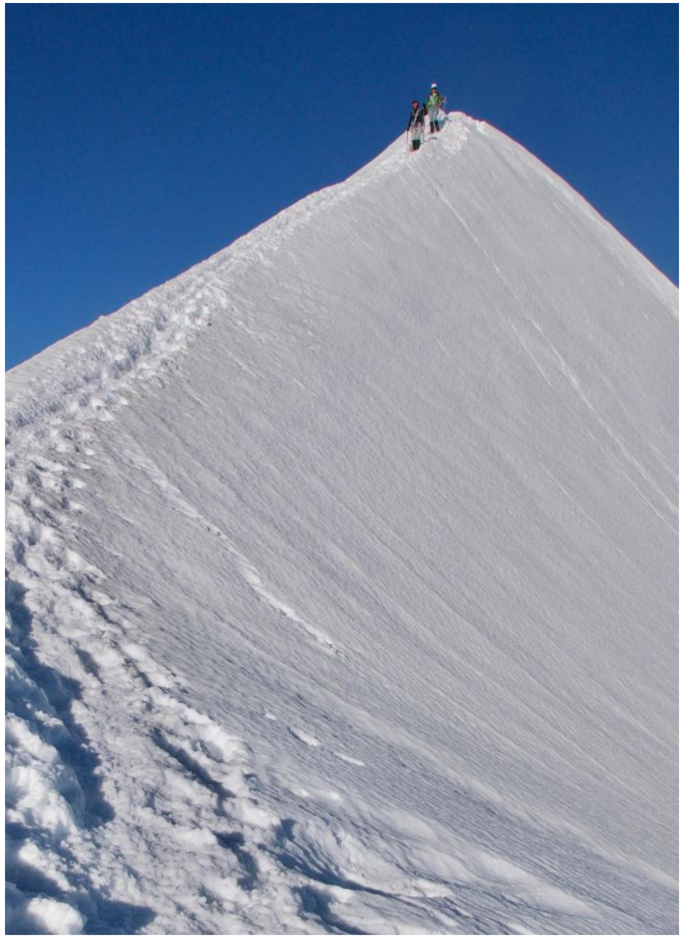
Generalized Hall current from index theorem for gapless edge fermions

