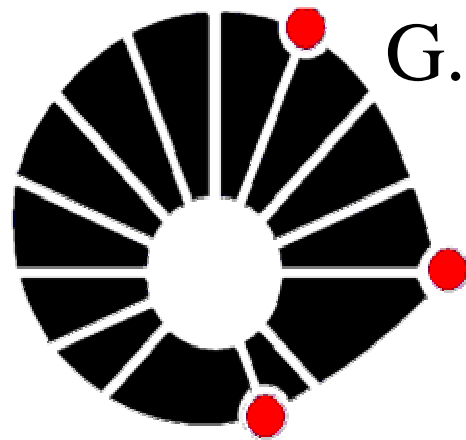


Hydrodynamics with 50 particles. What does it mean and how  
to think about it?



G.Torrieri



**UNICAMP**

Based on [2007.09224](#)



**This is a very speculative talk** so **don't** take any of my answers too seriously, for they could be wrong.

**But think** about the issues I am raising, for they are important!

**A lot of very useful context** in Pavel Kovtun's excellent talk,  
<https://m.youtube.com/watch?feature=youtu.be&v=s30XzAX-XnM>

Much of the same issues, but an "orthogonal" perspective!

Also a great workshop going on right now on these topics,  
<https://indico.ectstar.eu/event/94/>



- The necessity to redefine hydro
    - Small fluids and fluctuations
    - Statistical mechanicians and mathematicians
  - A possible answer:
    - Describing equilibrium at the operator level using the Zubarev operator
    - Defining non-equilibrium at the operator level using Crooks theorem
- Relationship to usual hydrodynamics analogous to "Wilson loops" vs "Chiral perturbation" regarding usual QCD
- Discussion, extensions, implementations etc.

Some experimental data warmup (Why the interest in relativistic hydro ?)  
(2004) Matter in heavy ion collisions seems to behave as a perfect fluid,  
characterized by a very rapid thermalization



### RHIC Scientists Serve Up 'Perfect' Liquid

New state of matter more remarkable than predicted — raising many new questions

April 18, 2005

TAMPA, FL — The four detector groups conducting research at the [Relativistic Heavy Ion Collider \(RHIC\)](#) — a giant atom "smasher" located at the U.S. Department of Energy's Brookhaven National Laboratory — say they've created a new state of hot, dense matter out of the quarks and gluons that are the basic particles of atomic nuclei, but it is a state quite different and even more remarkable than had been predicted. In peer-reviewed papers summarizing the first three years of RHIC findings, the scientists say that instead of behaving like a gas of free quarks and gluons, as was expected, the matter created in RHIC's heavy ion collisions appears to be more like a *liquid*.

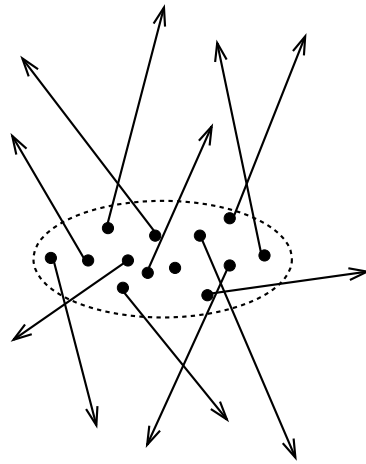
"Once again, the physics research sponsored by the Department of Energy is producing historic results," said Secretary of Energy Samuel Bodman, a trained chemical engineer. "The DOE is the principal federal funder of basic research in the physical sciences, including nuclear and high-energy physics. With today's announcement we see that investment paying off."

"The truly stunning finding at RHIC that the new state of matter created in the collisions of gold ions is more like a liquid than a gas gives us a profound insight into the earliest moments of the universe," said Dr. Raymond L. Orbach, Director of the DOE Office of Science.

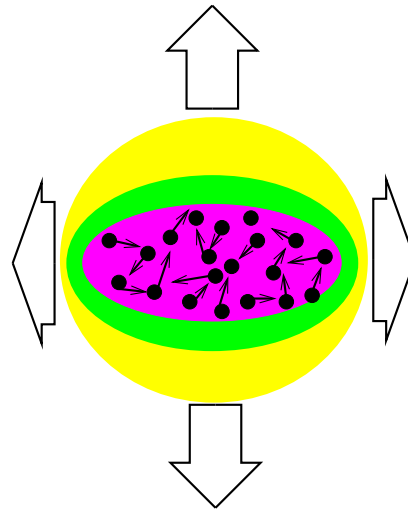


# The technical details

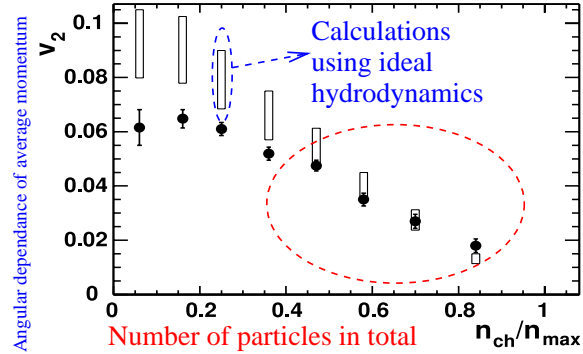
A "dust"  
 Particles ignore each other, their path is independent of initial shape



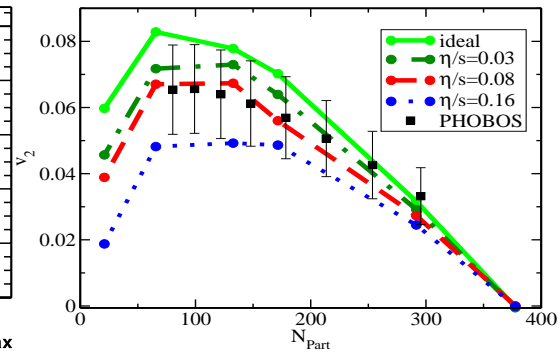
A "fluid"  
 Particles continuously interact. Expansion determined by density gradient (shape)



P.Kolb and U.Heinz,Nucl.Phys.A702:269,2002.



P.Romatschke,PRL99:172301,2007



## The conventional wisdom

Hydrodynamics is an "effective theory", built around coarse-graining and "fast thermalization". **Fast w.r.t. Gradients of coarse-grained variables**

If thermalization instantaneous, then isotropy, EoS enough to close evolution

$$T_{\mu\nu} = (e + P(e))u_\mu u_\nu + P(e)g_{\mu\nu}$$

In rest-frame at rest w.r.t.  $u^\mu$

$$T_{\mu\nu} = \text{Diag}(e(p), p, p, p)$$

(**NB:** For simplicity we assume no conserved charges,  $\mu_B = 0$  )

If thermalization not instantaneous,

$$T_{\mu\nu} = T_{\mu\nu}^{eq} + \Pi_{\mu\nu} \quad , \quad u_{\mu}\Pi^{\mu\nu} = 0$$

$$\sum_n \tau_{n\Pi} \partial_{\tau}^n \Pi_{\mu\nu} = -\Pi_{\mu\nu} + \mathcal{O}(\partial u) + \mathcal{O}((\partial u)^2) + \dots$$

A series whose "small parameter" (Barring phase transitions/critical points/... all of these these same order):

$$K \sim \frac{l_{micro}}{l_{macro}} \sim \frac{\eta}{sT} \nabla u \sim \frac{\text{Det}\Pi_{\mu\nu}}{\text{Det}T_{\mu\nu}} \sim \dots$$

and the transport coefficients calculable from asymptotic correlators of microscopic theory

Navier-Stokes  $\sim K$  , Israel-Stewart  $\sim K^2$  etc.

So hydrodynamics is an EFT in terms of  $K$  and correlators

$$\eta = \lim_{k \rightarrow 0} \frac{1}{k} \int dx \langle \hat{T}_{xy}(x) \hat{T}_{xy}(y) \rangle \exp [ik(x - y)] \quad , \quad \tau_\pi \sim \int e^{ikx} \langle TTT \rangle, \dots$$

**This is a classical theory** ,  $\hat{T}_{\mu\nu} \rightarrow \langle T_{\mu\nu} \rangle$  Higher order correlators  $\langle T_{\mu\nu}(x) \dots T_{\mu\nu} \rangle$  play role in transport coefficients, not in EoM (if you know equation and initial conditions, you know the whole evolution!)

