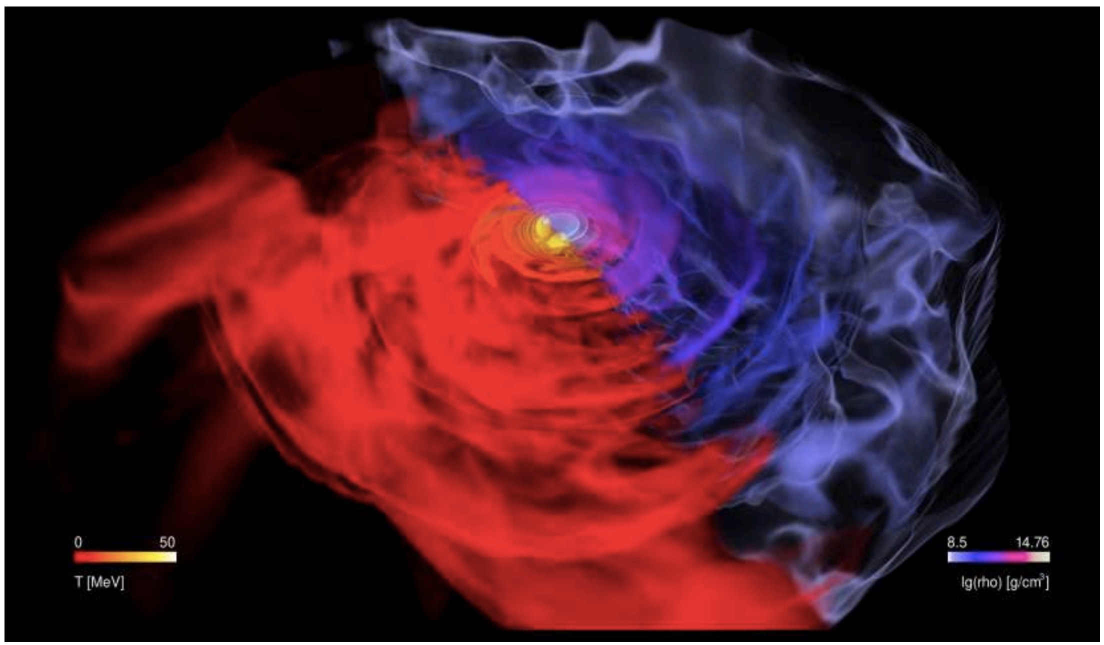



Figure by Most et al., PRL (2019)

# General-Relativistic Viscous Fluid Dynamics

Bemfica, Disconzi, JN, arxiv: 2009.11388

Conserved current "a la Eckart": $J^{\mu} = n u^{\mu}$

Energy-momentum tensor:

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})\, u^{\mu} u^{\nu} + (P + \Pi)\Delta^{\mu\nu} - 2\eta\sigma^{\mu\nu} + u^{\mu}\mathcal{Q}^{\nu} + u^{\nu}\mathcal{Q}^{\mu}$$

Out-of-equilibrium contributions

Contribution to energy density: $\mathcal{A} = \tau_{\varepsilon}\left[u^{\lambda}\nabla_{\lambda}\varepsilon + (\varepsilon + P)\nabla_{\lambda}u^{\lambda}\right]$

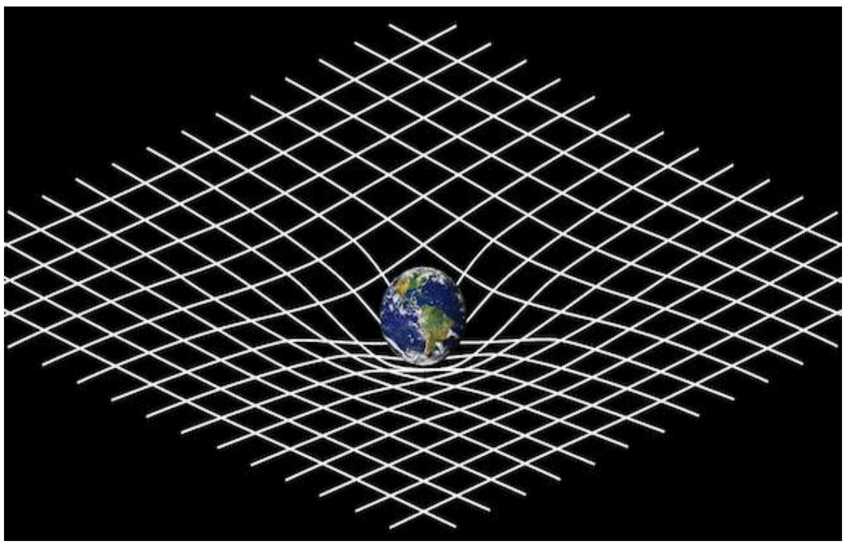
Contribution to pressure: $\Pi = -\zeta\nabla_{\lambda}u^{\lambda} + \tau_{P}\left[u^{\lambda}\nabla_{\lambda}\varepsilon + (\varepsilon + P)\nabla_{\lambda}u^{\lambda}\right]$

