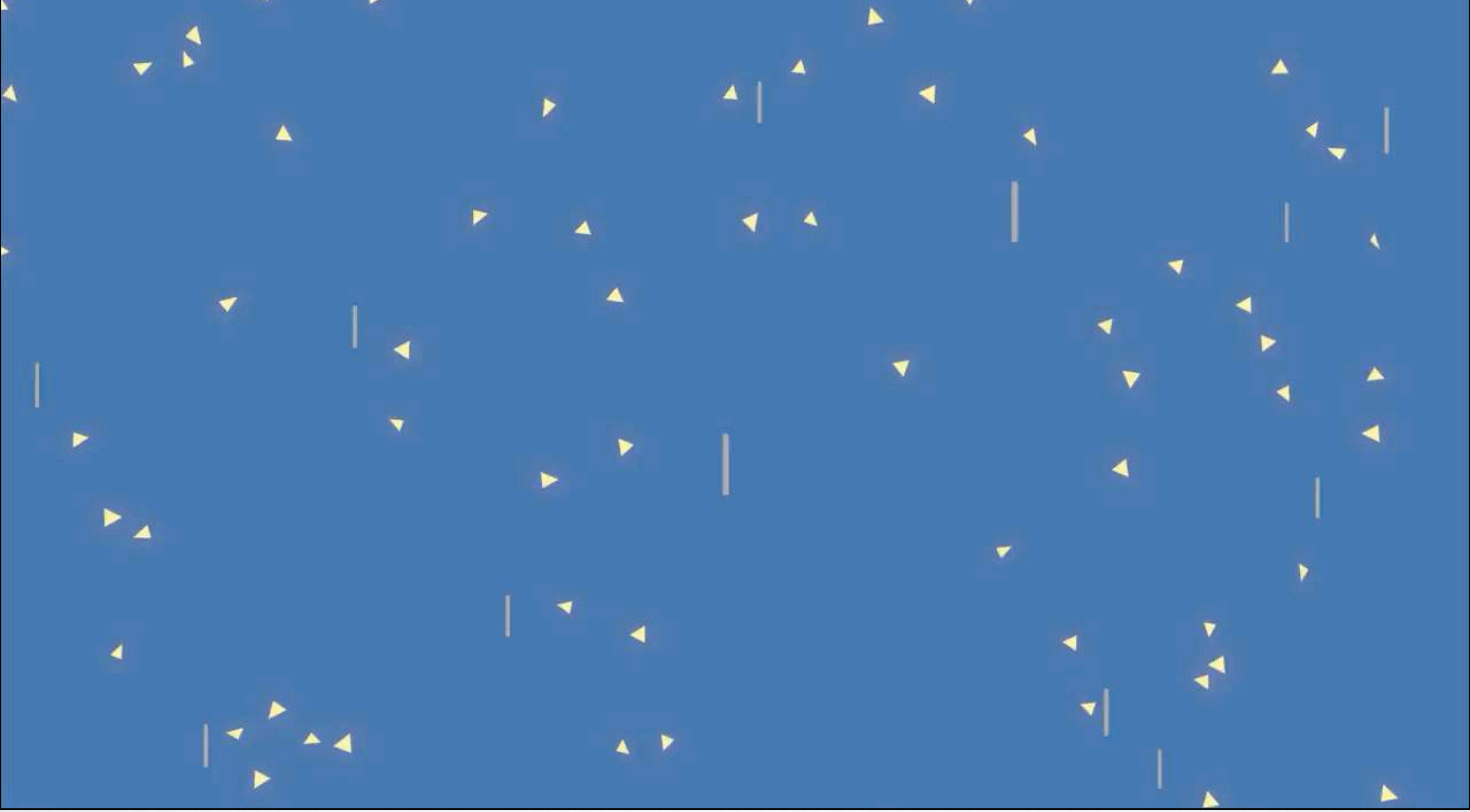

The Noise of Gravitons

Maulik Parikh
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Theoretical Physics Colloquium

The Noise of Photons

Electromagnetic fields are quantum fields whose quanta are photons

The photons reveal themselves in situations with low photon number

They appear as random fluctuations — noise where

$$\epsilon(x,\bar{x})=+1 \quad \text{for} \quad x^4<\bar{x}^4$$
$$=-1 \quad \text{for} \quad \bar{x}^4<x^4. \qquad (3.7)$$

Because of the infinite integrations in Eq. (3.6), it is clear that the additional term is either zero or infinite for a periodic motion of the system. Thus, for periodic motions the Frenkel 4-momentum either coincides with the canonical 4-momentum or gives infinite results. In our case of point charges in circular motion, the right-hand side of Eq. (3.6) vanishes, so that the Frenkel 4-momentum also leads to the energy given by Eq. (3.4).

Our system, characterized by Eqs. (3.1) to (3.5), can now be quantized by putting $L=n\hbar$. For either positronium (e electronic charge, $m=\bar{m}$ electron mass) or hydrogen (e, m electronic charge and mass, $\bar{m}$ proton mass), the resulting quantized motions are all nonrelativistic. They are the usual Bohr motions with small corrections for retardation and other relativistic effects and, in the case of hydrogen, with small corrections for the motion of the nucleus.

PHYSICAL REVIEW VOLUME 131, NUMBER 6 15 SEPTEMBER 1963

Coherent and Incoherent States of the Radiation Field*

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(Received 29 April 1963)

Methods are developed for discussing the photon statistics of arbitrary radiation fields in fully quantum-mechanical terms. In order to keep the classical limit of quantum electrodynamics plainly in view, extensive use is made of the coherent states of the field. These states, which reduce the field correlation functions to factorized forms, are shown to offer a convenient basis for the description of fields of all types. Although they are not orthogonal to one another, the coherent states form a complete set. It is shown that any quantum state of the field may be expanded in terms of them in a unique way. Expansions are also developed for arbitrary operators in terms of products of the coherent state vectors. These expansions are discussed as a general method of representing the density operator for the field. A particular form is exhibited for the density operator which makes it possible to carry out many quantum-mechanical calculations by methods resembling those of classical theory. This representation permits clear insights into the essential distinction between the quantum and classical descriptions of the field. It leads, in addition, to a simple formulation of a superposition law for photon fields. Detailed discussions are given of the incoherent fields which are generated by superposing the outputs of many stationary sources. These fields are all shown to have intimately related properties, some of which have been known for the particular case of blackbody radiation.

I. INTRODUCTION

FEW problems of physics have received more attention in the past than those posed by the dual wave-particle properties of light. The story of the solution of these problems is a familiar one. It has culminated in the development of a remarkably versatile quantum theory of the electromagnetic field. Yet, for reasons which are partly mathematical and partly, perhaps, the accident of history, very little of the insight of quantum electrodynamics has been brought to bear on the problems of optics. The statistical properties of photon beams, for example, have been discussed to date almost exclusively in classical or semiclassical terms. Such discussions may indeed be informative, but they inevitably leave open serious questions of self-consistency, and risk overlooking quantum phenomena which have no classical analogs. The wave-particle duality, which should be central to any correct treatment of photon statistics, does not survive the transition to the classical limit. The need for a more consistent theory has led us to begin the development of a fully quantum-mechanical approach to the problems of photon statistics. We have quoted several of the results of this work in a recent note,[1] and shall devote much of the present paper to explaining the background of the material reported there.