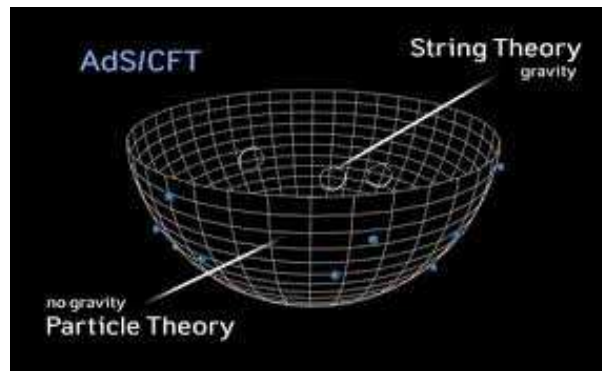
**As is the case with 99% of physics** we know how to calculate rigorously mostly in perturbative limit. **But 2nd law of thermodynamics** tells us that A Knudsen number of some sort can be defined in any limit as a thermalization timescale can always be defined **Strong coupling**  $\rightarrow$  lots of interaction  $\rightarrow$  "fast" thermalization  $\rightarrow$  "low"  $K$



e.g. “quantum lower limits” on viscosity? top-down answers

Danielewicz and Gyulassy used the uncertainty principle and Boltzmann equation

$$\eta \sim \frac{1}{5} \langle p \rangle n l_{mfp} \quad , \quad l_{mfp} \sim \langle p \rangle^{-1} \rightarrow \frac{\eta}{s} \sim 10^{-1}$$



**KSS and extensions** from AdS/CFT (actually any **classical** Gauge/gravity):

Viscosity  $\equiv$  Black hole graviton scattering  $\rightarrow \frac{\eta}{s} = \frac{1}{4\pi}$

Von Neuman QM (profound?) or Heisenberg's microscope (early step?)  
Both theories not realistic... in a similar way!

**Danielewitz+Gyulassy** In strongly coupled system the Boltzmann equation is inappropriate because **molecular chaos not guaranteed**

**KSS** UV-completion is conformal, planar, strong

Planar limit and molecular chaos has a surprisingly similar effect: decouple "macro" and "micro" DoFs. "number of microscopic DoFs infinite", "**large**" w.r.t. the coupling constant!

2011-2013 FLuid-like behavior has been observed down to very small sizes,  
 $p - p$  collisions of 50 particles



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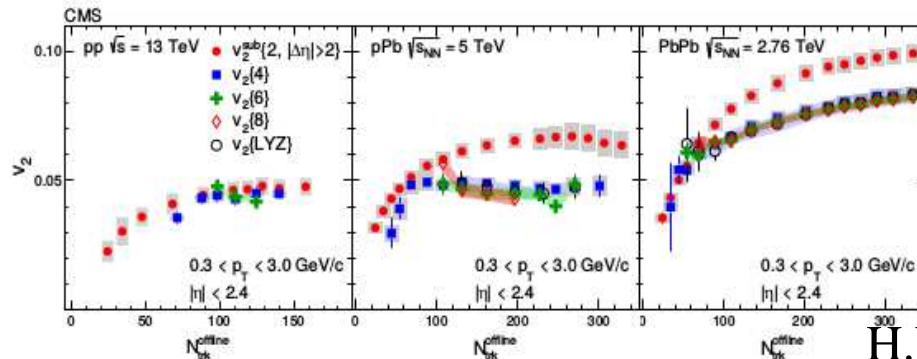
SCIENCE

## The LHC Might Have Created The Smallest Drop Of Liquid Ever

A tiny drop could have big implications for our understanding of particle collisions.

By Shaunacy Ferro May 8, 2013

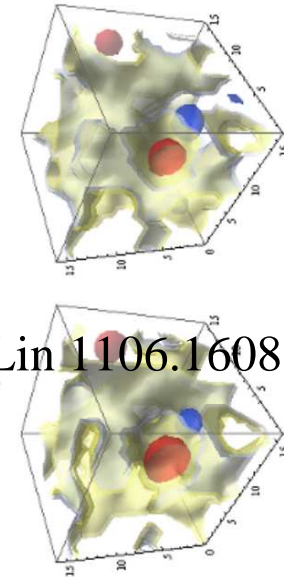
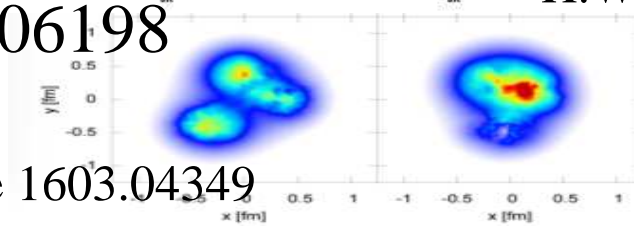




CMS 1606.06198

H.W.Lin 1106.1608

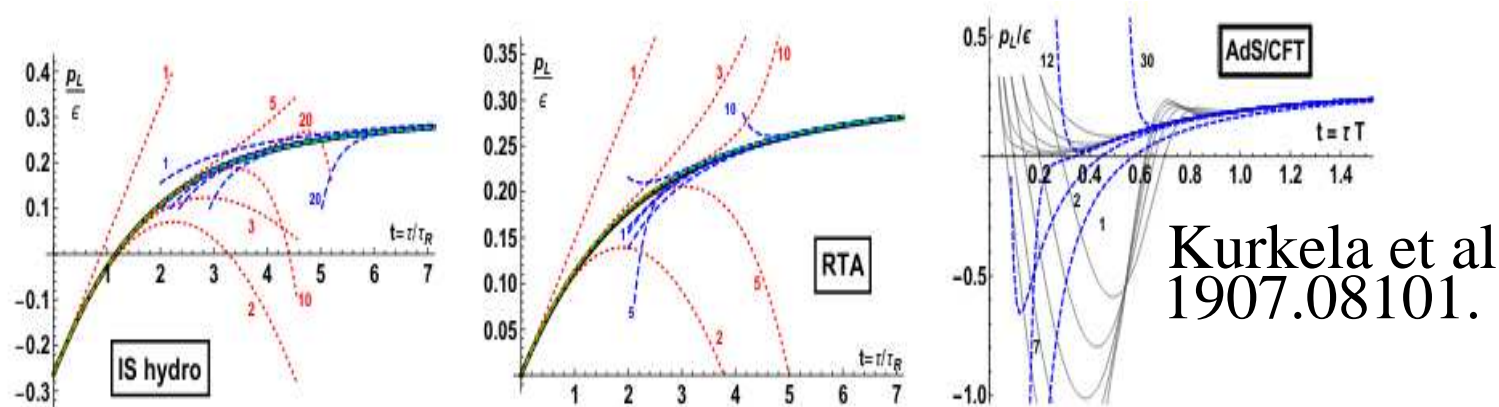
BSchenke 1603.04349



1606.06198 (CMS) : When you consider geometry differences, hydro with  $\mathcal{O}(20)$  particles "just as collective" as for 1000. Thermalization scale  $\ll$  color domain wall scale.

Little understanding of this in "conventional wisdom"

Hydrodynamics in small systems: “hydrodynamization” /” fake equilibrium”  
 A lot more work in both AdS/CFT and transport theory about  
 ”hydrodynamization” /” Hydrodynamic attractors”



Fluid-like systems far from equilibrium (**large gradients** )! Usually from 1D solution of Boltzmann and AdS/CFT EoMs! “hydrodynamics converges even at large gradients with no thermal equilibrium”

But I have a basic question: ensemble averaging!

- What is hydrodynamics if  $N \sim 50$  ...
  - **Ensemble averaging** ,  $\langle F(\{x_i\}, t) \rangle \neq F(\{\langle x_i \rangle\}, t)$   
suspect for any non-linear theory. **molecular chaos in Boltzmann,**  
**Large  $N_c$  in AdS/CFT, all assumed . But for  $\mathcal{O}(50)$  particles?!?!**
  - For water, a cube of length  $\eta/(sT)$  has  $\mathcal{O}(10^9)$  molecules,

$$P(N \neq \langle N \rangle) \sim \exp \left[ - \langle N \rangle^{-1} (N - \langle N \rangle)^2 \right] \ll 1$$

- How do microscopic, macroscopic and quantum corrections talk to each other? EoS is given by  $p = T \ln Z$  but  $\partial^2 \ln Z / \partial T^2, dP/dV??$

**NB: nothing to do with equilibration timescale** . Even "things born in equilibrium" locally via Eigenstate thermalization have fluctuations!

And there is more... How does dissipation work in such a “semi-microscopic system”?