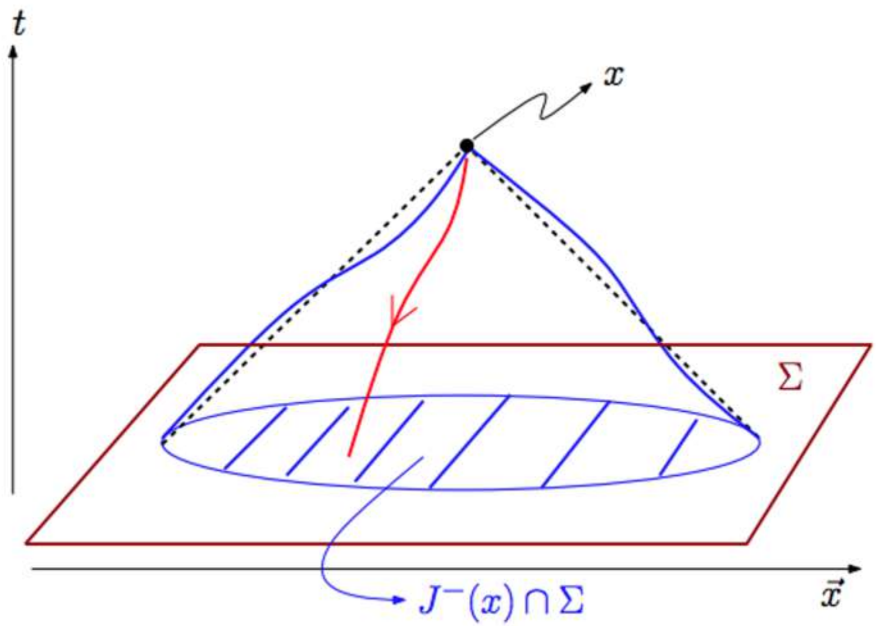
Energy flux: $\mathcal{Q}^{\nu} = \tau_{Q}(\varepsilon + P)u^{\lambda}\nabla_{\lambda}u^{\nu} + \beta_{\varepsilon}\Delta^{\nu\lambda}\nabla_{\lambda}\varepsilon + \beta_{n}\Delta^{\nu\lambda}\nabla_{\lambda}n$

# Causality

An essential property of relativity

Curved spacetime → Distorted light cone






**General Relativity**

Bemfica, Disconzi, JN, arxiv: 2009.11388

- System given by fluid + Einstein's equations is causal

## Causality

**Theorem I.** *Let* $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ *be a solution to* (2) *and* (12), *with* $u^\mu u_\mu = -1$, *defined in a globally spacetime* $(M, g_{\alpha\beta})$. *Assume that:*

*(A1)* $\rho = \varepsilon + P, \tau_\varepsilon, \tau_Q, \tau_P > 0$ *and* $\eta, \zeta, \sigma \geq 0$.

*Then, causality holds for* $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ *if, and only if, the following conditions are satisfied:*

$$\rho\tau_Q > \eta, \tag{20a}$$

$$\left[\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q\right]^2 \geq 4\rho\tau_\varepsilon\tau_Q\left[\tau_P\left(\rho c_s^2\tau_Q + \sigma\kappa_s\right) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right)\right] \geq 0, \tag{20b}$$