Most of the mathematical development of quantum electrodynamics to date has been carried out through the use of a particular set of quantum states for the field. These are the stationary states of the non-interacting field, which corresponds to the presence of a precisely defined number of photons. The need to use these states has seemed almost axiomatic inasmuch as nearly all quantum electrodynamical calculations have been carried out by means of perturbation theory. It is characteristic of electrodynamical perturbation theory that in each successive order of approximation it describes processes which either increase or decrease the number of photons present by one. Calculations performed by such methods have only rarely been able to deal with more than a few photons at a time. The

* Supported in part by the U. S. Air Force Office of Scientific Research under Contract No. AF 49(638)-589.

[1] R. J. Glauber, Phys. Rev. Letters 10, 84 (1963).

Dyson's Critique of the Detectability of Gravitons

Dyson argued that since there are $\sim 10^{37}$ gravitons within one cubic wavelength…

… and since we have only barely managed to detect gravitational waves

… we would have to improve sensitivity by ~37 orders of magnitude
in order to detect individual gravitons

"The Noise of Gravitons," arXiv:2005.07211

The Noise of Gravitons

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Abstract

We show that when the gravitational field is treated quantum-mechanically, it induces fluctuations – noise – in the lengths of the arms of gravitational wave detectors. The characteristics of the noise depend on the quantum state of the gravitational field, and can be calculated exactly in several interesting cases. For coherent states the noise is very small, but it can be greatly enhanced in thermal and (especially) squeezed states. Detection of this fundamental noise would constitute direct evidence for the quantization of gravity and the existence of gravitons.

This essay was awarded First Prize in the 2020 Essay Competition of the Gravity Research Foundation.

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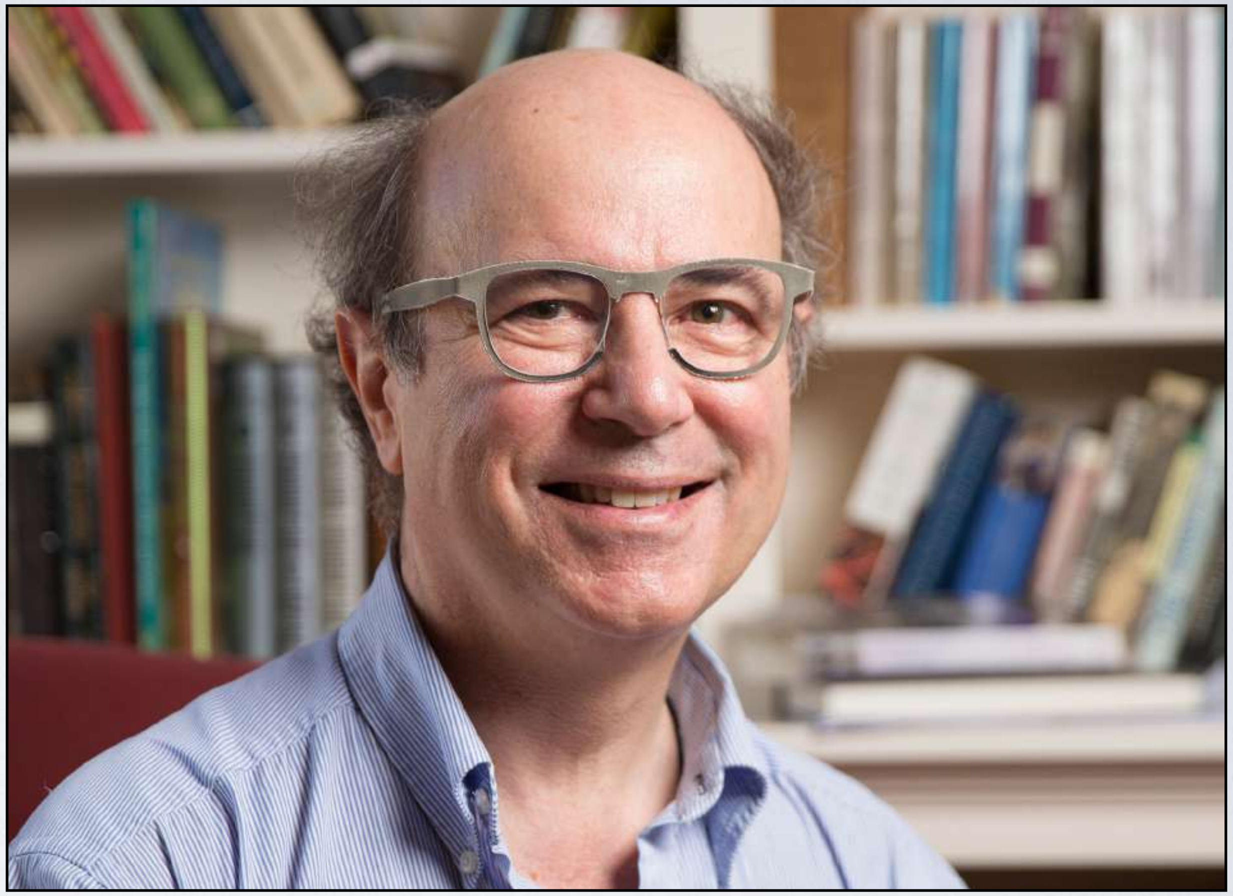
"Quantum Mechanics of Gravitational Waves," arXiv:2010.08205