Srimoyee Sen,
Iowa State University

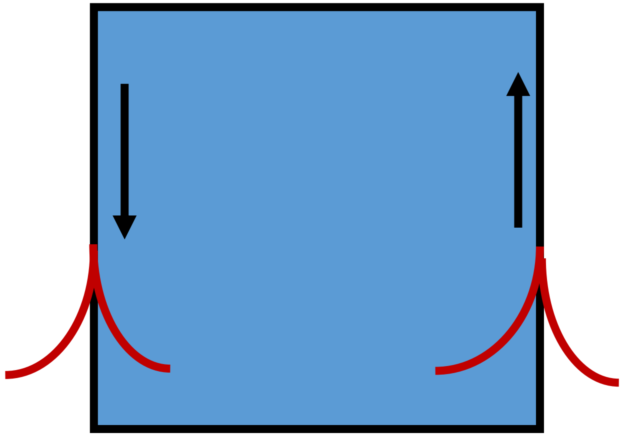


Paper in preparation with

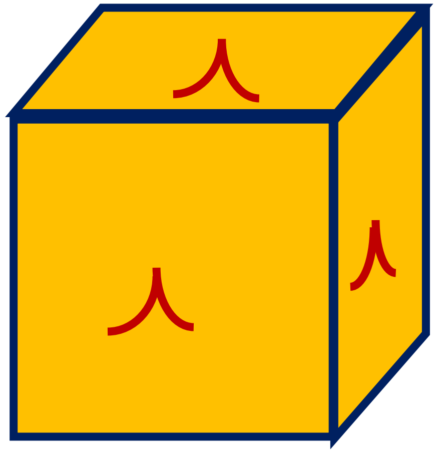
David Kaplan,
University of Washington

Theoretical Physics Colloquium @ASU, Nov 10, 2021

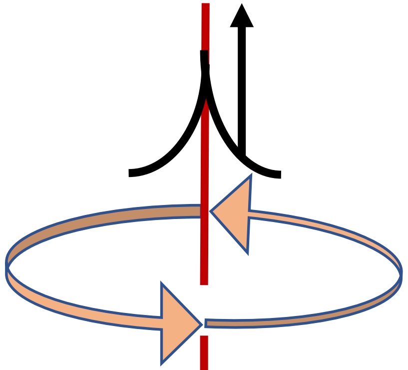
Fermion edge states



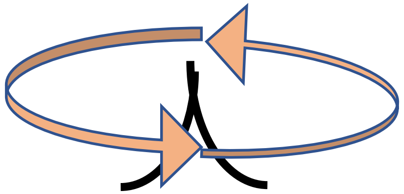
2+1 D bulk, 1+1 edge



3+1 D bulk, 2+1 surface states

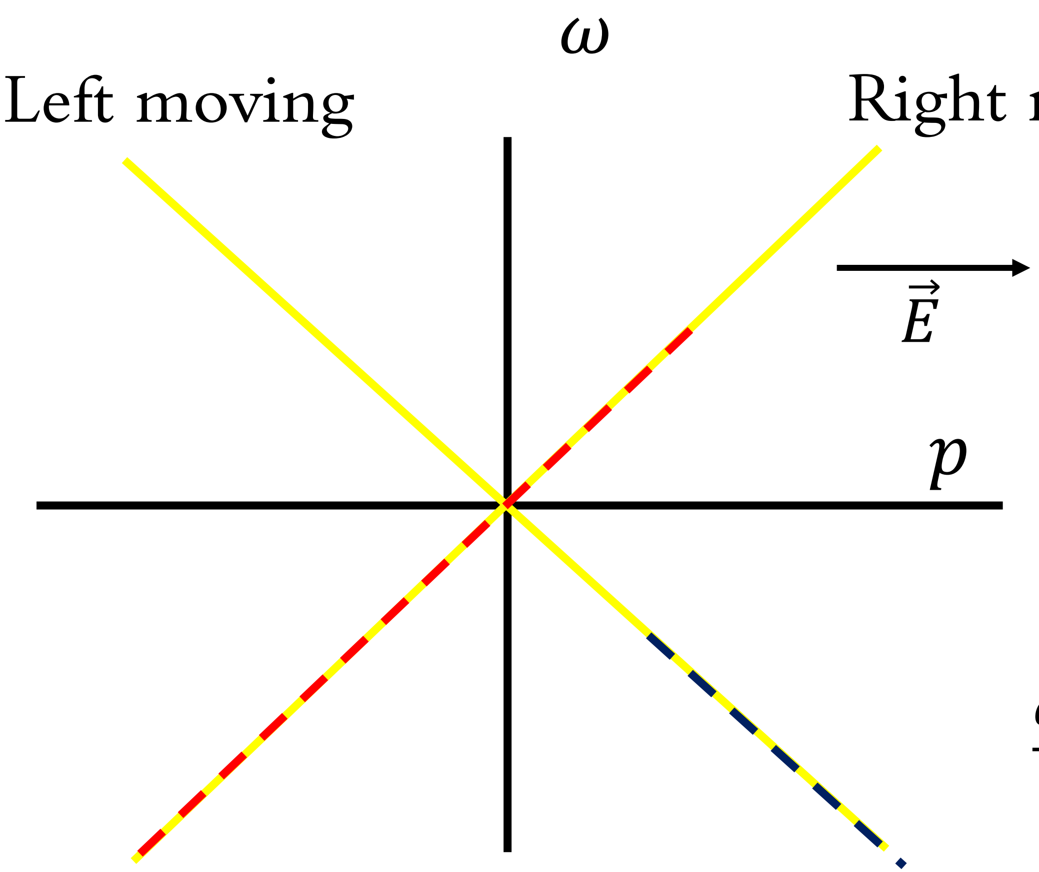


3+1 D bulk, 1+1 vortex string



2+1 D bulk, 0+1 vortex defect

Nice thing about even dimensional chiral fermions: chiral anomaly



$$\frac{dn_R}{dt} = \frac{eE}{2\pi}$$

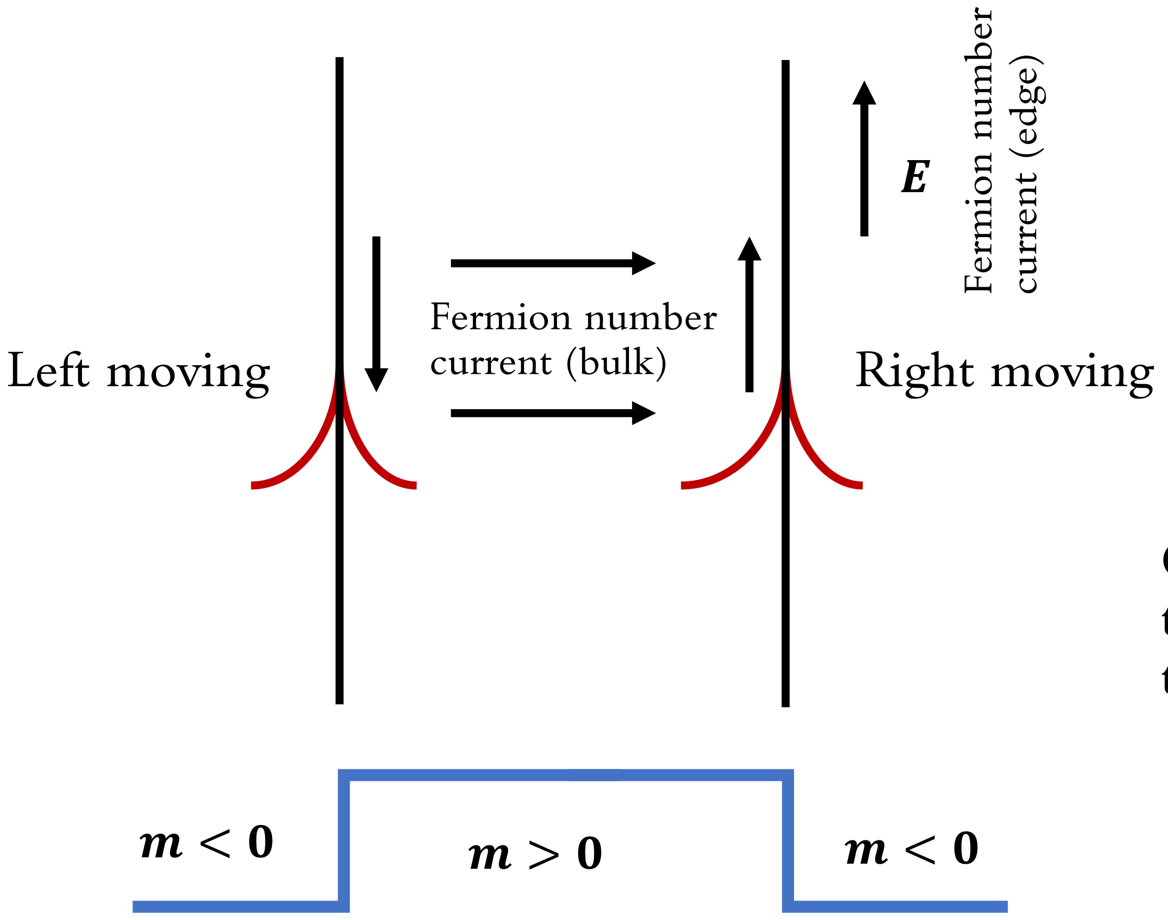
$$\frac{dn_L}{dt} = -\frac{eE}{2\pi}$$

$$\frac{d(n_R - n_L)}{dt} = \frac{eE}{\pi}$$

Also current $\propto E$

Place the anomalous theory in one higher dimensions: quantum Hall effect or domain wall fermions

Nice thing about even dimensional chiral fermions: Inflowing current



$$L_{2+1} = \bar{\psi}(i\gamma\partial - m)\psi$$

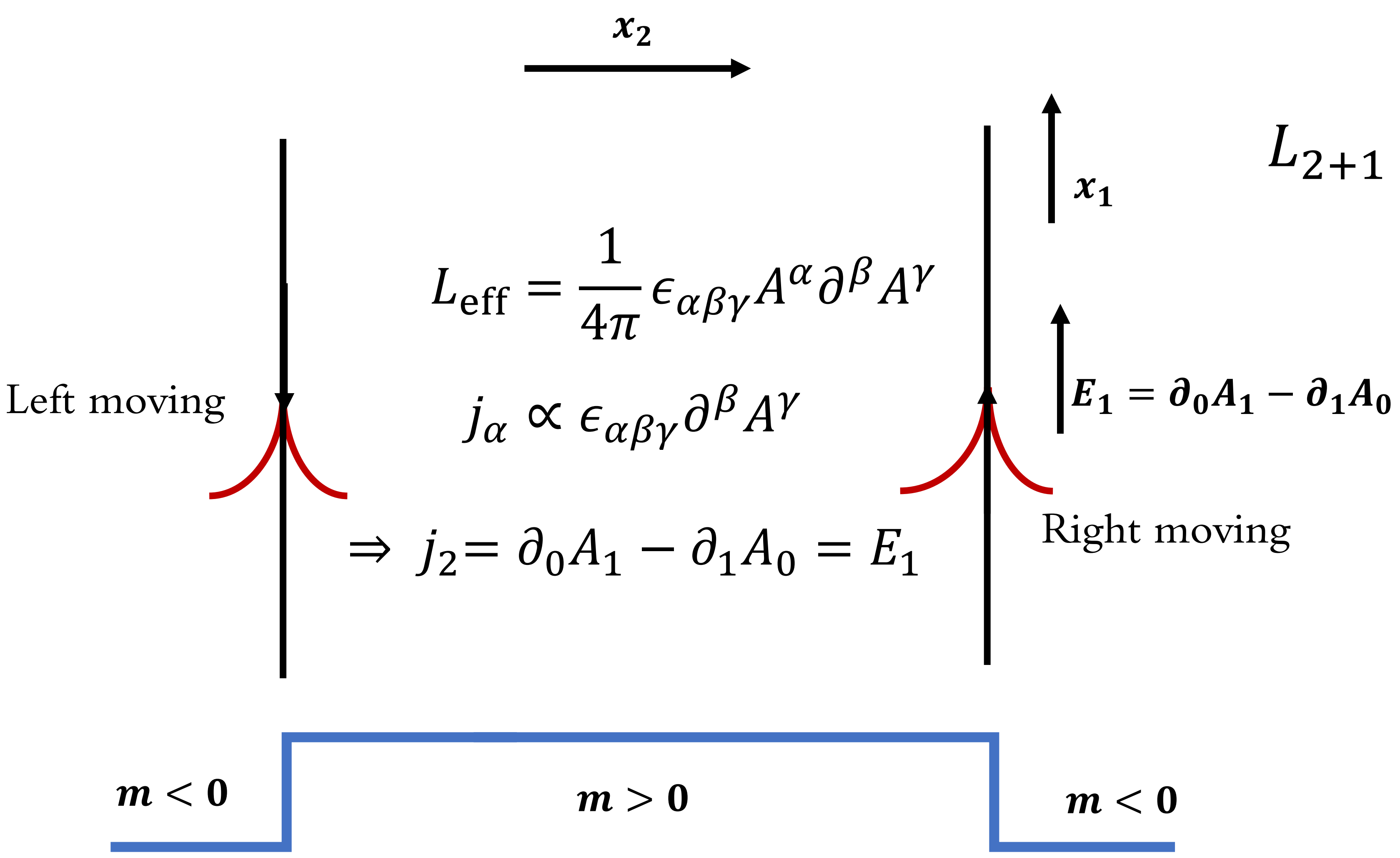
Step function in Dirac mass m in x_3 (domain walls)

Chern-Simons theory in the bulk: current flow to the wall

Chiral fermions on the edge: current along the wall

Current conservation

A bit more detail about inflowing current



$$L_{\text{eff}} = \frac{1}{4\pi} \epsilon_{\alpha\beta\gamma} A^{\alpha} \partial^{\beta} A^{\gamma}$$

$$j_{\alpha} \propto \epsilon_{\alpha\beta\gamma} \partial^{\beta} A^{\gamma}$$

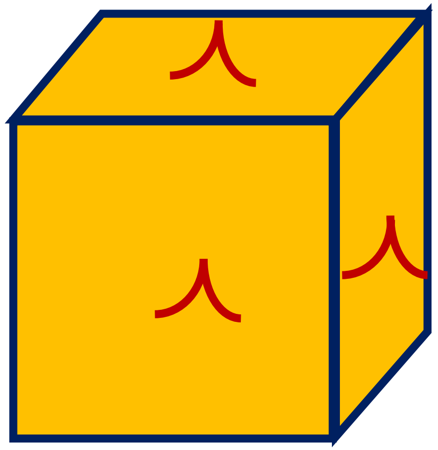
$$\Rightarrow\ j_2 = \partial_0 A_1 - \partial_1 A_0 = E_1$$

$$L_{2+1} = \bar{\psi}(i\gamma D - m)\psi$$

($D = \partial + i\,A$)

Step function in Dirac mass m in x_3 (domain walls)

Not so nice…



3+1 D bulk, 2+1 surface states

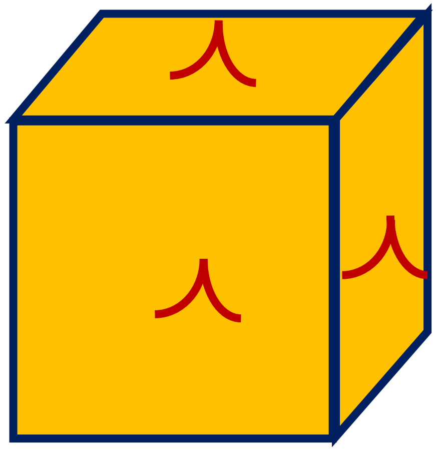
Odd dimensional defect fermions: No chiral anomaly, no current inflow.

Theories without continuous symmetries: No current to flow anywhere.

$$L_{2+1} = \bar{\psi}(\gamma\partial - m)\psi$$

Add a $\psi\psi$ term here: fermion number is broken down to Z_2 … so no current inflow. But the chiral edge states survive.