- What does local and global equilibrium mean there?
- If  $T_{\mu\nu} \rightarrow \hat{T}_{\mu\nu}$  what is  $\hat{\Pi}_{\mu\nu}$  Second law fluctuations? Sometimes because of a fluctuation entropy decreases!



**Bottom line:** Either hydrodynamics is not the right explanation for these observables (possible! But small/big systems similar! ) or we are not understanding something basic about what's behind the hydrodynamics! What do fluctuations do?

In "fireball" there might be "infinite correlated" DoFs ,  
but final entropy  $\ll \infty$



Landau and Lifshitz (also D.Rishke,B Betz et al): Hydrodynamics has three length scales

$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

**Weakly coupled:** Ensemble averaging in Boltzmann equation good up to  $\mathcal{O}((1/\rho)^{1/3} \partial_\mu f(\dots))$

**Strongly coupled:** classical supergravity requires  $\lambda \gg 1$  but  $\lambda N_c^{-1} = g_{YM} \ll 1$  so

$$\frac{1}{TN_c^{2/3}} \ll \frac{\eta}{sT} \quad \left( \text{or} \quad \frac{1}{\sqrt{\lambda T}} \right) \ll L_{macro}$$

**QGP:**  $N_c = 3 \ll \infty$ , so  $l_{micro} \sim \frac{\eta}{sT}$ . **Cold atoms:**  $l_{micro} \sim n^{-1/3} > \frac{\eta}{sT}$  ?

Why is  $l_{micro} \ll l_{mfp}$  necessary? Without it, microscopic fluctuations (which come from the finite number of DoFs and have nothing to do with viscosity) will drive fluid evolution.

$\Delta\rho/\rho \sim C_V^{-1} \sim N_c^{-2}$ , thermal fluctuations “too small” to be important!

Kovtun, Moore, Romatschke, 1104.1586 As  $\eta \rightarrow 0$  “infinite propagation of soundwaves” impacts “IR limit of Kubo formula”

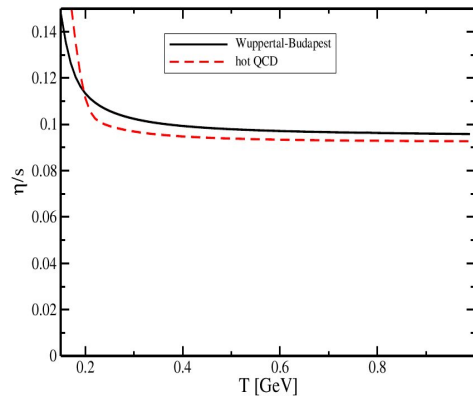
$$\lim_{\eta, k \rightarrow 0} \int d^3x e^{ikx} \langle T^{xyxy}(x) T^{xyxy}(0) \rangle \simeq -i\omega \frac{7T p_{max}}{60\pi^2 \gamma_\eta} + (i+1)\omega^{\frac{3}{2}} \frac{7T}{240\pi \gamma_\eta^{\frac{3}{2}}}$$

where  $p_{max}$  is the maximum momentum scale and  $\gamma_\eta = \eta/(e+p)$

Kovtun, Moore and Romatschke plug in  $p_{max}$  into viscosity

$$\eta^{-1} \sim \eta_{bare}^{-1} \left( 1 + \frac{p_{max}}{T} \right) \quad , \quad \frac{\eta}{s} \geq \frac{T}{p_{max}} \geq \frac{T}{s^{1/3}}$$

Away from planar limit relaxation time overwhelmed by “stochastic mode”,  
 $\sim w^{3/2}$



G.Moore,P.Romatschke

Phys.Rev.D84:025006,2011 arXiv:1104.1586

$N_c=3$ ,  $\eta/s=KSS$

This is interesting but makes the 50 particles problem worse! . And isn't assuming  $p_{max}$  “circular”? In fireball could be “many correlated” DoFs!

Can one go further? Away from critical point, hydro is non-renormalizable ( $p_{max}$  physical) but strongly coupled (anomalous dimensions...?)

If we could include both microscopic and collective modes minimal viscosity might be calculable just from hydrodynamics and, e.g., Lorentz symmetry and Quantum mechanics? after all,

$$\chi_{ijkl}(w) = \int dx \langle \hat{T}_{ij}(x) \hat{T}_{kl}(y) \rangle e^{ik(x-y)} \quad \left\{ \begin{array}{c} c_s^2 \\ \eta \end{array} \right\} \sim \lim_{w \rightarrow 0} w^{-1} \left[ \begin{array}{c} \text{Re}[\chi_{zzzz}] \\ \text{Im}[\chi_{xyxy}] \end{array} \right]$$

These include both microscopic and macroscopic fluctuations calculable in equilibrium! “Dyson-Schwinger equations”? Eigenstate thermalization hypothesis? (Delacretaz et al, 1805.04194)

$$\hat{\rho} \underbrace{\Rightarrow}_{fast \sim l_{micro}} \delta(E - E_i) \underbrace{\Rightarrow}_{cell} \exp \left[ -\frac{u_\mu \hat{H}}{T} \right]$$

Between Eigenstate thermalization, Kramers-Konig, the EoS

$$\left\{ \begin{array}{c} Re \\ Im \end{array} \right\} \chi_{ijkl} = \frac{1}{\pi} \int \frac{dw'}{w - w'} \left\{ \begin{array}{c} Im \\ -Re \end{array} \right\} \chi_{ijkl} \quad , \quad \langle \hat{T}_{\mu}^{\mu} \rangle \sim \langle e \rangle - 3 \langle p \rangle$$

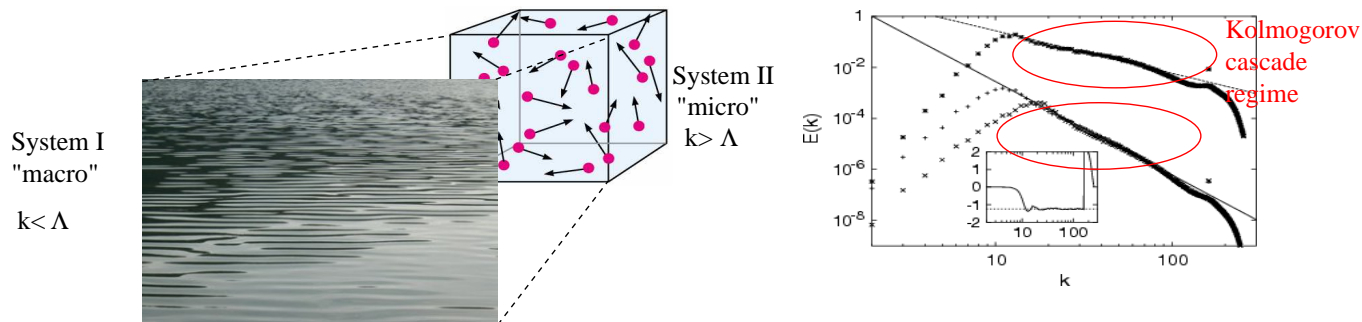
and **the Ward identity** (Lorentz invariance)  $\partial^{\alpha} \left\{ \left\langle \left[ \hat{T}_{\mu\nu}(x), \hat{T}_{\alpha\beta}(x') \right] \right\rangle - \right.$

$$\left. -\delta(x - x') \left( g_{\beta\mu} \langle \hat{T}_{\alpha\nu}(x') \rangle + g_{\beta\nu} \langle \hat{T}_{\alpha\mu}(x') \rangle - g_{\beta\alpha} \langle \hat{T}_{\mu\nu}(x') \rangle \right) \right\} = 0$$

We should be able to relate microscopic Lagrangian to “ideal” fluid limit **with fluctuations** in equilibrium!