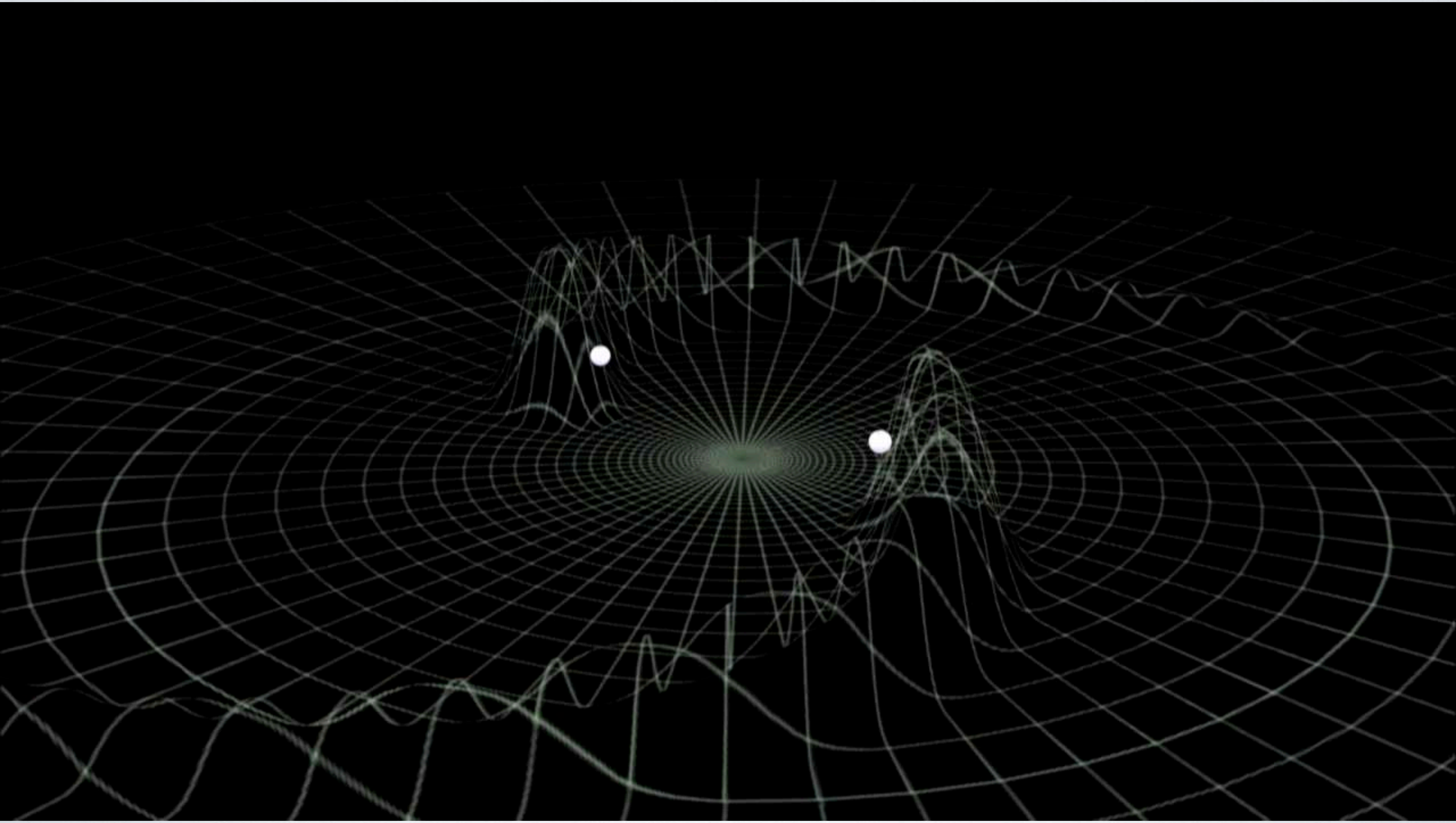
"Signatures of the Quantization of Gravity at Gravitational Wave Detectors," arXiv:2010.08208

Frank Wilczek



George Zahariade

Gravitational Waves



Gravitational Waves

Geodesic Deviation Equation

$$\ddot{\xi}^{\mu} = -R^{\mu}_{0\nu 0}\xi^{\nu}$$

curvature of spacetime

geodesic separation

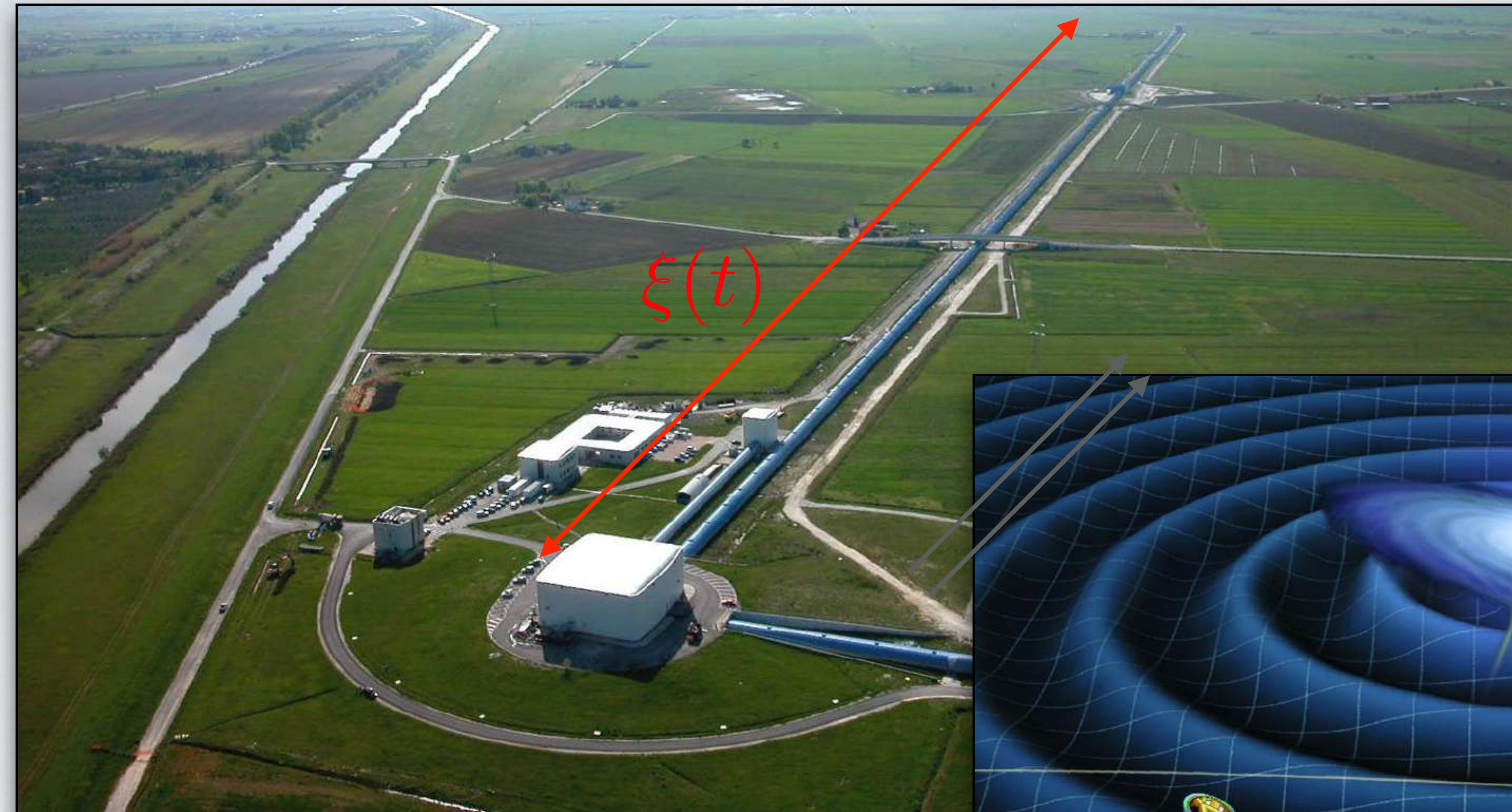
$$R_{i0j0}(t,0) = -\frac{1}{2}\ddot{h}_{ij}(t,0)$$