Not so nice…



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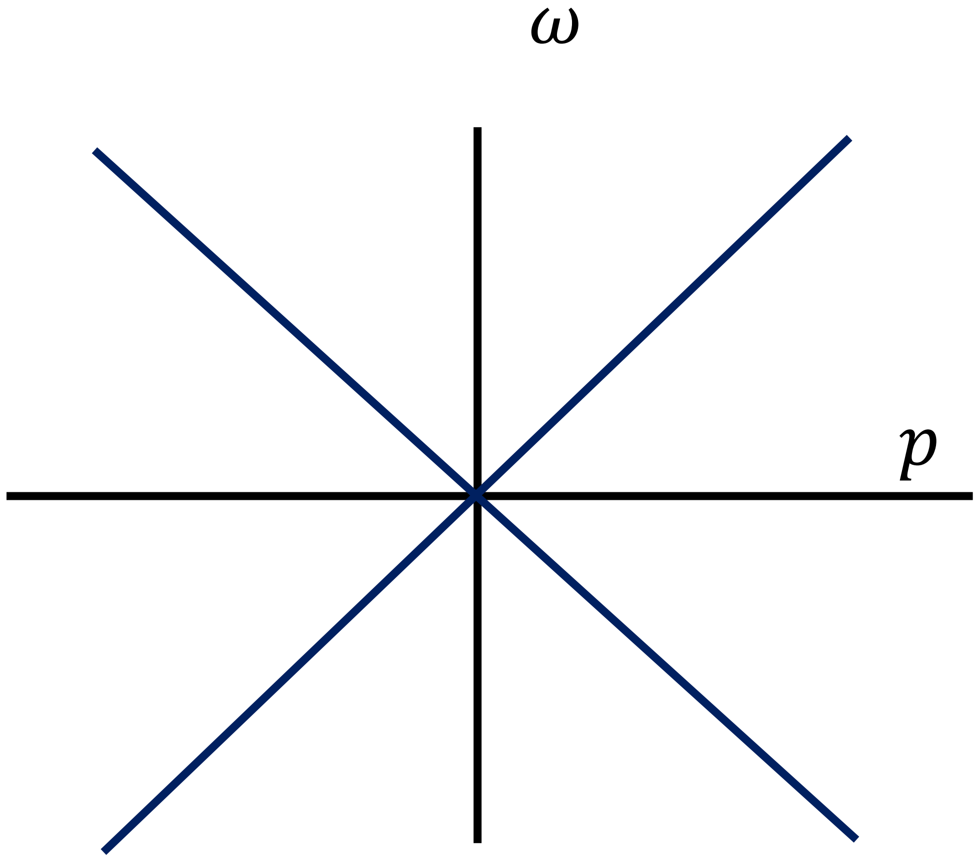
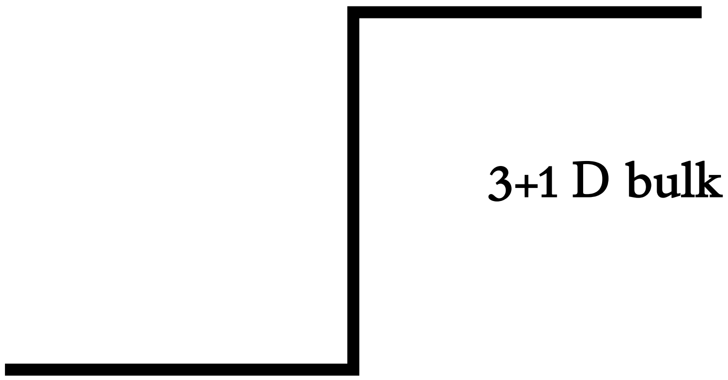
3+1 dimensional topological insulator with 2+1 surface states





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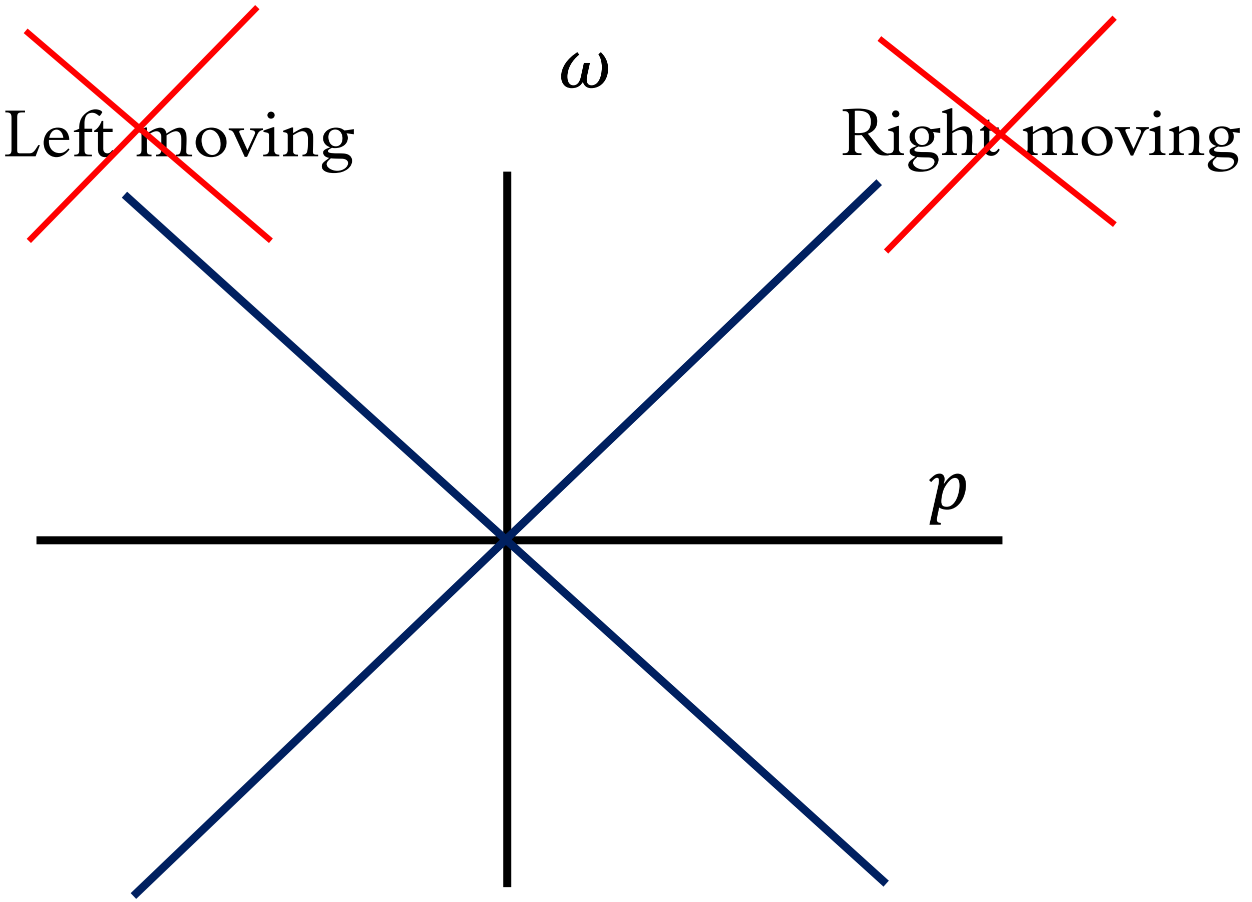
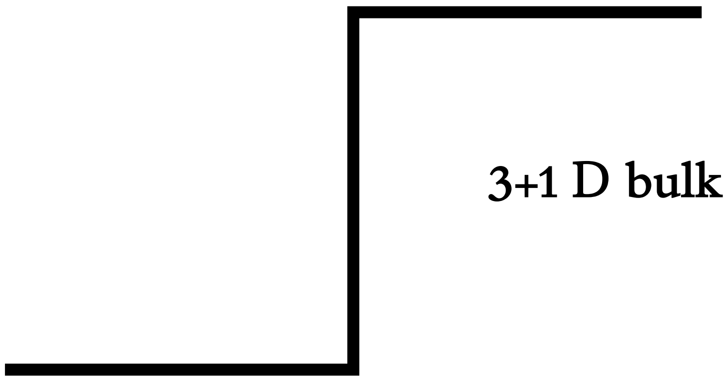
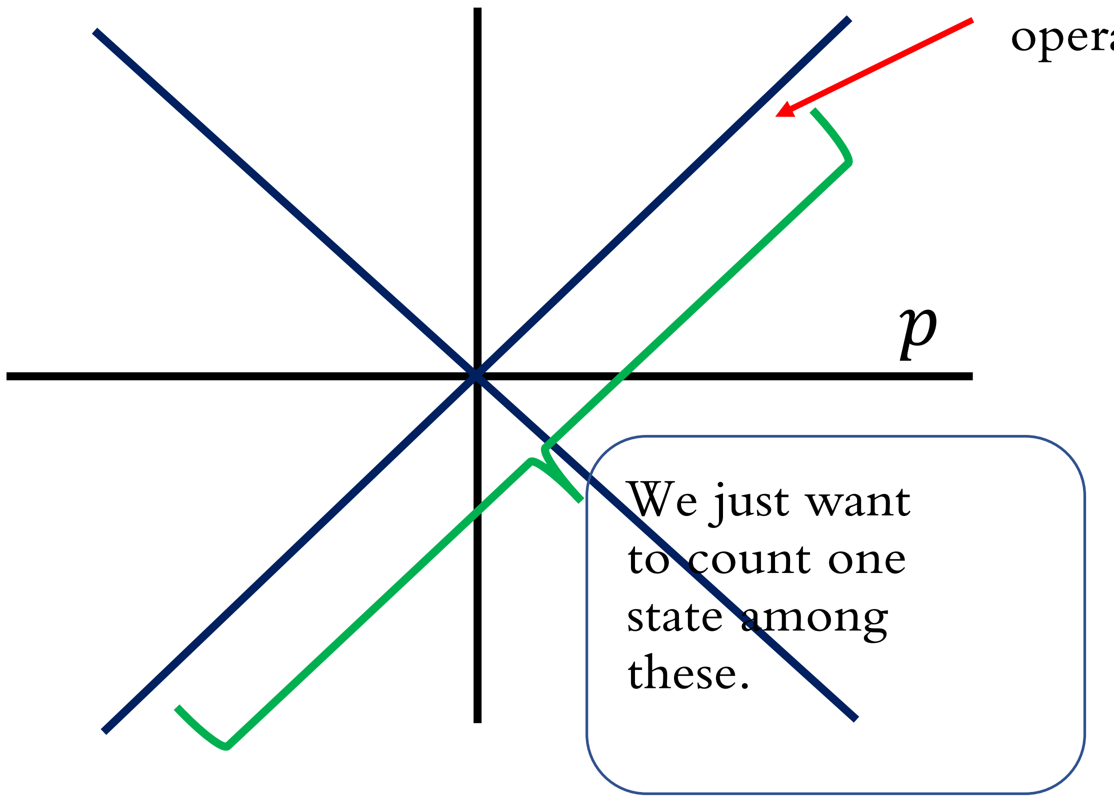