**Problem:** What happens when **macro** and **micro** talk to each other in a strongly coupled/turbulent regime?

**Turbulence** a strongly coupled fixed point, and **vortices** have no energy gap and don't propagate!  $\Rightarrow$  “non-perturbative” vacuum

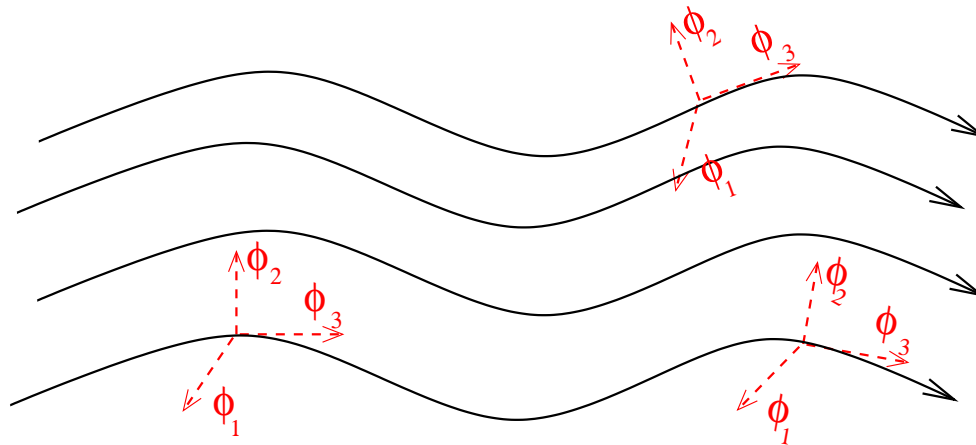


A classical low-viscosity fluid is turbulent. Typically, low- $k$  modes cascade into higher and higher  $k$  modes via sound and vortex emission (phase space looks more "fractal"). In a non-relativistic incompressible fluid

$$\eta/(sT) \ll L_{eddy} \ll L_{boundary} \quad , \quad E(k) \sim \left( \frac{dE}{dt} \right)^{2/3} k^{-5/3}$$

For a classical ideal fluid, no limit! since  $\lim_{\delta\rho \rightarrow 0, k \rightarrow \infty} \delta E(k) \sim \delta\rho k c_s \rightarrow 0$  but for quantum perturbations,  $E \geq k$  so conservation of energy has to cap cascade.

My previous attempt, from Endlich et al, 1011.6396 Continuous mechanics (fluids, solids, jellies,...) is written in terms of 3-coordinates  $\phi_I(x^\mu)$ ,  $I = 1\dots 3$  of the position of a fluid cell originally at  $\phi_I(t = 0, x^i)$ ,  $I = 1\dots 3$ .



The system is a **Fluid** if it's Lagrangian obeys some symmetries (Ideal hydrodynamics  $\leftrightarrow$  Isotropy in comoving frame)

$$L \rightarrow \ln \mathcal{Z} \quad \mathcal{Z} = \int \mathcal{D}\phi_i \exp \left[ -T_0^4 \int F(B(\phi_I)) d^4x \right], \langle \mathcal{O} \rangle \sim \frac{\partial \ln \mathcal{Z}}{\partial \dots}$$

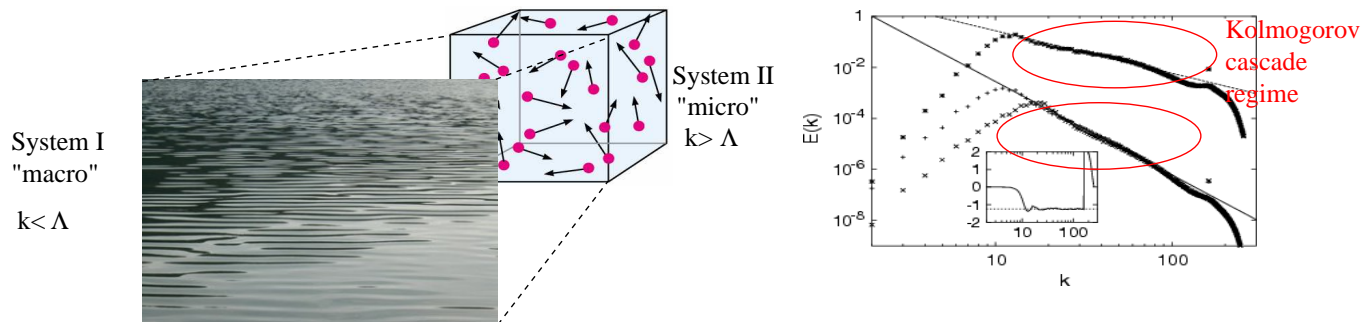
A lot of work on this 1903.08729

**Some accomplishments** EFT techniques, insights from Ostrogradski's theorem, extensions to polarization (minimum viscosity? )  
<https://indico.ectstar.eu/event/94/contributions/1879/>  
Polarization could provide a “soft dissipative gap” to vortices and stabilize hydrodynamics

**Some limitations** no clear way to incorporate microscopic fluctuations, functional integral hard , lattice regularization possible but limited to hydrostatic case (1502.05421 )

Using a volume cell as a DoF makes it hard to understand fluctuations within it!





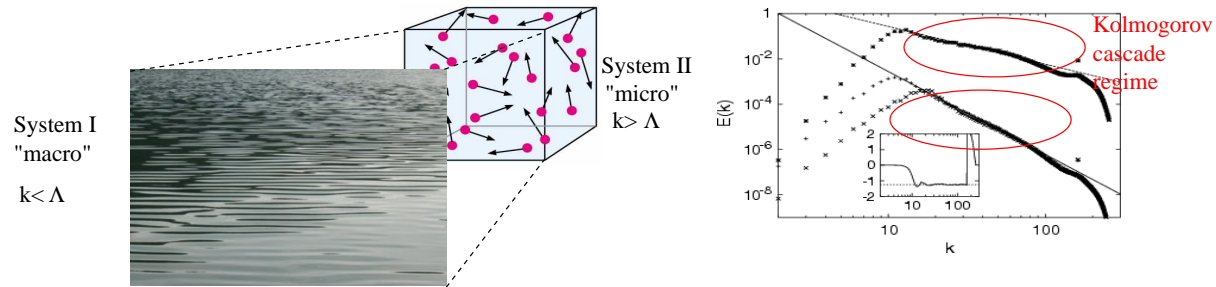
Consider effective Lagrangian approach of ideal fluid,  
 Endlich et al, 1011.6396

**Coupling constant**  $\sim k^\alpha$ ,

**For vortices** ideal Lagrangian has no kinetic term  $\mathcal{L} \sim \dot{\pi}$ ,  
 no S-matrix, non-perturbative!

**Lagrangian hydro on lattice** (T.Burch,GT,1502.05421) evidence for  
 phase transition between hydrostatic and turbulent phases

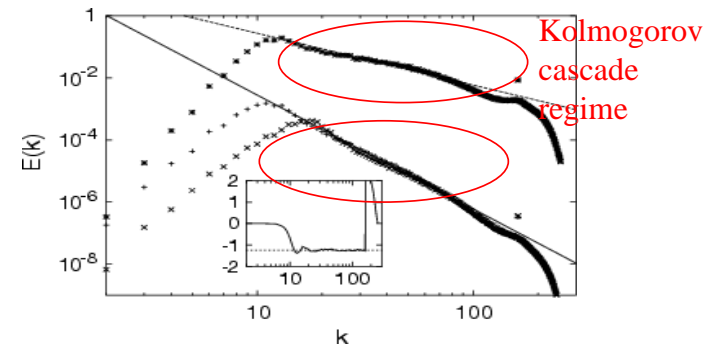
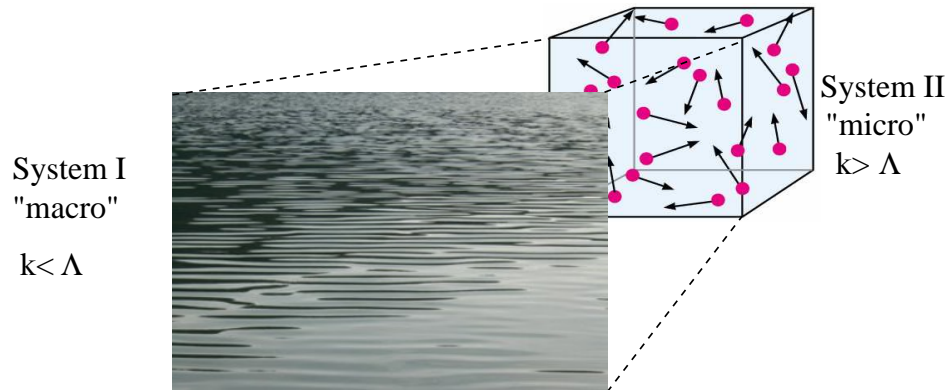
More fundamentally: Let us take a stationary slab of fluid at local equilibrium.



**Statistical mechanics:** This is a system in global equilibrium, described by a partition function  $Z(T, V, \mu)$ , whose derivatives give expectation values  $\langle E \rangle$ , fluctuations  $\langle (\Delta E)^2 \rangle$  etc. in terms of parameters representing conserved charges

**Fluid dynamics:** This is the state of a field in local equilibrium which can be perturbed in an infinity of ways. The perturbations will then interact and dissipate according to the Euler/N-S equations

More fundamentally: Let us take a stationary slab of fluid at local equilibrium.