gravitational wave

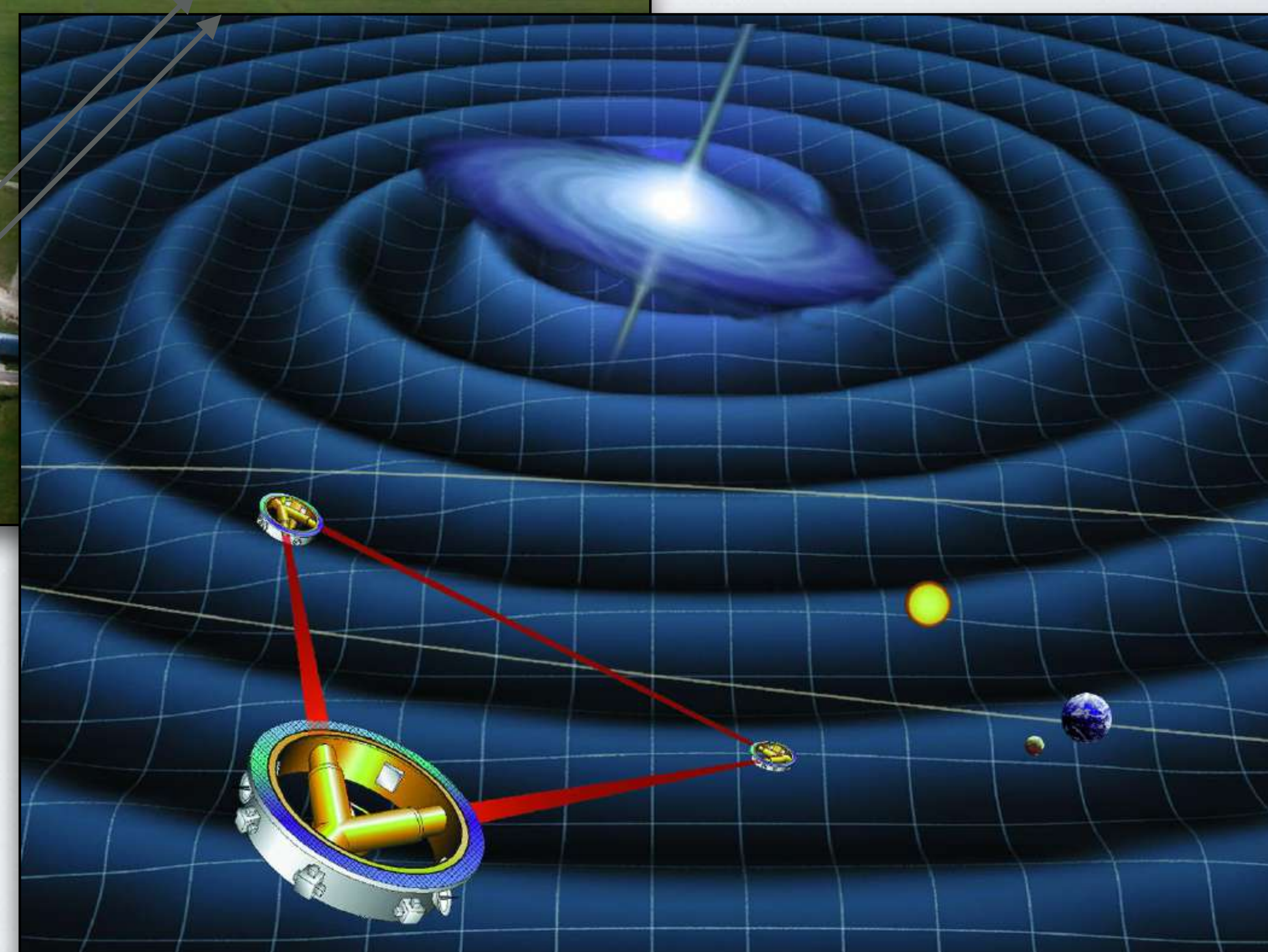
$$\ddot{\xi} = \frac{1}{2}\ddot{h}\xi$$

This equation describes the separation of free-falling particles when subject to a gravitational wave h(t)

Gravitational Wave Interferometers



LIGO



LISA

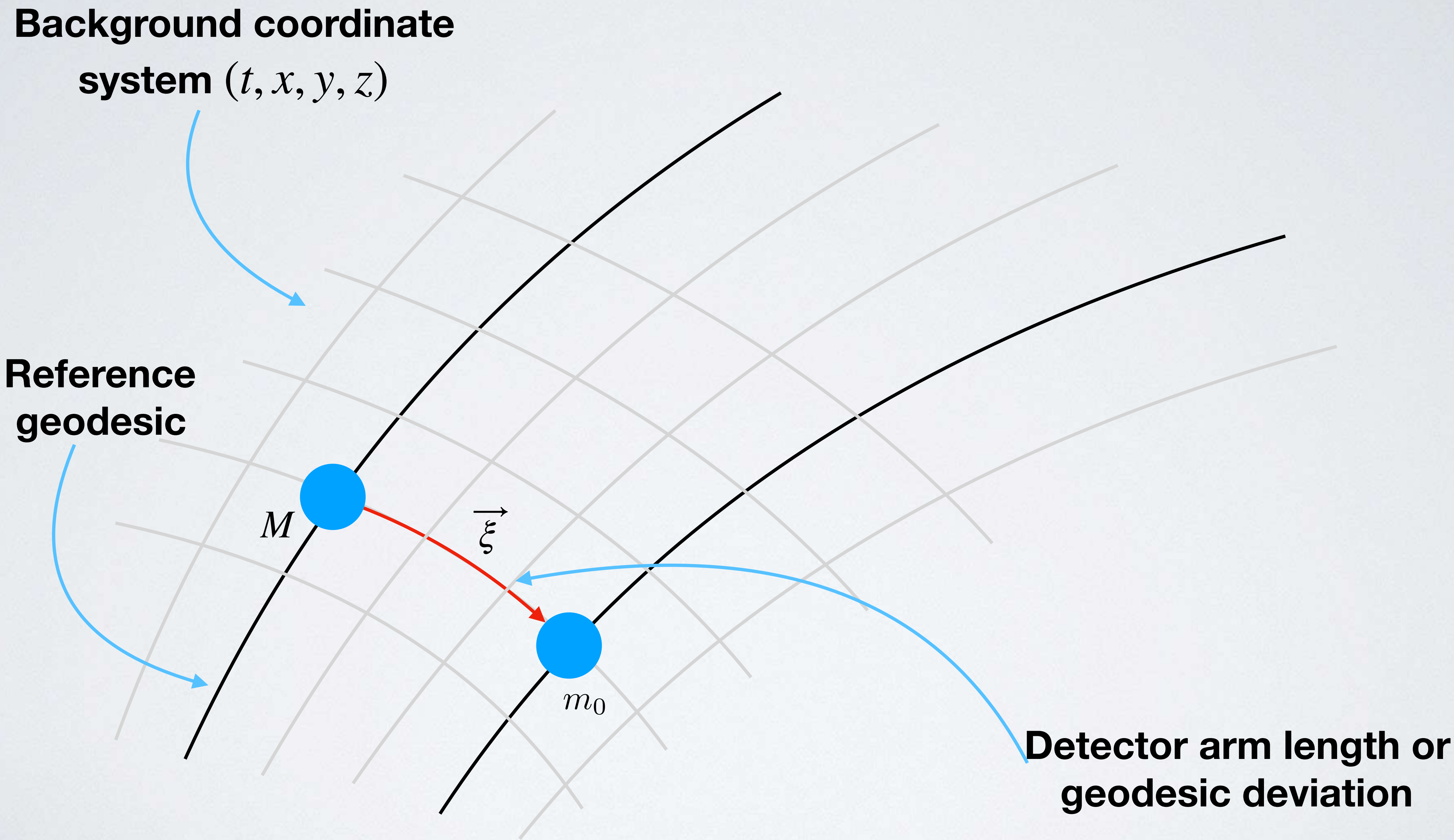
Quantum Geodesic Deviation Equation?

$$\ddot{\xi} = \frac{1}{2}\ddot{h}\xi$$

This is the geodesic deviation equation in the presence of a classical gravitational wave

What is the generalization of this equation when the spacetime metric is treated as a quantum field?