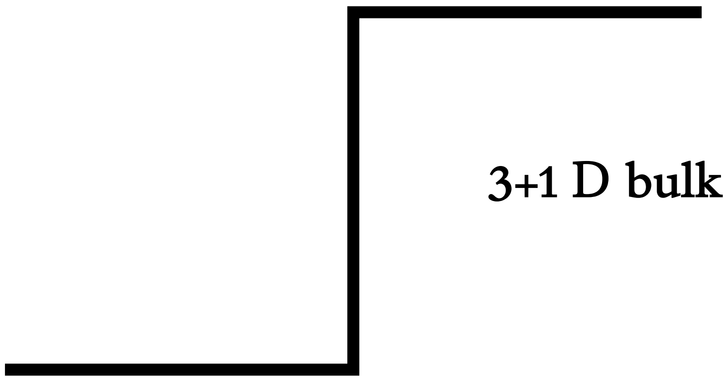


$$L_{3+1} = \bar{\psi}(i\gamma\partial - m)\psi$$

3+1 dimensional topological insulator with 2+1 surface states

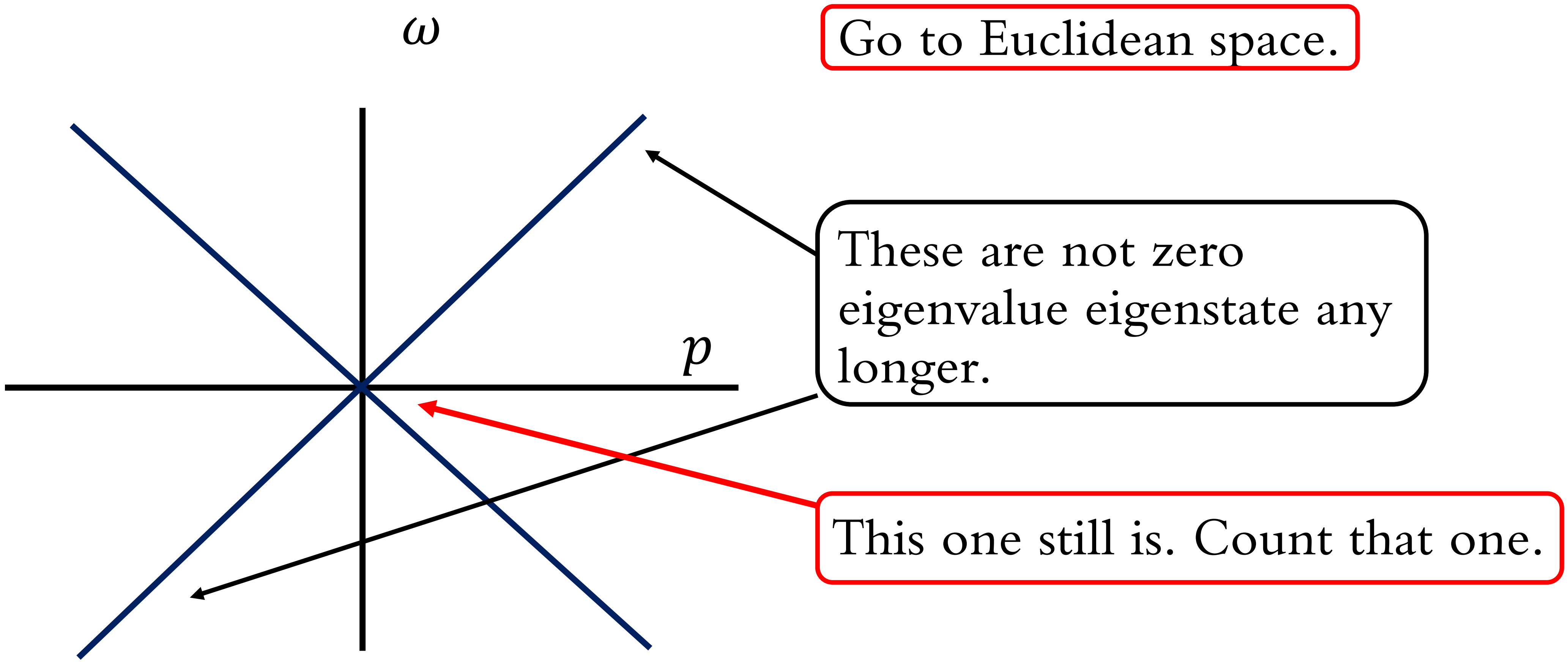
Pick a random state here: it's a zero eigen state of the Dirac operator (Minkowski).





$$L_{3+1} = \bar{\psi}(i\gamma\partial - m)\psi$$

3+1 dimensional topological insulator with 2+1 surface states and other such edge states



What do we want and how do we get it ?

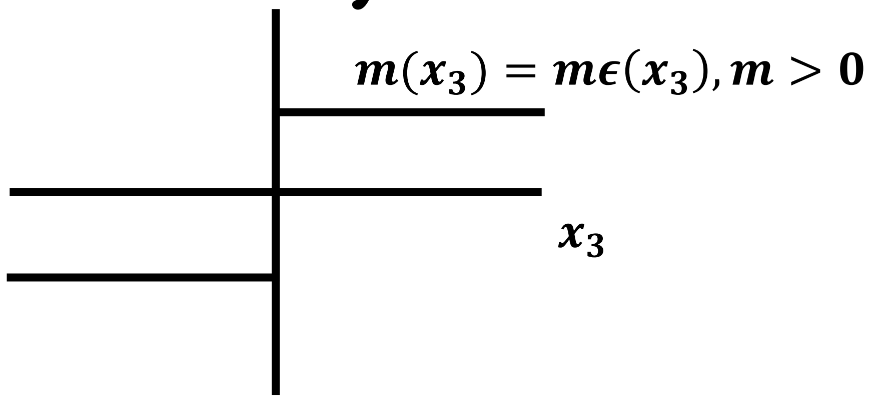
We want a current inflow picture for the not nice ones as well.

Use the index of the Euclidean Dirac operator.

We'll call this current a generalized Hall current.

Present irrespective of what symmetries the original theory has and whether there is a chiral anomaly on the defect.

Look at 1+1 domain walls in 2+1 more carefully



Massless edge states with

$$L = \bar{\psi}\big(i\gamma_\mu \partial_\mu + m(x_3)\big)\psi$$

Solve for massless edge states with:

$$\big(\mathrm{i}\,\gamma_\mu \partial_\mu + m(x_3)\big)\psi = 0$$

$$\Rightarrow \big(i\,\partial_1 - i\,\sigma_3 \partial_2 + i\sigma_2 \partial_3 + m(x_3)\big)\psi(x_1, x_2, x_3) = 0$$

$$\psi = \phi(x_1, x_2)\chi(x_3)$$

$= 0$ $\quad$ $= 0$

$$\begin{pmatrix} 0 & \partial_3 + m\epsilon(x_3) \\ -\partial_3 + m\epsilon(x_3) & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

$\partial_3 + m\epsilon(x_3) \rightarrow D$

$-\partial_3 + m\epsilon(x_3) \rightarrow = D^\dagger$

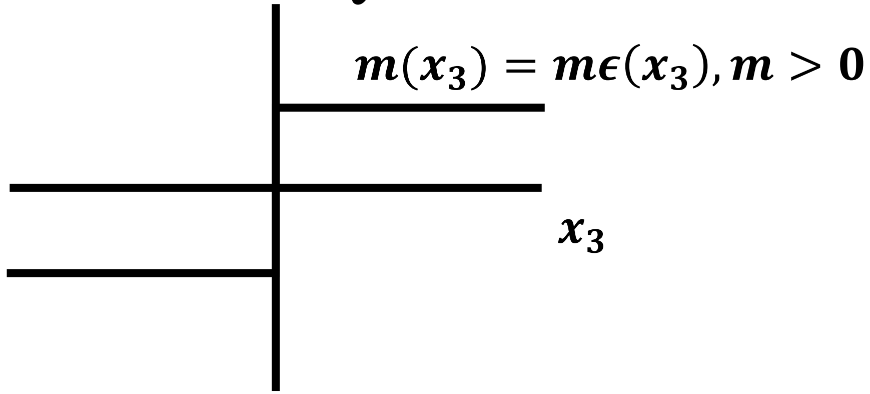
Solutions

$$\phi(x_1, x_2) = e^{ik_1x_1 + ik_2x_2},$$

$$\chi_2(x_3) = e^{-m|x_3|}$$

$$\chi_1 = 0$$

Look at 1+1 domain walls in 2+1 more carefully



Massless edge states with

$D\chi_2 = 0 \Rightarrow D$ has a one normalizable solution with zero eigenvalue. $D^\dagger$has none.