To what extent are these two pictures the same?

- Global equilibrium is also local equilibrium, if you forget fluctuations
- Dissipation scale in local equilibrium  $\eta/(Ts)$  , global equilibration timescale  $(Ts)/\eta$

## Some insight from maths

Millenium problem: existence and smoothness of the Navier-Stokes equations



Important tool are “weak solutions” , similar to what we call “coarse-graining” .

$$F\left(\frac{d}{dx}, f(x)\right) = 0 \Rightarrow F\left(\int \frac{d}{dx}\phi(x) \dots, f(x)\right) = 0$$

$\phi(x)$  “test function”, similar to coarse-graining!

Existence of Wild/Nightmare solutions and non-uniqueness of weak solutions shows this tension is non-trivial, coarse-graining “dangerous”



I am a physicist so I care little about the “existence of eternal solutions” to an approximate equation, **Turbulent regime and microscopic local equilibria need to be consistent**

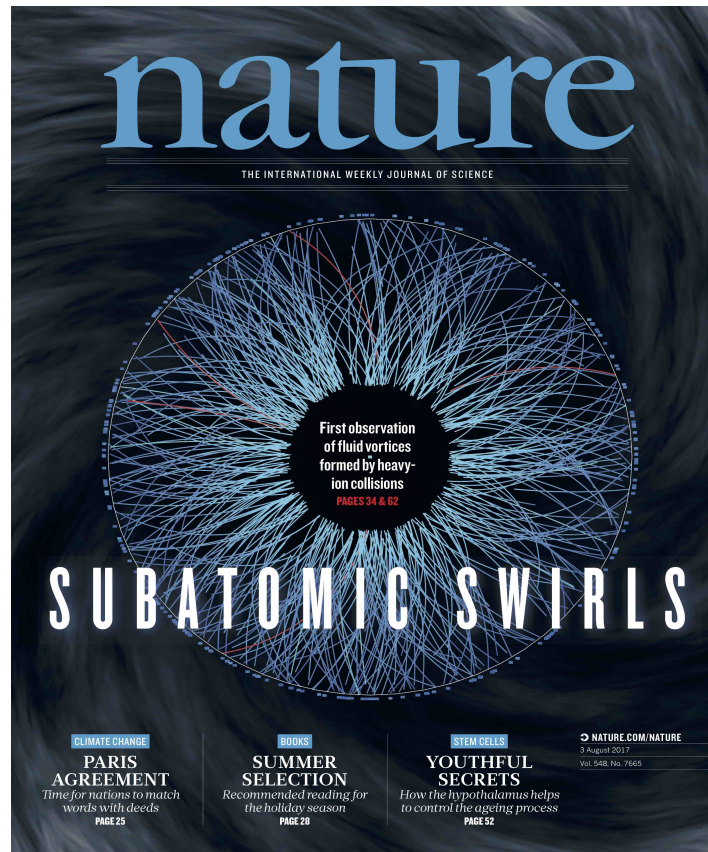
Thermal fluctuations could both “stabilize” hydrodynamics and “accelerate” local thermalization

PS: All highly theoretical but transfer of micro to macro DoFs  
experimentally proven!

STAR  
collaboration  
1701.06657

NATURE  
August 2017

Polarization by vorticity  
in heavy ion collisions



# Our proposal

Every statistical theory needs a "state space" and an "evolution dynamics"  
The ingredients

**State space: Zubarev hydrodynamics** Mixes micro and macro DoFs

**Dynamics: Crooks fluctuation theorem** provides the dynamics via a definition of  $\Pi_{\mu\nu}$  from fluctuations

$\hat{T}^{\mu\nu}$  is an operator, so any decomposition, such as  $\hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$  must be too!



Zubarev partition function for local equilibrium: think of Eigenstate thermalization...

Let us generalize the GC ensemble to a co-moving frame  $E/T \rightarrow \beta_\mu T_\nu^\mu$

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp \left[ - \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

$Z$  is a partition function with a field of Lagrange multipliers  $\beta_\mu$ , with microscopic and quantum fluctuations included.

Effective action from  $\ln[Z]$ . Correction to Lagrangian picture?

All normalizations diverge but hey, it's QFT! (Later we resolve this! )

This is perfect global equilibrium. What about imperfect local?

- Dynamics is not clear. [Becattini et al, 1902.01089](#): Gradient expansion in  $\beta_\mu$ . Reproduces Euler and Navier-Stokes, but...
  - 2nd order Gradient expansion (Navier stokes) non-causal perhaps...
  - Use Israel-Stewart,  $\Pi_{\mu\nu}$  arbitrary perhaps...
  - Foliation  $d\Sigma_\mu$  arbitrary but not clear how to link to Arbitrary  $\Pi_{\mu\nu}$
- What about fluctuations? Coarse-graining and fluctuations mix? How does one truncate?

## An operator formulation

$$\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}_{\mu\nu}$$

and  $\hat{T}_0^{\mu\nu}$  truly in equilibrium!

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp \left[ - \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

describes all cumulants and probabilities

$$\langle T_0^{\mu\nu}(x_1) T_0^{\mu\nu}(x_2) \dots T_0^{\mu\nu}(x_n) \rangle = \prod_i \frac{\delta^n}{\delta \beta_\mu(x_i)} \ln Z$$

and also the full energy-momentum tensor

$$\langle T^{\mu\nu}(x_1)T^{\mu\nu}(x_2)\dots T^{\mu\nu}(x_n)\rangle = \prod_i \frac{\delta^n}{\delta g_{\mu\nu}(x_i)} \ln Z$$

What this means

- Equilibrium at "probabilistic" level

$$\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$$

- KMS Condition obeyed by "part of density matrix" in equilibrium, "expand" around that! An operator constrained by KMS condition is still an operator!  $\equiv$  time dependence in interaction picture

Does this make sense

$$\hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu} \quad , \quad \hat{\rho}_{T_{\mu\nu}} = \frac{\hat{\rho}_{T_0} + \hat{\rho}_{\Pi_0}}{\text{Tr}(\hat{\rho}_{T_0} + \hat{\rho}_{\Pi_0})} \simeq \hat{\rho}_{T_0} (1 + \delta\hat{\rho})$$

For any flow field  $\beta_\mu$  and lagrangian we can define

$$Z_{T_0}(J(y)) = \int \mathcal{D}\phi \exp \left[ - \int_0^{T^{-1}(x_i^\mu)} d\tau' \int d^3x (L(\phi) + J(y)\phi) \right] \propto$$
$$\propto \exp \left[ -\beta^0 \hat{T}_{00} \right] \Big|_{\beta_\mu = (T^{-1}(x,t), \vec{0})}$$

E.g. Nishioka, 1801.10352  $\langle x | \rho | x' \rangle =$

$$= \frac{1}{Z} \int_{\tau=-\infty}^{\tau=\infty} \int [\mathcal{D}\phi, \mathcal{D}y(\tau) \mathcal{D}y'(\tau)] e^{-iS(\phi, y, y')} \underbrace{\delta [y(0^+) - x'] \delta [y'(0^-) - x]}_{\frac{\delta J_i(y(0^+))}{\delta J_i(x')} \frac{\delta J_j(y(0^-))}{\delta J_j(x)}}$$

$$\Rightarrow \frac{\delta^2}{\delta J_i(x) \delta J_j(x')} \ln [Z_{T_0}(T^{\mu\nu}, J) \times Z_{\Pi}(J)]_{J=J_1(x)+J_2(x')}$$

$J_1(x) + J_2(x')$  chosen to respect Matsubara conditions!

**Any**  $\rho$  can be separated like this for any  $\beta_\mu$ . The question is, is this a good approximation? **“Close enough to equilibrium”**

**The source**  $J$  related to the smearing in “weak solutions”. Pure maths angle?

## Entropy/Deviations from equilibrium

- In quantum mechanics Entropy function of density matrix

$$s = \text{Tr}(\hat{\rho} \ln \hat{\rho}) = \frac{d}{dT} (T \ln Z)$$

Conserved in quantum evolution, not coarse-graining/gradient expansion

- In IS entropy function of the dissipative part of E-M tensor

$$n^\nu \partial_\nu (s u^\mu) = n^\mu \frac{\Pi^{\alpha\beta}}{T} \partial_\alpha \beta_\beta \quad , \quad \geq 0$$

$n_\mu = d\Sigma_\mu / |d\Sigma_\mu|$ ,  $\Pi_{\mu\nu}$  arbitrary. How to combine coarse-graining? **if vorticity non-zero**  $n_\mu u^\mu \neq 0$

## What about fluctuations

$$n^\nu \partial_\nu (su^\mu) = n^\mu \frac{\Pi^{\alpha\beta}}{T} \partial_\alpha \beta_\beta \quad , \quad \geq 0$$

- If  $n_\mu$  arbitrary cannot be true for “any” choice
- 2nd law is true for “averages” anyways, sometimes entropy can decrease

We need a fluctuating formulation!