Model Gravitational Wave Detector



Action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R - M\int d\lambda\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}} - m_0\int d\tau\sqrt{-g_{\mu\nu}\frac{dY^\mu}{d\tau}\frac{dY^\nu}{d\tau}}$$

Einstein-Hilbert action + relativistic action for two free-falling particles

Use Fermi normal coordinates, putting heavy particle on classical trajectory

$$X^\mu = (t, \vec{0})$$

Let the other particle be at

$$Y^\mu = (t, \vec{\xi})$$

Action

Next, insert metric in Fermi normal coordinates into particle action:

$$
\begin{aligned}
g_{00}(t,\xi) &= -1 - R_{i0j0}(t,0)\xi^i\xi^j + O(\xi^3) \\
g_{0i}(t,\xi) &= -\frac{2}{3}R_{0jik}(t,0)\xi^j\xi^k + O(\xi^3) \\
g_{ij}(t,\xi) &= \delta_{ij} - \frac{1}{3}R_{ikjl}(t,0)\xi^k\xi^l + O(\xi^3)\ .
\end{aligned}
$$

Expand action to lowest order in metric perturbation in transverse-traceless gauge:

$$
S = -\frac{1}{64\pi G}\int d^4x\ \partial_\mu h_{ij}\partial^\mu h^{ij} + \int dt \frac{1}{2} m_0 \left(\delta_{ij}\dot{\xi}^i\dot{\xi}^j - \dot{h}_{ij}\dot{\xi}^i\xi^j\right)
$$

Strategy

We wish to calculate the effect of a quantized gravitational field on the gravitational wave detector. We have the action. We can quantize the theory.

Suppose the gravitational field is initially in state $|\Psi\rangle$

We don't know what the final state of the field is

We wish to calculate the transition probability of the detector to go from state A to state B in time T

Quantum Mechanics

Thus we wish to calculate

$$P_\Psi(A \to B) = \sum_{|f\rangle} |\langle f, B|\hat{U}(T)|\Psi, A\rangle|^2$$

Here the amplitudes are evaluated as path integrals using the action we have found

The relatively simple form of the action allows the calculation to be performed exactly

Aside: Influence Functionals

Feynman and Vernon (1963) considered a very general problem in quantum mechanics

Suppose we have two interacting systems but we only have access to or interest in one of them

Then the effect of the system we are not interested in on the system we are interested in is completely encoded by the influence functional

The influence functional is a double path integral

Integrating Out Gravity

In our context, we wish to integrate out gravity to see the effect on the detector

$$P_\Psi(A \to B) \sim \int \mathscr{D}\xi \mathscr{D}\xi' e^{\frac{i}{\hbar}\int_0^T dt \frac{1}{2}m(\dot{\xi}^2 - \dot{\xi}'^2)} F_\Psi[\xi, \xi']$$

Here the influence functional F is

$$F_\Psi[\xi, \xi'] = \sum_{|f\rangle} \int \mathcal{D}h \, \mathcal{D}h' \, e^{\frac{i}{\hbar}(S_{h,\xi} - S_{h',\xi'})}$$

Mode Decomposition

$$S = -\frac{1}{64\pi G}\int d^4x\ \partial_\mu h_{ij}\partial^\mu h^{ij} + \int dt \frac{1}{2} m_0 \left(\delta_{ij}\dot{\xi}^i\dot{\xi}^j - \dot{h}_{ij}\dot{\xi}^i\xi^j\right)$$

Write metric perturbation in Fourier modes:

$$h_{ij}(t,\vec{x}) = \frac{1}{\sqrt{\hbar G}}\sum_{\vec{k},s} q_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}}\epsilon^s_{ij}(\vec{k})$$

$$S = \int dt \frac{1}{2} m_0 \dot{\xi}^2 + \int dt \sum_{\vec{k},s}\frac{1}{2} m \left(\dot{q}^2_{\vec{k},s} - \omega^2_{\vec{k}} q^2_{\vec{k}}\right) - g\int dt \sum_{\vec{k},s}\dot{q}_{\vec{k},s}\epsilon^s_{ij}(\vec{k})\dot{\xi}^i\xi^j$$

where

$$m \equiv \frac{L^3}{16\pi\hbar G^2} \qquad g \equiv \frac{m_0}{2\sqrt{\hbar G}}$$

Detector Arm Length Interacting with Graviton Mode

free particle

$$S_\omega = \int dt \left(\frac{1}{2} m(\dot{q}^2 - \omega^2 q^2) + \frac{1}{2} m_0 \dot{\xi}^2 - g\dot{q}\dot{\xi}\xi \right)$$

simple harmonic oscillator

cubic interaction term

Influence Functional for Gravity

A calculation in ordinary quantum mechanics
yields the influence functional for gravity

$$|F_\Psi| = \exp\left[-\frac{m_0^2}{32\hbar^2}\int_0^T\int_0^T dt\,dt'\,A_\Psi(t-t')\,(X(t)-X'(t))\,(X(t')-X'(t'))\right]$$

where

$$X(t) = \frac{d^2}{dt^2}(\xi^2)$$

A Mathematical Trick

We now exploit the fact that exponentials can be written as Gaussian integrals:

$$e^{\frac{b^2}{4a}} \sim \int dy\, e^{-ay^2+by}$$

The infinite-dimensional generalization of this is a path integral:

$$\exp\left[-\frac{m_0^2}{32\hbar^2}\int_0^T\int_0^T dt\,dt'\,A(t,t')\,(X(t)-X'(t))\,(X(t')-X'(t'))\right]$$
$$=\int \mathcal{D}N\exp\left[-\frac{1}{2}\int_0^T\int_0^T dt\,dt'\,A^{-1}(t,t')N(t)N(t')+\frac{i}{\hbar}\int_0^T dt\frac{m_0}{4}N(t)\,(X(t)-X'(t))\right]$$

N(t) is a zero-mean Gaussian stochastic function with auto-correlation A(t-t'): noise!