Number of chiral fermions on wall ⟺ # of zero eigenvalues of D −# of zero eigenvalues of $D^\dagger$

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Solve for massless edge states with:

$$\left(\mathrm{i}\,\gamma_\mu\partial_\mu + m(x_3)\right)\psi = 0$$

$$\Rightarrow \left(i\,\partial_1 - i\,\sigma_3\partial_2 + i\sigma_2\partial_3 + m(x_3)\right)\psi(x_1, x_2, x_3) = 0$$

$$\psi = \phi(x_1, x_2)\chi(x_3)$$

$(i\,\partial_1 - i\,\sigma_3\partial_2)\phi = 0$, $\quad (i\sigma_2\partial_3 + m(x_3))\chi = 0$

$$\begin{pmatrix} 0 & \partial_3 + m\epsilon(x_3) \\ -\partial_3 + m\epsilon(x_3) & 0 \end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = 0$$

$\partial_3 + m\epsilon(x_3) = D$

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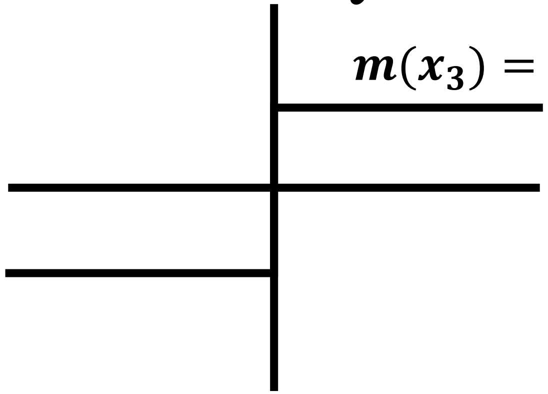
Look at 1+1 domain walls in 2+1 more carefully



Massless edge states with

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Solve for massless edge states with:

$$\left(\mathrm{i}\,\gamma_\mu \partial_\mu + m(x_3)\right)\psi = 0$$

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$= 0$ $\quad$ $= 0$

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$\partial_3 + m\epsilon(x_3) \to D$

$-\partial_3 + m\epsilon(x_3) \to\, = D^\dagger$

$D\chi_2 = 0 \Rightarrow D$ has a one normalizable solution with zero eigenvalue. $D^\dagger$has none.

Index of *D*

Number of chiral fermions on wall ⇔ # of zero eigenvalues of D – # of zero eigenvalues of $D^\dagger$

Index

Index of D, $\mathrm{Lim}_{M\to 0}\, I(M) = \mathrm{Lim}_{M\to 0}\, \mathrm{Tr}\left[\frac{M^2}{D^\dagger D + M^2} - \frac{M^2}{DD^\dagger + M^2}\right]$

$$= \mathrm{Lim}_{M\to 0}\, \mathrm{Tr}\left[\Gamma_\chi \frac{M}{M+K}\right]$$

$$K = \begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix} \qquad \Gamma_\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

What if we use a Euclidean Fermion operator for D ?

Use that D, the edge states of which we are interested in.

Index to doubled theory

Index of D, $\text{Lim}_{M\to 0}\, I(M) = \text{Lim}_{M\to 0}\text{Tr}\left[\Gamma_\chi \frac{1}{M+K} M\right]$

Looks like a fermion propagator

$$S = \int d^{d+1}x\, \bar{\Psi}(K+M)\Psi$$

← Spinors double the size of the original spinors: has its own fermion number symmetry and chiral symmetry in the $M \to 0$ limit.

In this new theory $I(M) = M \int d^{d+1}x \langle \bar{\Psi}(x)\Gamma_\chi \Psi(x)\rangle$

Index of D

Just because the Minkowski theory has massless edge states, does not imply the Euclidean Dirac operator will have nonzero index.

No reason to expect a normalizable zero mode.

We will turn on diagnostic background fields needed to localize the state to turn it into a zero mode.

Thus, every time there is a massless edge state in the Minkowski theory, there will be a corresponding index in the Euclidean theory.

Doubled theory

$$S = \int d^{d+1}x\, \bar{\Psi}(K + M)\Psi$$

This doubled theory amounts to introducing an extra dimension… But the fields don't depend on it.

Gamma matrices of this theory:

$$\frac{\delta(K\Psi)}{\delta\, \partial_\mu \Psi} = \Gamma_\mu \Psi$$

Define an axial current: $J_\mu = \bar{\Psi}\Gamma_\mu \Gamma_\chi \Psi$

Satisfies a Ward-Takahashi identity $\partial_\mu J_\mu^\chi = 2M\, \bar{\Psi}\Gamma_\chi \Psi + \mathcal{A}$ → Variance of the measure of the path integral (ends up being zero)

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What we want! The divergence of a current.

Satisfies a Ward-Takahashi identity

$$\partial_\mu J_\mu^\chi = 2M\, \bar{\Psi}\Gamma_\chi \Psi + \mathcal{A}$$

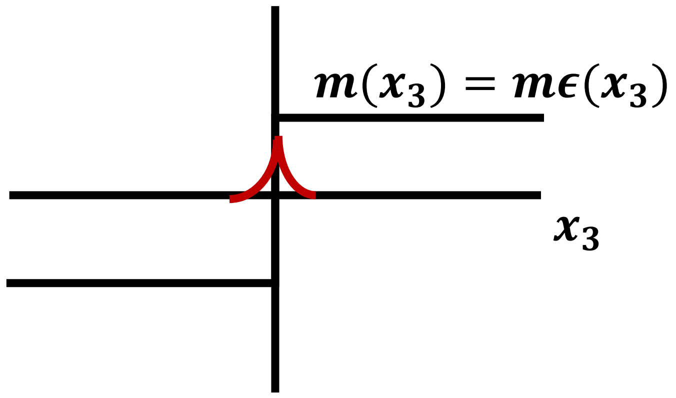
Variance of the measure of the path integral (ends up being zero)

Divergence of the current

$$\text{Lt}_{M\to 0}\partial_\mu J_\mu^\chi = \text{Lt}_{M\to 0}\left(2M\,\bar{\Psi}\Gamma_\chi\Psi\right)$$
$$= \text{Lt}_{M\to 0}2I(M) = 2I(0)$$

The index of the relevant Dirac operator is being written in terms of the divergence of an axial current in the doubled theory.

Example: vanilla domain wall fermion in a doubled theory



$$D = \begin{pmatrix} -\partial_3 - m\epsilon(x_3) & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_3 - m\epsilon(x_3) \end{pmatrix}$$

$$D^\dagger = \begin{pmatrix} \partial_3 - m\epsilon(x_3) & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & -\partial_3 - m\epsilon(x_3) \end{pmatrix}$$

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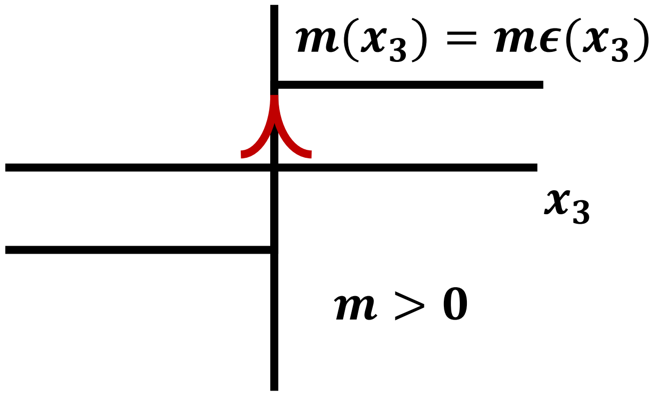