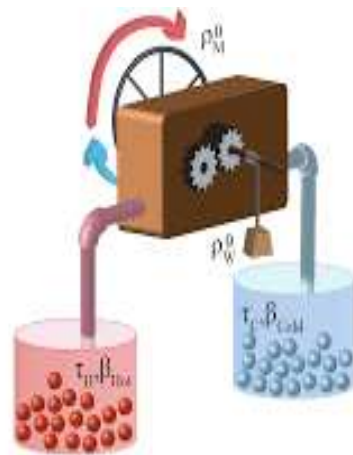
- “Statistical” (probability depends on “local microstates”)
- Dynamics with fluctuations, time evolution of  $\beta_\mu$  distribution



So we need

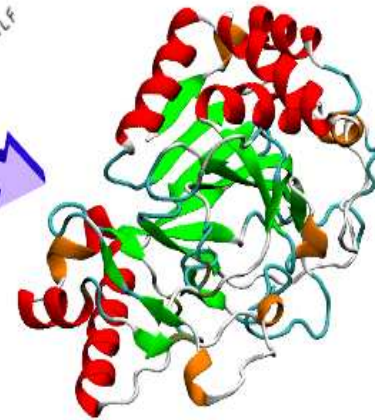
- a similarly probabilistic definition of  $\hat{\Pi}^{\mu\nu} = \hat{T}^{\mu\nu} - \hat{T}_0^{\mu\nu}$  as an operator!!
- Probabilistic dynamics, to update  $\hat{\Pi}_{\mu\nu}, \hat{T}_{\mu\nu}$  !

Crooks fluctuation theorem!



From talk  
Gabriel Landi

LSIEDFTDAFGMTPTAFSAALPRRWKQONLKKKGLF



Relates fluctuations, entropy in small fluctuating systems (Nano,proteins )

## Crooks fluctuation theorem!

$$P(W)/P(-W) = \exp [\Delta S]$$

**P(W)** Probability of a system doing some work in its usual thermal evolution

**P(-W)** Probability of the same system “running in reverse” and decreasing entropy due to a thermal fluctuation

$\Delta S$  Entropy produced by  $P(W)$

Looks obvious but...

**Is valid** for systems very far from equilibrium (nano-machines, protein folding and so on)

**Proven** for Markovian processes and fluctuating systems in contact with thermal bath

**Leads to irreducible** fluctuation/dissipation: TUR (more later!)

Applying it to locally equilibrium systems within Zubarev's formalism is straight-forward . Since ratios of probabilities, divergences are resolved!

How is Crooks theorem useful for what we did? Guarnieri et al, arXiv:1901.10428 (PRX) derive Thermodynamic uncertainty relations from

$$\hat{\rho}_{ness} \simeq \hat{\rho}_{les}(\lambda) e^{\hat{\Sigma}} \frac{Z_{les}}{Z_{ness}} \quad , \quad \hat{\rho}_{les} = \frac{1}{Z_{les}} \exp \left[ -\frac{\hat{H}}{T} \right]$$

$\hat{\rho}_{les}$  is Zubarev operator while  $\Sigma$  is calculated with a Kubo-like formula

$$\hat{\Sigma} = \delta_{\beta} \Delta \hat{H}_+ \quad , \quad \hat{H}_+ = \lim_{\epsilon \rightarrow 0^+} \epsilon \int dt e^{\epsilon t} e^{-\hat{H}t} \Delta \hat{H} e^{\hat{H}t}$$

Relies on

$$\lim_{w \rightarrow 0} \left\langle \left[ \hat{\Sigma}, \hat{H} \right] \right\rangle \rightarrow 0 \equiv \lim_{t \rightarrow \infty} \left\langle \left[ \Sigma(t), \hat{H}(0) \right] \right\rangle \rightarrow 0$$

This “infinite” is “small” w.r.t. hydro gradients.  $\equiv$  **Markovian** as in Hydro with  $l_{mfp} \rightarrow \partial$  but with operators  $\rightarrow$  carries all fluctuations with it!

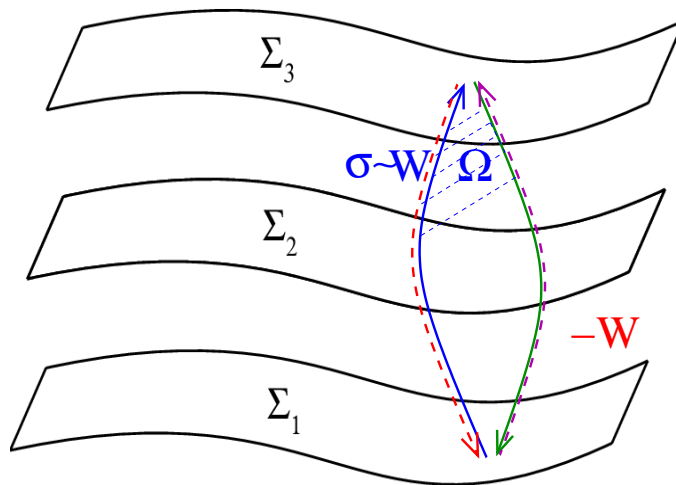
$$P(W)/P(-W) = \exp[\Delta S] \quad \forall s \quad S_{eff} = \ln Z$$

**KMS condition** reduces the functional integral to a Metropolis type weighting,  $\equiv$  periodic time at rest with  $\beta_\mu$

**Markovian systems** exhibit Crooks theorem, two adjacent cells interaction outcome probability proportional to **number of ways of reaching outcome**. The normalization divergence is resolved since ratios of probabilities are used. “instant decoherence/thermalization” within each step

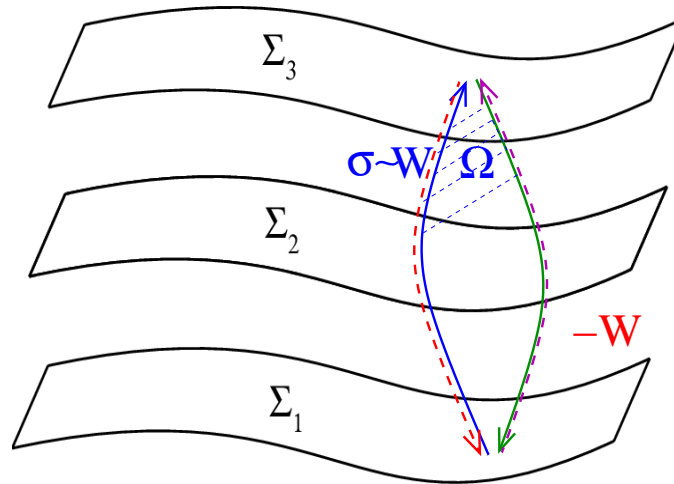
**Relationship** to gradient expansion similar to relationship between Wilson loop coarse-graining ( [Jarzynski's theorem, used on lattice](#) , Caselle et al, 1604.05544) with hadronic EFTs

## Applying Crooks theorem to Zubarev hydrodynamics: Stokes theorem



$$- \int_{\Sigma(\tau_0)} d\Sigma_\mu \left( \hat{T}^{\mu\nu} \beta_\nu \right) = - \int_{\Sigma(\tau')} d\Sigma_\mu \left( \hat{T}^{\mu\nu} \beta_\nu \right) + \int_{\Omega} d\Omega \left( \hat{T}^{\mu\nu} \nabla_\mu \beta_\nu \right),$$

true for “any” fluctuating configuration.