Transition Probability

Putting everything together we obtain the detector transition probability

$$P_\Psi(A \to B) \sim \int D\xi D\xi' e^{\frac{i}{\hbar}\int dt \frac{1}{2} m_0 (\dot{\xi}^2 - \dot{\xi'}^2)} \left\langle e^{\left(\frac{i}{\hbar}\int_0^T dt \frac{m_0}{4} N(t)\left(X(t) - X'(t)\right)\right)} \right\rangle_N$$

Since the detector is well-approximated as classical, we can take a saddle point to obtain its effective equation of motion

Langevin Equation

$$\ddot{\xi} = \frac{1}{2}\left(\ddot{h} + \ddot{N}_{\Psi} - \frac{m_0 G}{c^5}\frac{d^5}{dt^5}\xi^2\right)\xi$$

classical gravitational wave

quantum noise

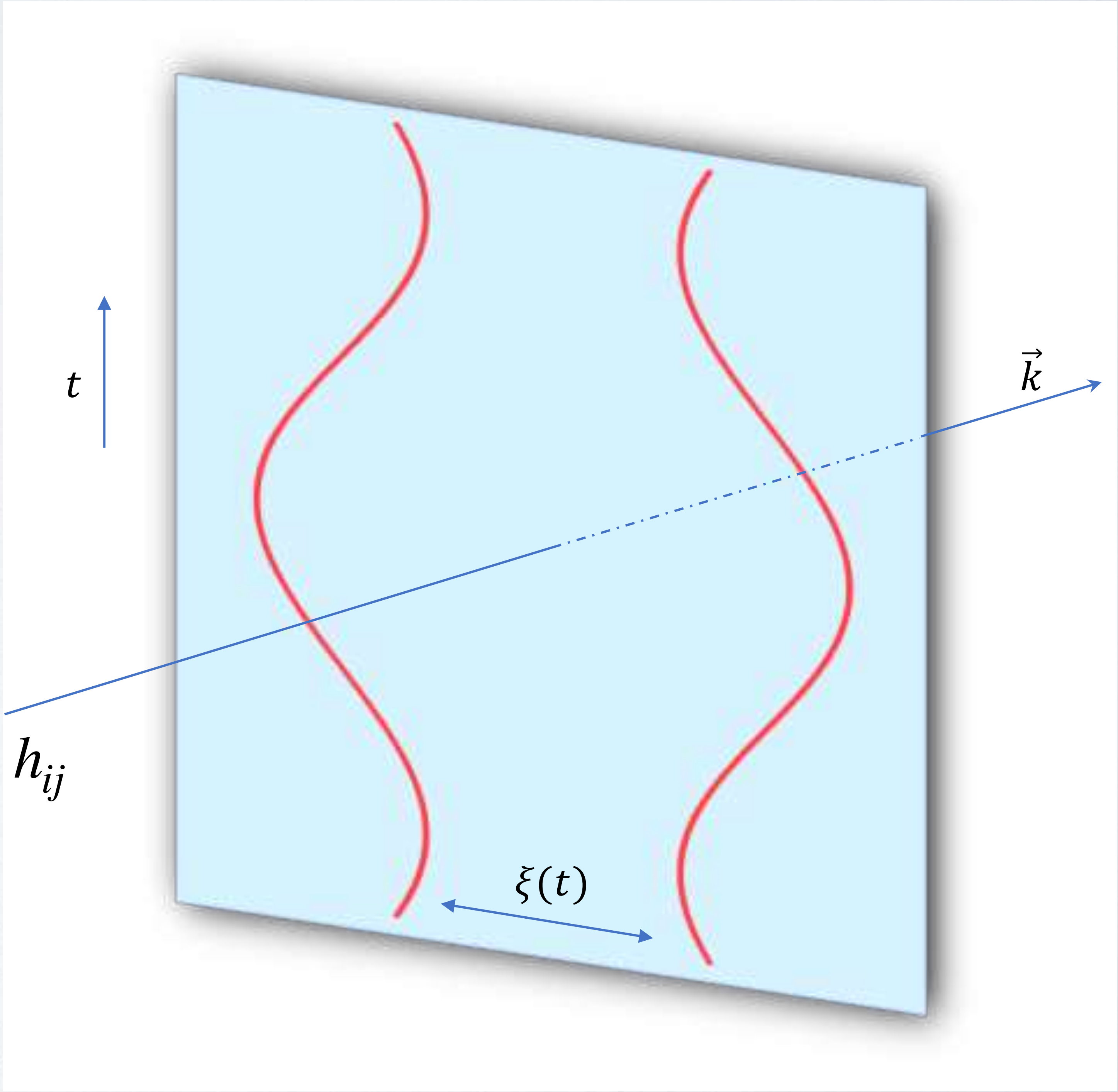
radiation reaction

This is effectively the generalization of the classical geodesic deviation equation when the spacetime metric is a *quantum* field

Because of the noise term, it is no longer a deterministic equation but a stochastic equation