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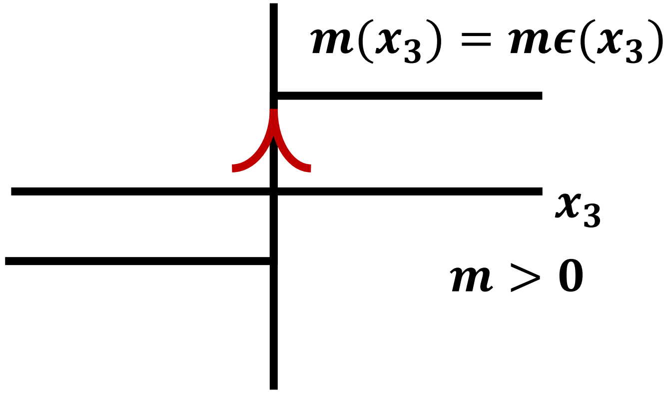
Both have zero eigenvalue solutions

$$D\chi = \begin{pmatrix} -\partial_3 - m\epsilon(x_3) & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_3 - m\epsilon(x_3) \end{pmatrix} \begin{pmatrix} e^{-m|x_3|} \\ 0 \end{pmatrix} \qquad D^\dagger\chi' = \begin{pmatrix} \partial_3 - m\epsilon(x_3) & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & -\partial_3 - m\epsilon(x_3) \end{pmatrix} \begin{pmatrix} 0 \\ e^{-m|x_3|} \end{pmatrix}$$

$$\Gamma_\chi \chi = \chi, \qquad \Gamma_\chi \chi' = -\chi'$$

So, the index will be zero… we are unable to capture the edge fermion

Domain wall fermion in a doubled theory – diagnostic fields



Note that neither solution normalizable in the x_1, x_2 directions.

$$\chi = \begin{pmatrix} 0 \\ e^{-m|x_3|} \end{pmatrix} \qquad \chi' = \begin{pmatrix} e^{-m|x_3|} \\ 0 \end{pmatrix}$$

The solutions are not localized in x_1, x_2

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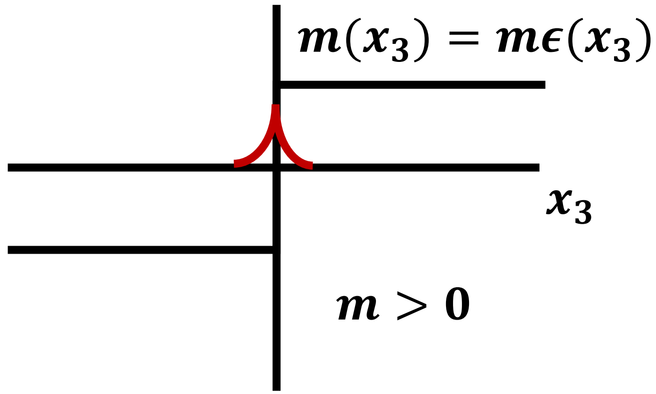

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The solutions are not localized in x_1, x_2

Turn on A_1, A_2 background gauge fields.

$$D = \begin{pmatrix} -\partial_3 - m\epsilon(x_3) & \partial_1 + A_2 - i(\partial_2 - A_1) \\ \partial_1 - A_2 + i(\partial_2 + A_1) & \partial_3 - m\epsilon(x_3) \end{pmatrix} \qquad D^\dagger = \begin{pmatrix} \partial_3 - m\epsilon(x_3) & -(\partial_1 + A_2) + i(\partial_2 - A_1) \\ -(\partial_1 - A_2) - i(\partial_2 + A_1) & -\partial_3 - m\epsilon(x_3) \end{pmatrix}$$

Diagnostic fields (Use the 2D index theorem)

$$A_2 \sim -\frac{1}{r}\cos\theta,\ A_1 \sim \frac{1}{r}\sin\theta$$

$$D = \begin{pmatrix} -\partial_3 - m\epsilon(x_3) & \partial_1 + A_2 - i(\partial_2 - A_1) \\ \partial_1 - A_2 + i(\partial_2 + A_1) & \partial_3 - m\epsilon(x_3) \end{pmatrix}$$

✓

Has one normalizable zero mode

$$D^\dagger = \begin{pmatrix} \partial_3 - m\epsilon(x_3) & -(\partial_1 + A_2) + i(\partial_2 - A_1) \\ -(\partial_1 - A_2) - i(\partial_2 + A_1) & -\partial_3 - m\epsilon(x_3) \end{pmatrix}$$

✗

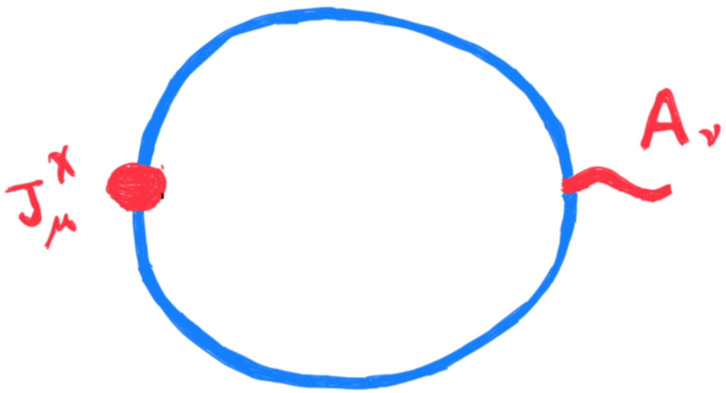
No normalizable zero mode

☺

We have localized the solution: the index is nonzero

$$\mathrm{Lim}_{M \to 0} \partial_\mu J_\mu^\chi = 2I(0) \neq 0$$

Now we can compute the current diagrammatically.



The original theory and the doubled one

$\gamma_i D_i$

$$L_3 = \bar{\psi}\, D\, \psi, D = D + m(x_3), D_i = \partial_i - ie\, A_i, \gamma_i = \sigma_i, m(x_3) = m_0 \frac{x_3}{|x_3|}$$

Original theory

$$L = \bar{\psi}\, (K + M)\psi, K = K_\mu \Gamma_\mu + M, D_i = \partial_\mu - ie\, A_\mu, \mu = 1,2,3,4$$

Doubled theory

$$\Gamma_i = \sigma_1 \otimes \gamma_i, \Gamma_4 = \sigma_2 \otimes 1, \Gamma_\chi = \sigma_3 \otimes 1$$

Doubled theory gamma matrices

The current

The doubled Dirac operator looks like $K = i\,\Gamma_\mu l_\mu + m\Gamma_4$

So, domain wall Dirac mass in the original theory acts like a background gauge field in the doubled theory.

$$J_i^\chi = -i\,q\int \frac{d^3l}{(2\pi)^3} tr\left[\Gamma_\mu\Gamma_\chi \frac{1}{i(\not{l}+\not{p}+m\Gamma_4)+M}\Gamma_\nu\frac{1}{i(\not{l}+m\Gamma_4)+M}\right]A_\nu(p)$$

$$= \mathrm{i}\,\frac{\mathrm{qm}}{2\pi}\epsilon_{ijk}p_jA_k(p)2\frac{\mathrm{Arctan}\left(\frac{p}{2|m|}\right)}{p}$$

And the divergence

$$\int d^2x\,J_3^\chi = \mathrm{Lim}_{\mathrm{p}\to 0}\mathrm{i}\frac{\mathrm{qm}}{2\pi}\epsilon_{ijk}p_jA_k(p)2\frac{\mathrm{Arctan}\left(\frac{p}{2|m|}\right)}{p} = \frac{\frac{q}{2\pi}m(x_3)}{|m(x_3)|}\int d^2x\,\tilde{F}$$

$$\frac{1}{2}\int d^2x\,dx_3\partial_3J_3^\chi = \frac{q}{2\pi}\int d^2x\,\tilde{F} = I(0) = 1$$

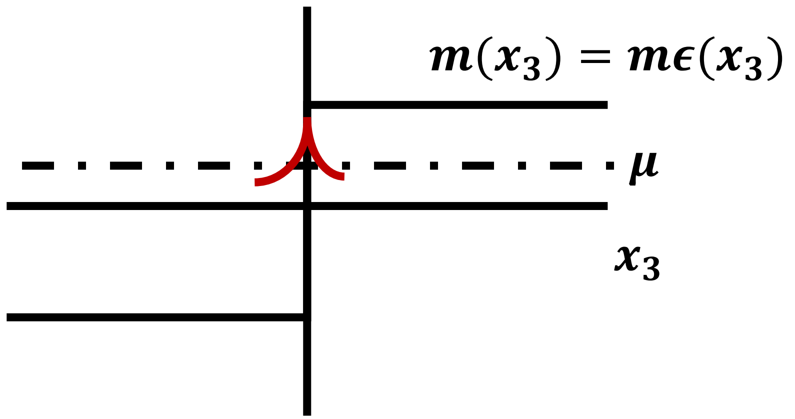
Another example DWF with a Majorana mass (2+1 D)

$$L_{2+1} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - i\frac{\mu}{2}\,\psi^T\sigma_2\psi - \frac{i\mu^*}{2}\,\bar{\psi}\sigma_2\,\bar{\psi}^T$$

So, no continuous Fermion number symmetry and no continuous chiral symmetry.

So, no question of a continuous current inflow to a domain wall since there is no conserved current.