Let us now invert one foliation so it goes “backwards in time” assuming  
Crooks theorem means

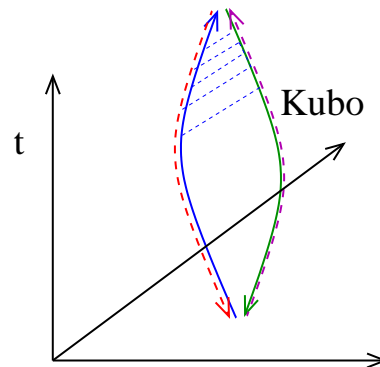
$$\frac{\exp \left[ - \int_{\sigma(\tau)} d\Sigma_{\mu} \beta_{\nu} \hat{T}^{\mu\nu} \right]}{\exp \left[ - \int_{-\sigma(\tau)} d\Sigma_{\mu} \beta_{\nu} \hat{T}^{\mu\nu} \right]} = \exp \left[ \frac{1}{2} \int_{\Omega} d\Omega_{\mu}^{\mu} \left[ \frac{\hat{\Pi}^{\alpha\beta}}{T} \right] \partial_{\beta} \beta_{\alpha} \right]$$

**Small loop limit**  $\left\langle \exp \left[ \oint d\Sigma_\mu \omega^{\mu\nu} \beta^\alpha \hat{T}_{\alpha\nu} \right] \right\rangle = \left\langle \exp \left[ \int \frac{1}{2} d\Sigma_\mu \beta^\mu \hat{\Pi}^{\alpha\beta} \partial_\alpha \beta_\beta \right] \right\rangle$

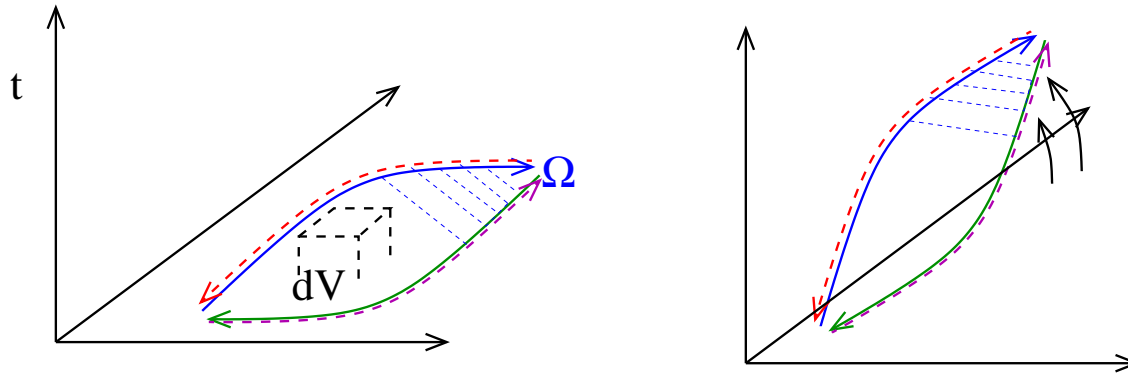
A non-perturbative operator equation, divergences cancel out...

$$\frac{\hat{\Pi}^{\mu\nu}}{T} \Big|_\sigma = \left( \frac{1}{\partial_\mu \beta_\nu} \right) \frac{\delta}{\delta \sigma} \left[ \int_{\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu} - \int_{-\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu} \right]$$

Note that a time-like contour produces a Kubo-formula



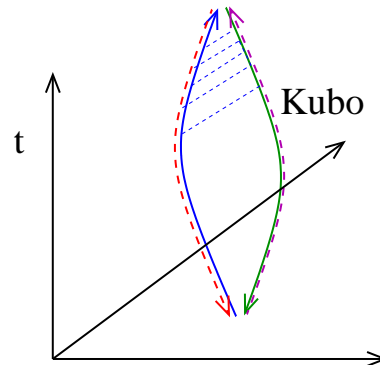




A sanity check: For an equilibrium spacelike  $d\Sigma_\mu = (dV, \vec{0})$  (left-panel) we recover Boltzmann's

$$\Pi^{\mu\nu} \Rightarrow \Delta S = \frac{dQ}{T} = \ln \left( \frac{N_1}{N_2} \right)$$

## A sanity check



When  $\eta \rightarrow 0$  and  $s^{-1/3} \rightarrow 0$  (the first two terms in the hierarchy), Crooks fluctuation theorem gives  $P(W) \rightarrow 1$   $P(-W) \rightarrow 0$   $\Delta S \rightarrow \infty$  so Crooks theorem reduces to  $\delta$ -functions of the entropy current

$$\delta(d\Sigma_\mu(su^\mu)) \Rightarrow n^\mu \partial_\mu(su^\mu) = 0$$

We therefore recover conservation equations for the entropy current, a.k.a. ideal hydro

## Crooks theorem: thermodynamic uncertainty relations

Andr M. Timpanaro, Giacomo Guarnieri, John Goold, and Gabriel T. Landi  
Phys. Rev. Lett. 123, 090604

$$\frac{\langle (\Delta Q)^2 \rangle}{\langle Q \rangle^2} \geq \frac{2}{\Delta S(W)}$$

Valid locally in time!

$$\frac{d}{d\tau} \Delta S \geq \frac{1}{2} \frac{d}{d\tau} \frac{\langle Q \rangle^2}{\langle (\Delta Q)^2 \rangle}$$

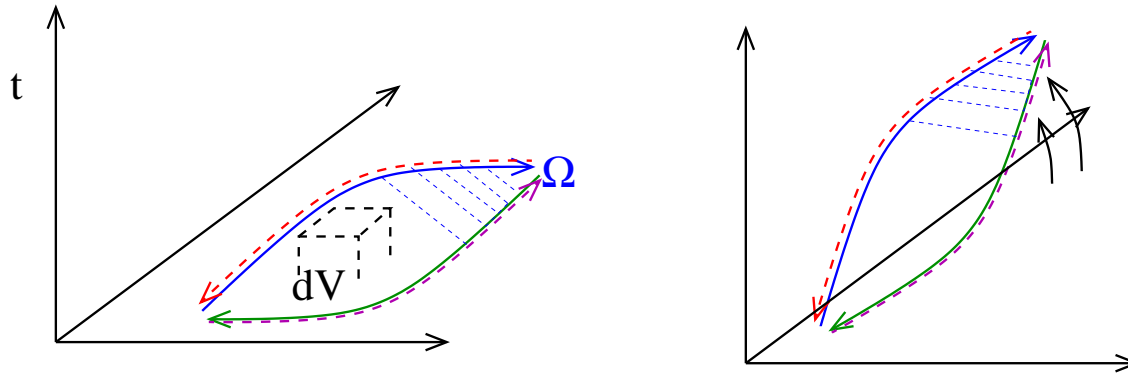
Relates thermal fluctuations and dissipation, producing an irreducible uncertainty. **Non-dissipative nano-engines fluctuate like crazy, produces “dissipation” anyway**

Consequences: Hydro-TUR? Separate flow into potential and vortical part

$$\beta_\mu = \partial_\mu \phi + \zeta_\mu \quad , \quad n_\mu \rightarrow T \partial_\mu \phi \quad , \quad \omega_{\mu\nu} = g_{\mu\nu}|_{\text{comoving}}$$

A likely TUR is

$$\frac{\langle [T_{\mu\gamma}, T_\nu^\gamma] \rangle}{\langle T^{\mu\nu} \rangle^2} \geq \frac{C \epsilon_{\mu\gamma\kappa} \langle T^{\gamma\kappa} \rangle \beta^\mu}{\Pi^{\alpha\beta} \partial_\beta \zeta_\alpha} \quad , \quad C \sim \mathcal{O}(1)$$

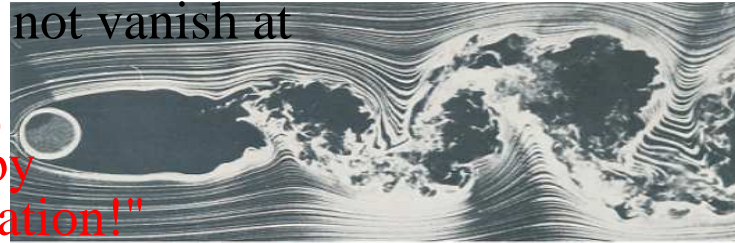


Deform the equilibrium contour and get Kubo formula! (right panel)

$$\mathcal{C} = \lim_{w \rightarrow 0} \frac{\text{Re}[F(w)]}{\text{Im}[F(w)]}, \quad F(w) = \int d^3x dt \langle T^{xy}(x) T^{xy}(0) \rangle e^{i(kx - wt)}$$

–dissipation does not vanish at zero viscosity

Vlad Vicol (talk)  
"will be proven by a different generation!"