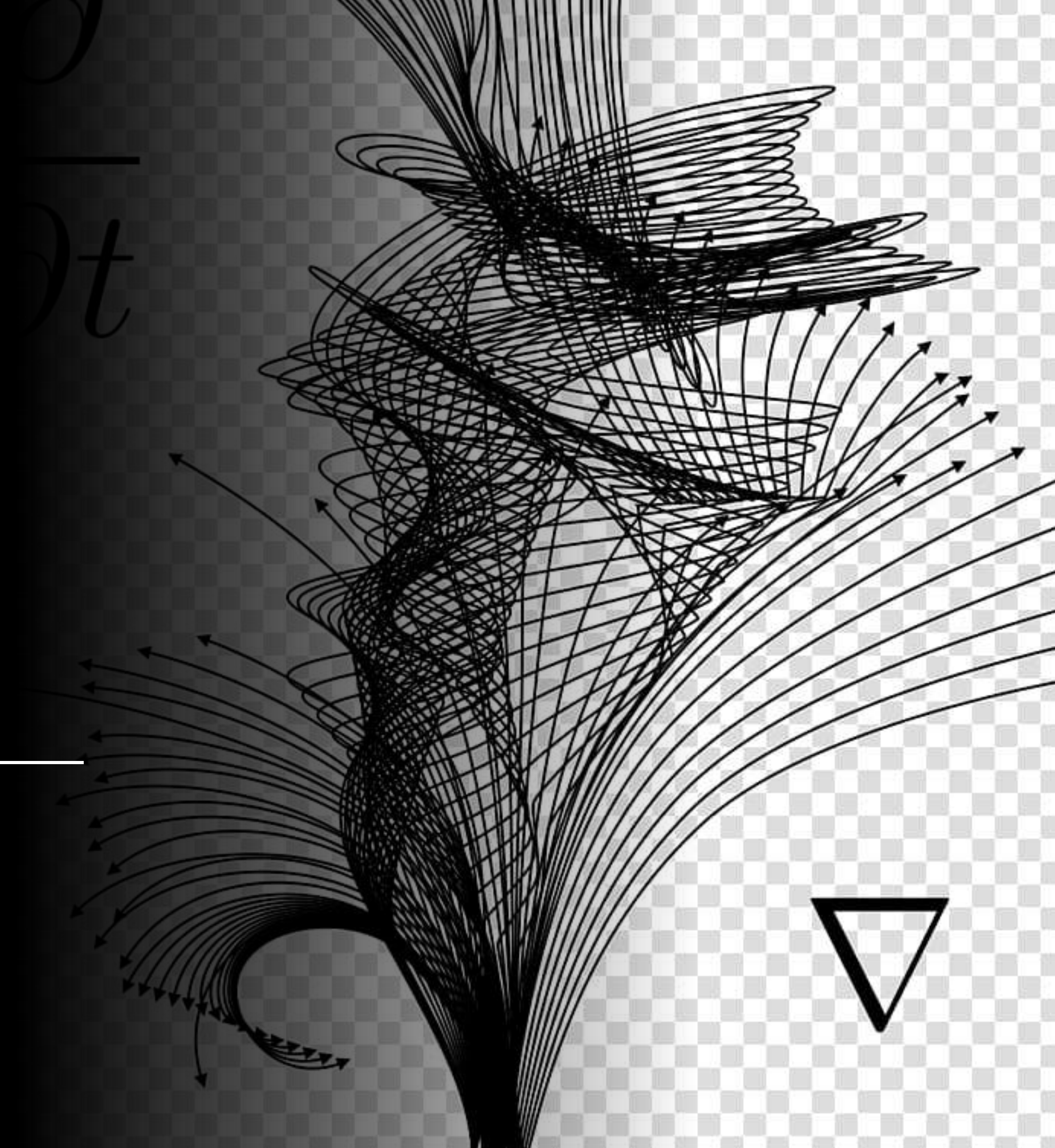
$$2\rho\tau_\varepsilon\tau_Q > \tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q \geq 0, \tag{20c}$$

$$\rho\tau_\varepsilon\tau_Q + \sigma\kappa_s\tau_P > \tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q(1 - c_s^2) + \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right). \tag{20d}$$

**True**

- Requires knowledge about the characteristics (nonlinear regime).
- Brute force determination of characteristics here is hopeless.
- Since our first work, we developed a new geometric approach to determine the characteristics and, hence, causality.

# Well-Posedness



# Local Well-Posedness

Hadamard

A system of PDEs is locally well-posed if:

- Given the initial data, a local solution exists.
- The local solution is unique.
- The solution depends continuously on initial data.

http://www.math.ucla.edu/~tao/Dispersive/

Bemfica, Disconzi, JN, arxiv: 2009.11388

- Fluid + Einstein's equations are causal and strongly hyperbolic, and the initial-value problem is well-posed.