Classical Geodesic Separation by Gravitational Waves

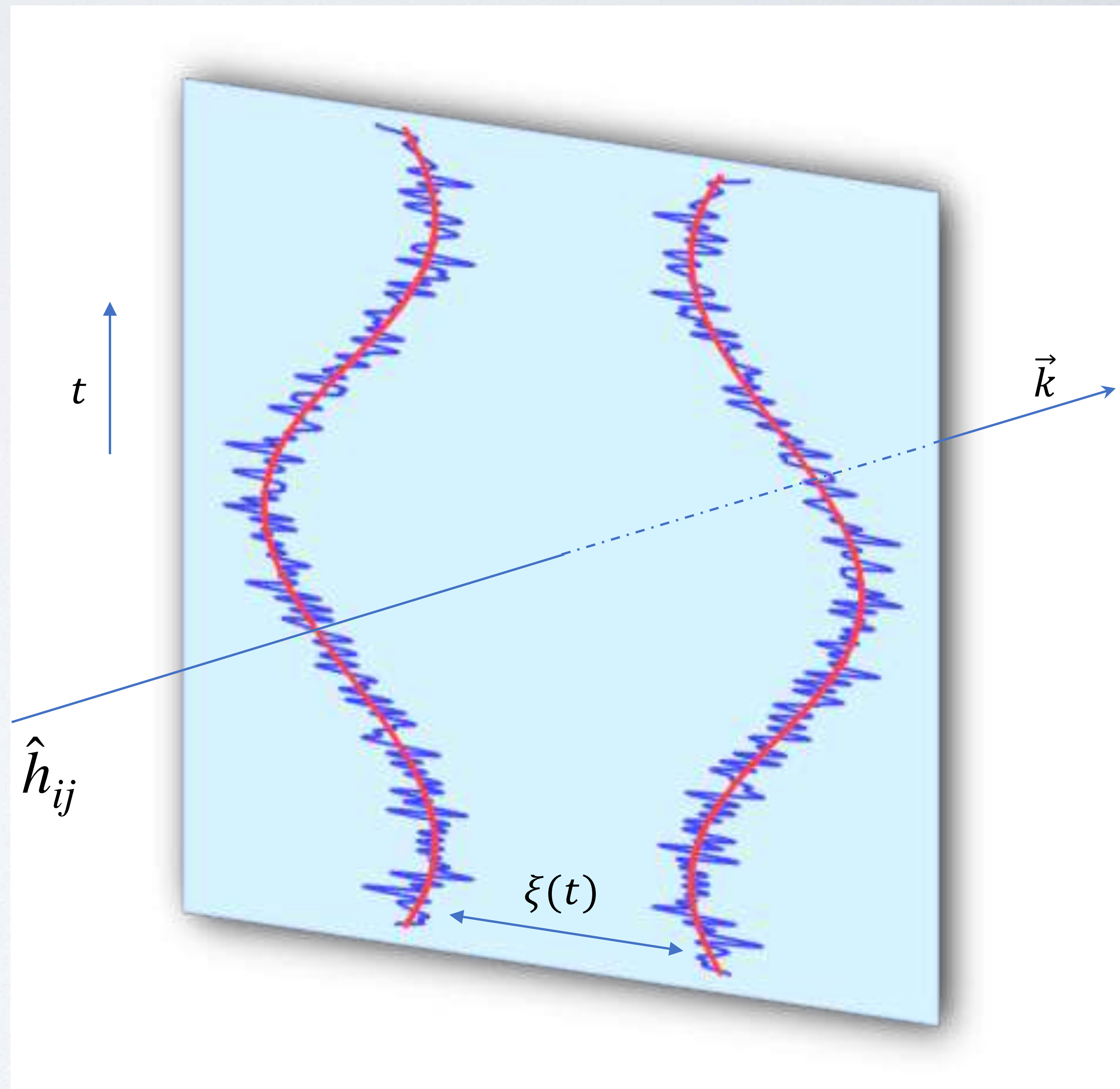
$$\ddot{\xi} = \frac{1}{2}\ddot{h}\xi$$



classical gravitational wave

The Noise of Gravitons

$$\ddot{\xi} \approx \frac{1}{2}\left(\ddot{h} + \ddot{N}_{\Psi}\right)\xi$$



quantized gravitational wave

Is The Noise Detectable?

For the noise to be detectable:

1. Its amplitude should not be too small

2. Its spectrum should be distinguishable from other sources of noise

Noise Spectrum

The power spectrum is explicitly calculable for many classes of quantum states

Moreover, the noise appears to be correlated between multiple detectors

Noise for Quantum States of Gravitational Field

The magnitude of the noise depends on the quantum state as well as on the detector

$$\sigma^2 \sim \xi_0^2 \int_0^{\omega_{\max}} d\omega S(\omega)$$

variance of fluctuation

arm length

detector sensitivity

noise power spectrum

For the vacuum state and for coherent states (classical gravitational waves from weak sources)

$$\sigma_0 \sim \ell_P \xi_0 \omega_{\max}/c \lesssim 10^{-36} - 10^{-38} \mathrm{m}$$

For thermal states (cosmic gravitational background, evaporating black holes)

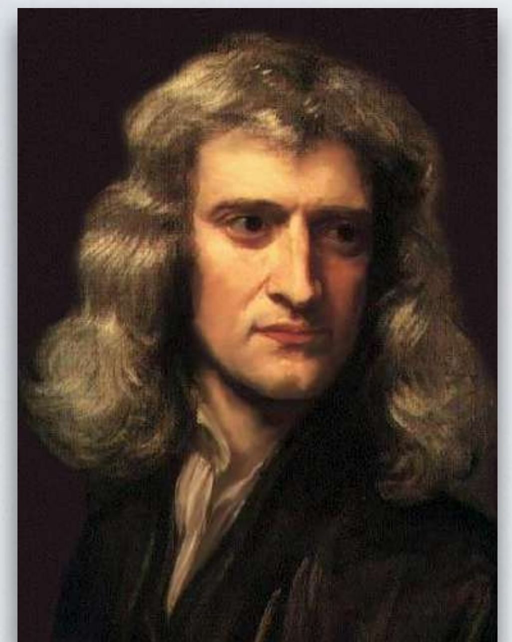
$$\sigma \sim \sigma_0 \sqrt{k_B T/\hbar\omega_{\max}} \lesssim 10^{-30} - 10^{-34} \mathrm{m}$$

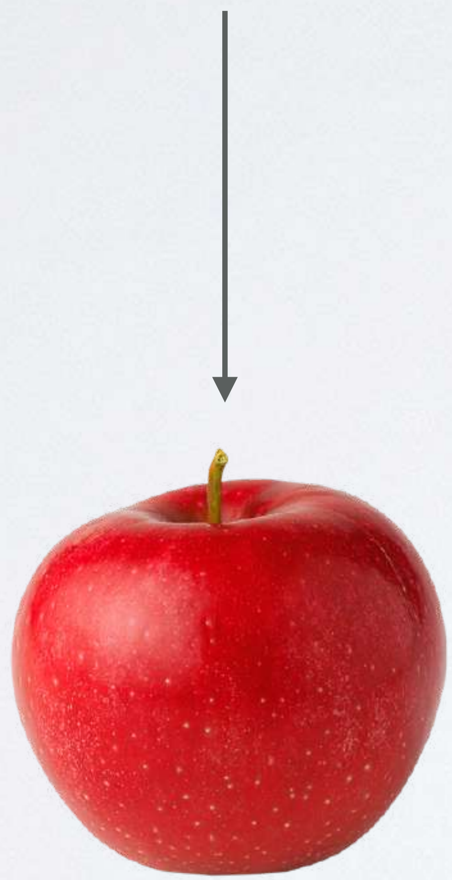
For squeezed states (cosmology, non-linear gravitational waves)

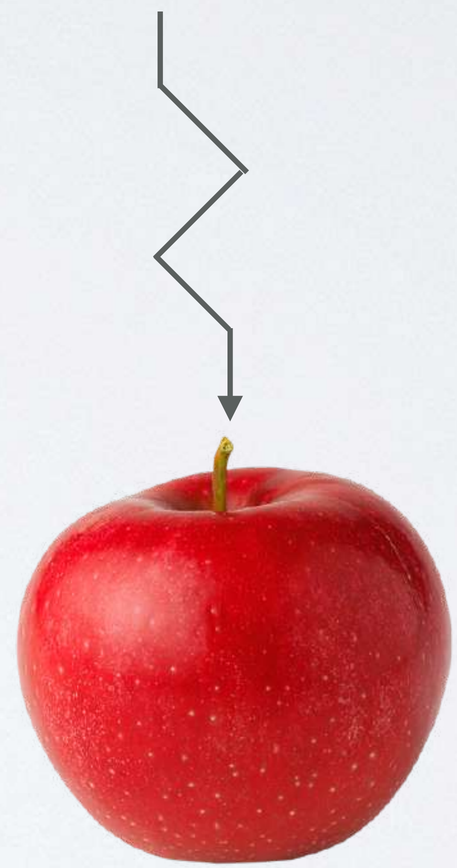
$$\sigma \sim e^{r/2} \sigma_0$$

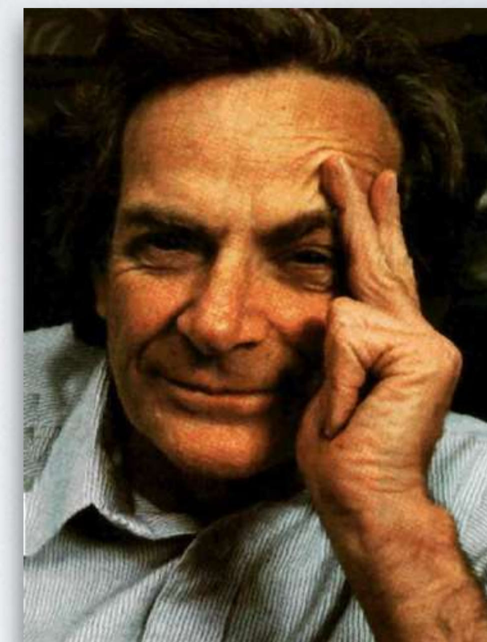
exponential enhancement in squeezing parameter