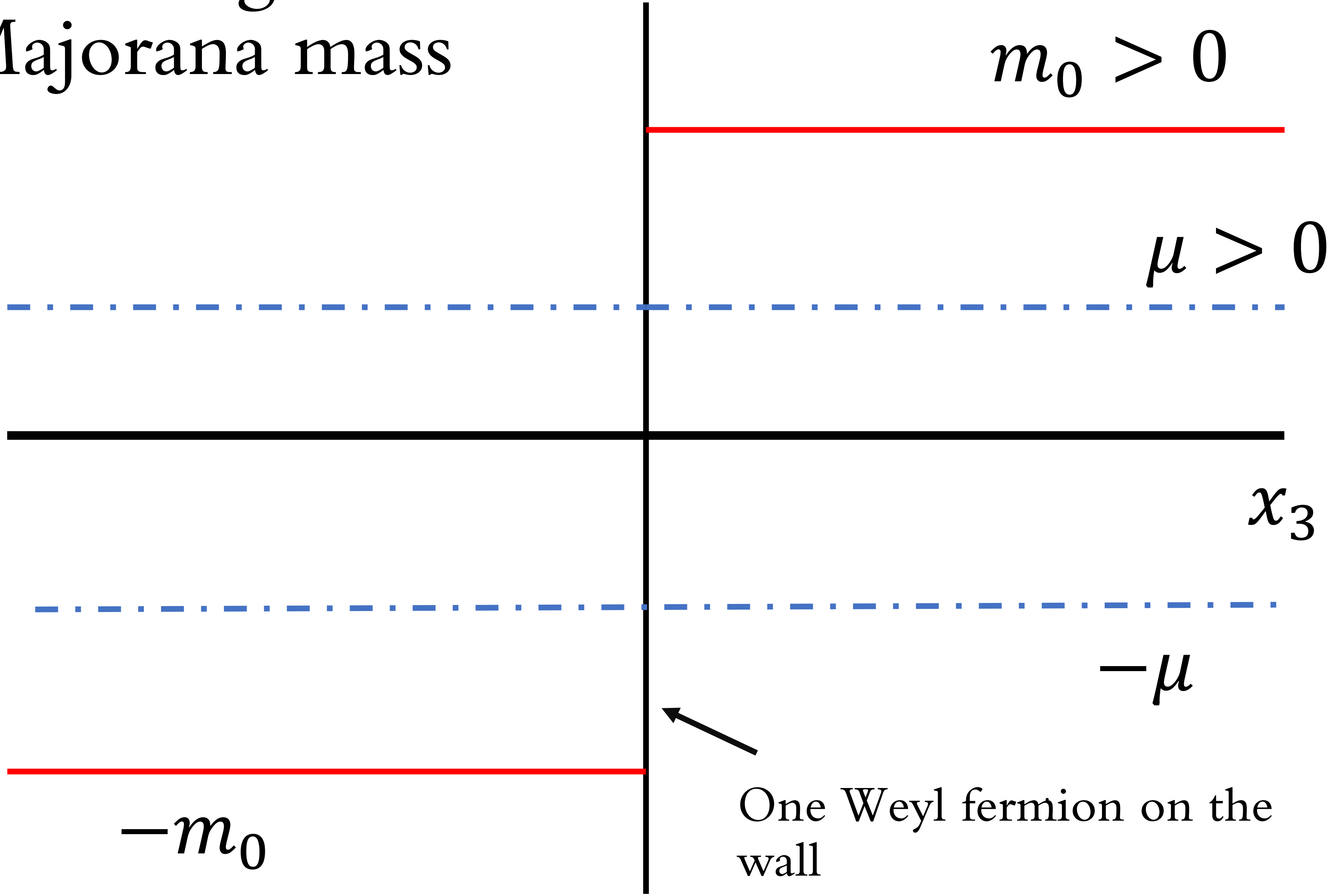
DWF with a Majorana mass (2+1 D)



For a small enough majorana mass nothing changes for the chiral edge state. It's stable. Yet, no current inflow picture for any nonzero μ.

$$\psi(x) = \begin{pmatrix} i\, c_{1,>}\, e^{-(m-\mu)x_3} + c_{2,>}\, e^{-(m+\mu)x_3} \\ 0 \end{pmatrix} \theta(x_3) + \begin{pmatrix} i\, c_{2,>}\, e^{(m-\mu)x_3} + i\, c_{1,>}\, e^{(m+\mu)x_3} \\ 0 \end{pmatrix} \theta(-x_3)$$

DWF edge states with Majorana mass



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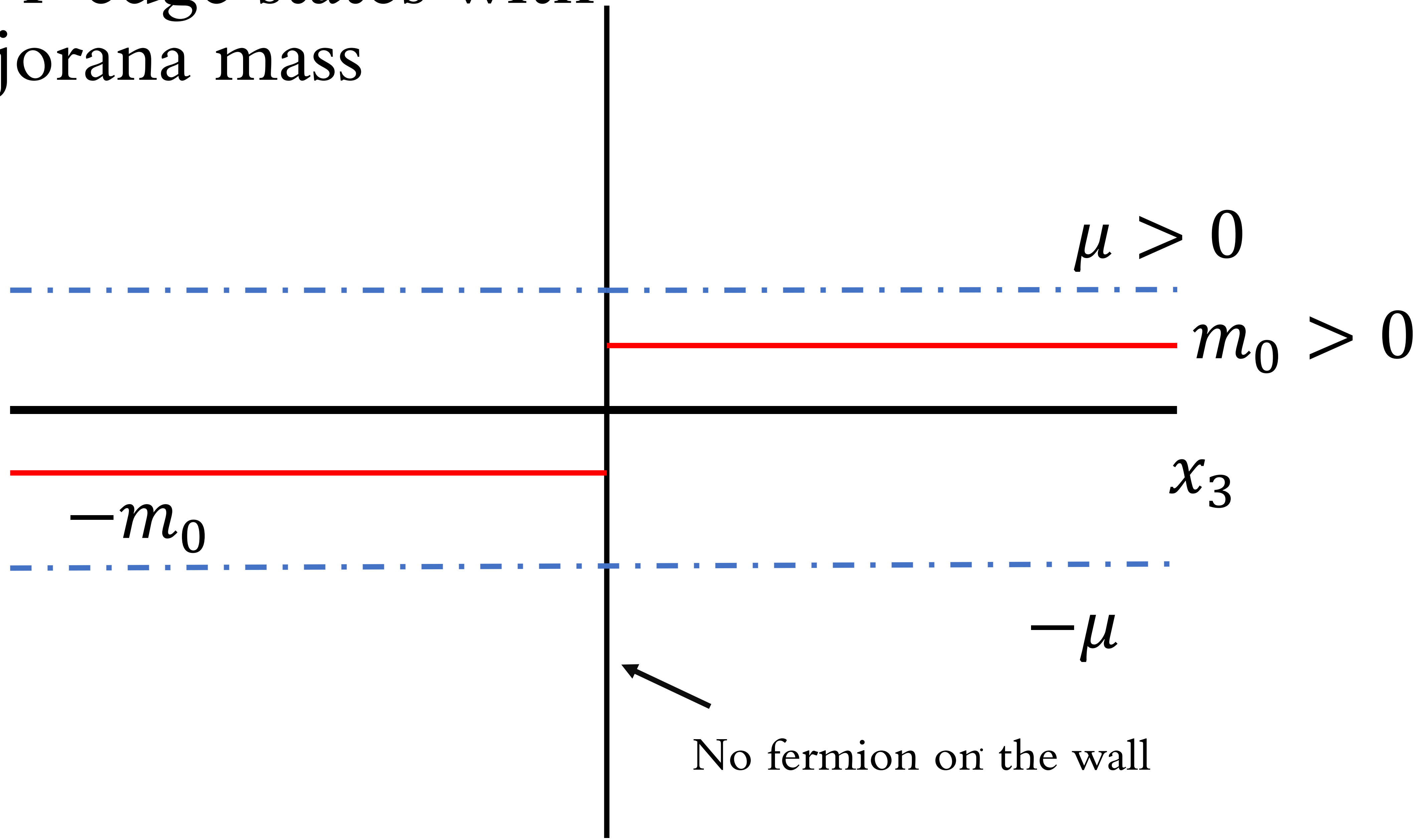
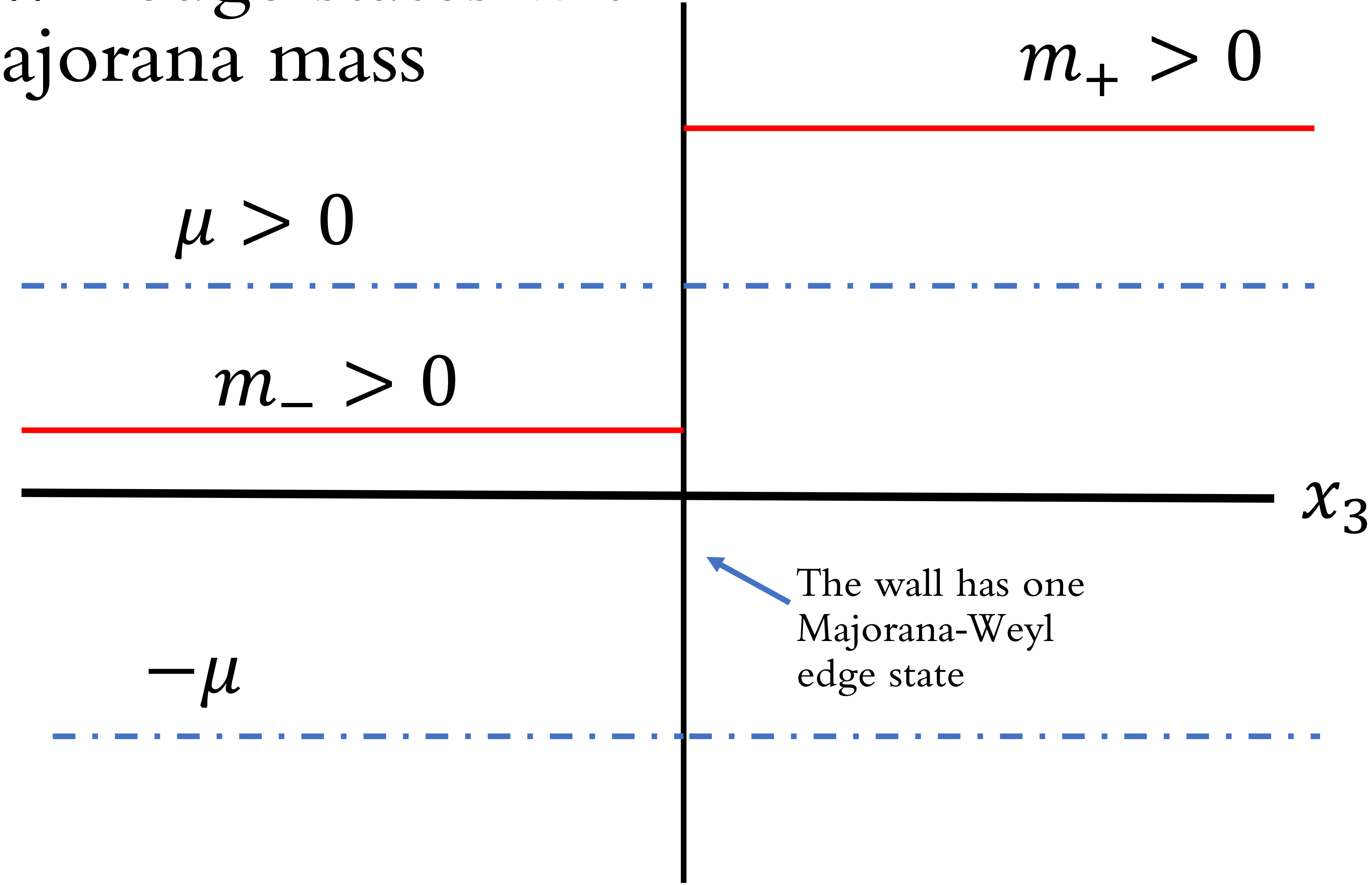