## Strong Hyperbolicity and Well-Posedness

**Theorem II.** *Let $(\Sigma, \mathring{g}_{\alpha\beta}, \widehat{\kappa}_{\alpha\beta}, \mathring{\varepsilon}, \widehat{\varepsilon}, \mathring{n}, \widehat{n}, \mathring{u}^\alpha, \widehat{u}^\alpha)$ be an initial-data set for the system comprised of Einstein's e*
*(1) and $\nabla_\mu J^\mu = 0$, where $T_{\alpha\beta}$ and $J^\mu$ are given in (10). Assume that $\mathring{u}^\mu \mathring{u}_\mu = -1$, $\mathring{n} \geq C > 0$, where*
*constant, and that $\nabla_\mu J^\mu = 0$ holds for the initial data. Assume (A1) with $\eta > 0$ and suppose that (20) of Th*
*hold in strict form and that the transport coefficients are analytic functions of their arguments. Finally, assu*
*$\mathring{g}_{\alpha\beta}, \mathring{\varepsilon}, \mathring{n}, \mathring{u}^\alpha \in H^N(\Sigma)$ and that $\widehat{\kappa}_{\alpha\beta}, \widehat{\varepsilon}, \widehat{n}, \widehat{u}^\alpha \in H^{N-1}(\Sigma)$, $N \geq 5$, where $H^N$ is the Sobolev space. Then, there exists a globally hyperbolic development of the initial data. This globally hyperbolic development is unique if taken to be the maximum globally hyperbolic development of the initial data.*

True

- Nonlinear equations in first-order form $\mathfrak{A}^\alpha \partial_\alpha U = R,$

- Prove that all eigenvalues are real and the set of eigenvectors form a complete set.

- Combine this result with advanced PDE techniques.

Bemfica, Disconzi, JN, arxiv: 2009.11388

- Fluid + Einstein's equations is causal and strongly hyperbolic, and the initial-value problem is well-posed.

## Strong Hyperbolicity and Well-Posedness

**Theorem II.** *Let $(\Sigma, \mathring{g}_{\alpha\beta}, \widehat{\kappa}_{\alpha\beta}, \mathring{\varepsilon}, \widehat{\varepsilon}, \mathring{n}, \widehat{n}, \mathring{u}^\alpha, \widehat{u}^\alpha)$ be an initial-data set for the system comprised of Einstein's e (1) and $\nabla_\mu J^\mu = 0$, where $T_{\alpha\beta}$ and $J^\mu$ are given in (10). Assume that $\mathring{u}^\mu \mathring{u}_\mu = -1$, $\mathring{n} \geq C > 0$, where constant, and that $\nabla_\mu J^\mu = 0$ holds for the initial data. Assume (A1) with $\eta > 0$ and suppose that (20) of Th hold in strict form and that the transport coefficients are analytic functions of their arguments. Finally, assu $\mathring{g}_{\alpha\beta}, \mathring{\varepsilon}, \mathring{n}, \mathring{u}^\alpha \in H^N(\Sigma)$ and that $\widehat{\kappa}_{\alpha\beta}, \widehat{\varepsilon}, \widehat{n}, \widehat{u}^\alpha \in H^{N-1}(\Sigma)$, $N \geq 5$, where $H^N$ is the Sobolev space. Then, there exists a globally hyperbolic development of the initial data. This globally hyperbolic development is unique if taken to be the maximum globally hyperbolic development of the initial data.*

True

- Equilibrium states are stable.

**Theorem III.** *Let (72) have a set of $N$ linearly independent real eigenvectors $\{\mathfrak{r}_1, \cdots, \mathfrak{r}_N\}$. If (68) is causal and stable in the local rest frame $\mathcal{O}$, then it is also stable in any other Lorentz frame $\mathcal{O}'$ connected to $\mathcal{O}$ by a Lorentz transformation.*

True

Based on our experience with heavy-ions …

Ideal fluids → Viscous fluids

in flat spacetime

Led to a paradigm shift and many new insights:

- Nearly perfect fluidity of the quark-gluon plasma
- Characterization of initial state (e.g. gluon saturation)
- Emergence of hydrodynamics far from equilibrium
- Connection to other fields: AdS/CFT, cold atoms

The inclusion of viscous effects in mergers will force us

# Conclusions

- QGP formed in heavy-ions forces us to explore relativistic fluids far from equilibrium.

- New constraints reveal challenges to far-from-equilibrium relativistic hydrodynamics.

- Neutron star mergers raise the possibility to determine the out-of-equilibrium properties of hot ultradense matter.

- New first-order formulation paves the way for describing for the first time all viscous fluids effects in general relativity.