Quantum Free-Fall









classical free-fall

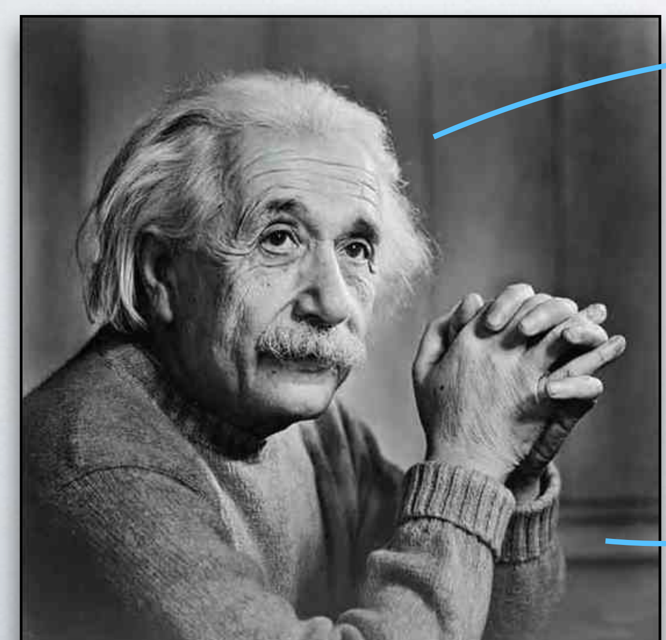
quantum free-fall

Summary

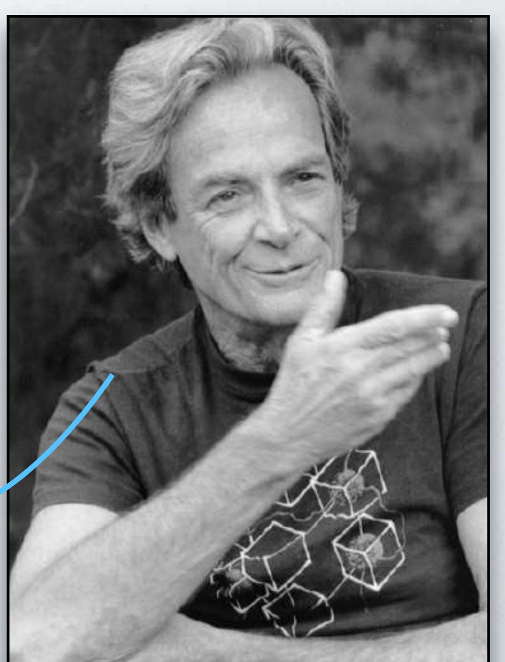
We have developed a formalism to consider general relativity in which the spacetime metric is treated as a quantum field

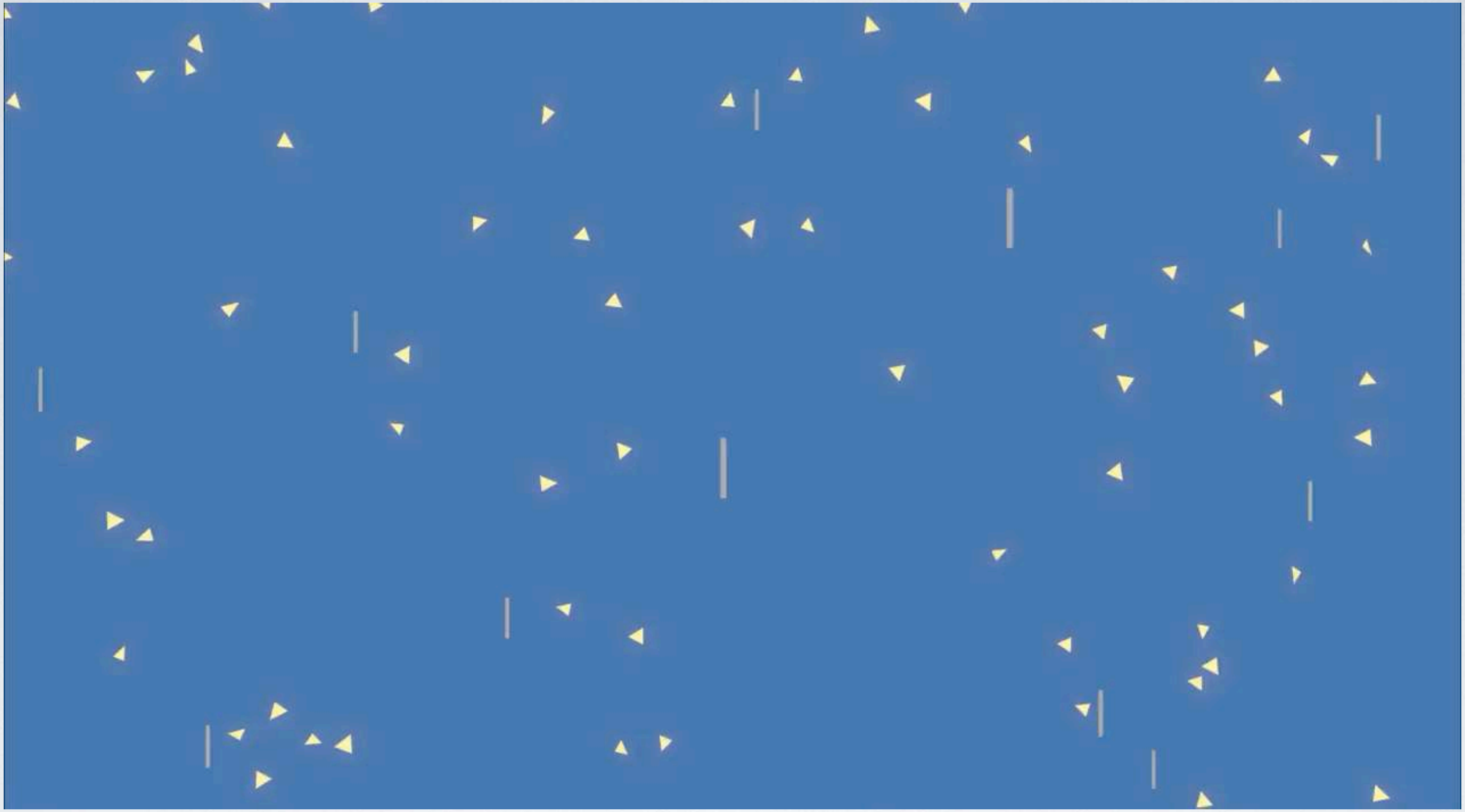
There are potentially observable quantum gravity effects even here on Earth

Falling objects don't just fall straight down but instead experience quantum jitters whose form depends on the quantum state of the gravitational field



The subject brings together gravitational waves, Brownian motion, and quantum mechanics





Thank you