DWF edge states with Majorana mass



The doubled theory

$$\psi(x) = \begin{pmatrix} i\, c_{1,>}\, e^{-(m_+-\mu)x_3} + c_{2,>}\, e^{-(m_++\mu)x_3} \\ 0 \end{pmatrix} \theta(x_3) + \begin{pmatrix} c_{2,>}\, e^{(m_--\mu)x_3} + i\, c_{1,>}\, e^{(m_-+\mu)x_3} \\ 0 \end{pmatrix} \theta(-x_3)$$

Real and imaginary parts of the solution fall off at different rates set by $|m-\mu|$ and $|m+\mu|$.

So, write the Lagrangian in terms of real fields

$$\psi = \frac{\chi_1 + i\,\chi_2}{\sqrt{2}} \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

4 by 4 Gamma matrices

$$L_E = \frac{1}{2}\chi^T\, \tilde{C}\, D\, \chi$$

The doubled theory

The K matrix has 8 by 8 gamma matrices

$$K = \begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix}$$

Compute the current with background gauge field

$$J_\mu^\chi = -\frac{\eta}{2\pi}\epsilon_{\mu\nu\rho}F_{\nu\rho}$$

$$\eta = \begin{cases} 1, & |\mu| < m \\ 0, & -|\mu| < m < |\mu| \\ -1, & m < -|\mu| \end{cases}$$

The index

$$\mathcal{I}(0) = \nu \times \begin{cases} 2 & m_\pm > |\mu| \\ 1 & m_- < |\mu| \text{ and } m_+ > |\mu| \\ 1 & m_- > |\mu| \text{ and } m_+ < |\mu| \\ 0 & m_\pm < |\mu| \end{cases}$$

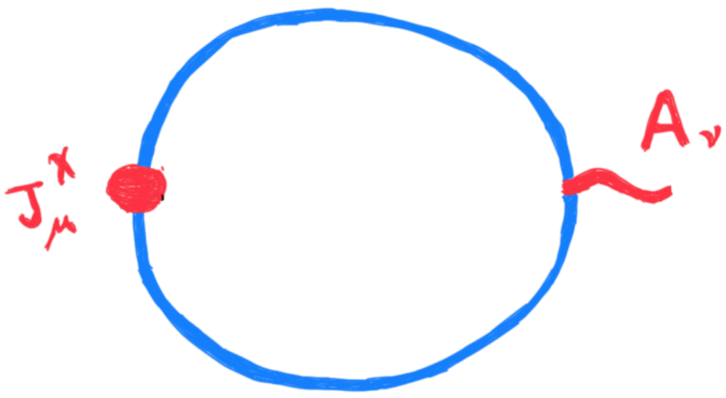
Counting Majorana-Weyl edge states

Two Majorana-Weyl edge state: One Weyl fermion

Other examples

- 0+1 dimensional domain wall in 1+1 dimensional fermionic theory.
- 2+1 dimensional domain wall in 3+1 dimensional fermionic theory: surface states of three dimensional topological insulators.
- 0+1 dimensional vortex edge state in 2+1 dimensional fermion +Higgs theory.

Why do Feynman diagrams work ?

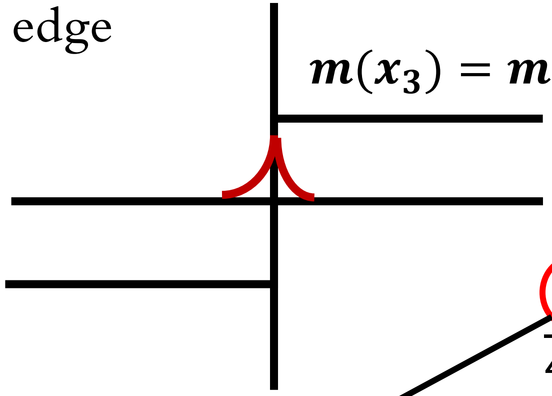


The index of D is an integer. Why are Feynman diagrams producing integers ?

Similar question in the case of quantum Hall effect or domain wall fermions.

One edge state

$\boldsymbol{m(x_3) = m\epsilon(x_3), m > 0}$



$\boldsymbol{x_3}$

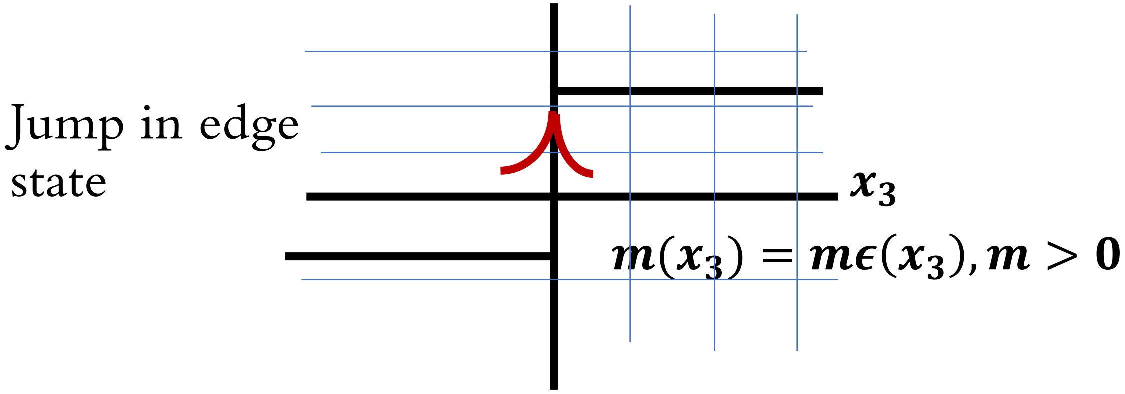
Level one CS theory $k = 1$

One unit of current.

$$\boldsymbol{\frac{k}{4\pi}\epsilon_{\alpha\beta\gamma}A^\alpha\partial^\beta A^\gamma}$$

Comes out of a Feynman diagram

Why do Feynman diagrams work ?



Number of edge states jump and so do the Chern-Simons level.

$$\frac{k}{4\pi}\epsilon_{\alpha\beta\gamma}A^{\alpha}\partial^{\beta}A^{\gamma}$$

k jumps between integers on the lattice. Why ?

Answer in Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs Phys. Rev. Lett. **49**, 405 and Golterman, Jansen, Kaplan Phys.Lett. B301 (1993) 219-223

Feynman diagram is computing the winding number of some map (Torus to sphere in the case of lattice regularizaion). Same is true for the construction here.

What next

Lattice studies of odd dimensional edge states with a current inflow picture.

We have worked out examples. Can all types of fermion edge states be counted this way?

Edge states of higher order topological insulators.