(a) Visualizing turbulent cylinder wake at Re = 10000  
[Courtesy: Thomas Corke and Hassan Nagib; from *An Album of Fluid Motion* by van Dyke (1982)]



(b) a closer look at Re = 2000 - patterns are identical as in (a)  
[Courtesy: ONERA pic. Werle & Gallon (1972) from *An Album of Fluid Motion* by van Dyke (1982)]

$$\frac{\langle [T_{\mu\gamma}, T_{\nu}^{\gamma}] \rangle}{\langle T_{\mu\nu} \rangle^2} \geq \frac{\mathcal{O}(1) \epsilon_{\mu\gamma\kappa} \langle T^{\gamma\kappa} \rangle \beta^{\mu}}{\Pi^{\alpha\beta} \partial_{\beta} \zeta_{\alpha}}$$

Fluctuations + Low viscosity  $\Rightarrow$  Turbulence  $\Rightarrow$  high vorticity  $\Rightarrow$  dissipation!  
(usually mathematicians consider incompressible fluids, non-relativistic!)

Towards equations: Gravitational Ward identity!

$$\partial^\alpha \left\{ \left\langle \left[ \hat{T}_{\mu\nu}(x), \hat{T}_{\alpha\beta}(x') \right] \right\rangle - \right.$$

$$\left. -\delta(x - x') \left( g_{\beta\mu} \left\langle \hat{T}_{\alpha\nu}(x') \right\rangle + g_{\beta\nu} \left\langle \hat{T}_{\alpha\mu}(x') \right\rangle - g_{\beta\alpha} \left\langle \hat{T}_{\mu\nu}(x') \right\rangle \right) \right\} = 0$$

Small change in  $T_{\mu\nu}$  related to infinitesimal shift! Conservation of momentum!

Can be used to fix one component of  $\beta_\mu = u_\mu/T$ , so  $u_\mu u^\mu = -1$  and  $(\beta_\mu \beta^\mu)^{-1/2} = T$  weights  $\hat{\Pi}^{\mu\nu}$  in a way that conserves  $\hat{\Pi}^{\mu\nu} + \hat{T}_0^{\mu\nu}$

## Putting everything together: Dynamics at $Z$ level

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta \ln Z}{\delta g^{\mu\nu}} = \langle T_0 \rangle^{\mu\nu} + \Pi^{\mu\nu}$$

$$\langle T_0^{\mu\nu} \rangle = \frac{\delta^2 \ln Z}{\delta \beta_\mu \delta n_\nu} \quad , \quad \langle \Pi^{\mu\nu} \rangle = \frac{1}{\partial_\mu \beta_\nu} \partial_\gamma \frac{d}{d \ln(\beta_\alpha \beta^\alpha)} [\beta^\gamma \ln Z]$$

$$\partial_\alpha \left[ \frac{2}{\sqrt{-g}} \frac{\delta^2 \ln Z}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} - \delta(x - x') \frac{2}{\sqrt{-g}} \left( g_{\beta\mu} \frac{\delta \ln Z}{\delta g_{\alpha\nu}} + g_{\beta\nu} \frac{\delta \ln Z}{\delta g_{\alpha\mu}} - g_{\beta\alpha} \frac{\delta \ln Z}{\delta g_{\nu\mu}} \right) \right] = 0$$

and, finally, Crook's theorem

$$\frac{\delta^2}{\delta g^{\mu\nu} \delta g^{\alpha\beta}} \ln Z = \frac{\sqrt{-g}}{2} \frac{\beta_\kappa}{2\omega^{\mu\nu} \beta^\alpha} \partial_\beta n^\kappa \partial_\gamma \frac{d}{d \ln(\beta_\alpha \beta^\alpha)} [\beta^\gamma \ln Z]$$



Ito process

$$\hat{T}_{\mu\nu}(t) = \hat{T}_{\mu\nu}(t_0) + \int \Delta^{\alpha\beta} [\hat{T}_{\mu\alpha} \hat{T}_{\beta\nu}] + \int \frac{1}{2} d\Sigma_{\mu\beta\nu} \hat{\Pi}_{\alpha\beta} \partial^\alpha \beta^\beta$$

$$\ln Z|_{t+dt} = \int \mathcal{D}g_{\mu\nu}(x) T^{\mu\nu}|_{t+dt} \quad , \quad \beta_\mu|_{t+dt} = \frac{\delta \ln Z|_{t+dt}}{\delta T_{\mu\nu}} n_\nu$$

At every point in a foliation, dynamics is regulated by a stochastic term and a dissipation term. Can be done numerically with montecarlo with an ensemble of configurations at every point in time...

**Need:** Euclidean correlator in equilibrium  $\langle T_{\mu\nu}(x) T_{\mu\nu} \rangle (x')$

## A numerical formulation

**Define a field**  $\beta_\mu$  field and  $n_\mu$

**Generate** an ensemble of

$$\ln Z|_{t+dt} = \int \mathcal{D}g_{\mu\nu}(x) T^{\mu\nu}|_{t+dt} \quad , \quad \beta_\mu|_{t+dt} = \frac{\delta \ln Z|_{t+dt}}{\delta T_{\mu\nu}} n_\nu$$

According to a Metropolis algorithm ran via Crooks theorem

**Reconstruct** the new  $\beta$  and  $\Pi_{\mu\nu}$  . The Ward identity will make sure  
 $\beta_\mu \beta^\mu = -1/T^2$

**Computationally intensive** (an ensemble at every timestep), but who knows?

## A semiclassical limit?

$$\partial_\mu \langle \hat{T}^{\mu\nu} \rangle = 0 \quad , \quad \partial_\mu \langle \hat{T}_0^{\mu\nu} \rangle = -\partial_\mu \langle \hat{\Pi}^{\mu\nu} \rangle$$

Integrating by parts the second term over a time scale of many  $\Delta_{\mu\nu}$  gives, in a frame comoving with  $d\Sigma_\mu$

$$\int_0^\tau d\tau' \langle \hat{\Pi}_{\mu\nu} \rangle \partial^\mu \beta^\nu \sim \beta^\mu \partial_\mu \langle \hat{\Pi}_{\mu\nu} \rangle + \langle \hat{\Pi}_{\mu\nu} \rangle = F(\partial^{n \geq 1} \beta_\mu, \dots)$$

where  $F(\beta_\mu)$  is independent of  $\Pi_{\mu\nu}$ . (Because local entropy is maximized at vanishing viscosity  $F()$  depends on gradients. **Israel-Stewart**)

**However**, results of, e.g., Gavassino 2006.09843 and Shokri 2002.04719 suggest that fluctuations with decreasing entropy have a role at first order in gradient!

# What next?



## A 1D example

$\beta, \Pi$  are numbers and there is no vorticity so **no**  $\Sigma$  either!

$$T_0^{\mu\nu} = U^{-1} \begin{pmatrix} e & 0 \\ 0 & -p(E) \end{pmatrix} U, \quad U = (1 - \beta^2)^{-1} \begin{pmatrix} 1 & \beta \\ \beta & -1 \end{pmatrix}$$

$$\Pi^{\mu\nu} = \begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix}, \quad \Sigma^\mu \propto \beta^\mu$$

**Random matrix** distribution of  $\{\beta, e\} \leftrightarrow \{\pi\}$

$$\frac{P(\{e\} + d\beta\beta w)}{P(\{e\} - d\beta\beta w)} \propto \exp [\{\Pi\} \beta^{-1} \{\partial\beta\}] \delta(\text{Ward}[e, \beta, \pi])$$

Ward identity can fix  $\{\beta\}$  from  $\{\pi\}$ , rest is Markov chain

Polarization, Chemical potential, rotations, accelerations,...

$$\beta_\mu T^{\mu\nu} \rightarrow \beta_\mu T^{\mu\nu} + \mu N^\mu + \mathcal{W} \mathcal{J}^\mu$$

F. Becattini et al, 2007.08249, Prokhorov et. al. 1911.04545: Global equilibrium under general “passive” non-inertial transformation

**A paradox:** State in “Global equilibrium” (Maximum entropy) but generally does not obey **KMS conditions** Stationarity/stability!

**Global/local equilibrium** not the same.

**2nd law of thermodynamics** defined locally, “entropy” frame dependent in non-inertial fluctuating system

How do you translate all this to dynamics?

Polarization, Chemical potential, rotations, accelerations,...

$$\beta_\mu T^{\mu\nu} \rightarrow \beta_\mu T^{\mu\nu} + \mu N^\mu + \mathcal{W} \mathcal{J}^\mu$$

Crooks Approach allows us to resolve these ambiguities straight-forwardly.

- System evolves to a state where KMS condition obeyed by proper time in the local foliation, ensemble foliation-independent
- Gauge potentials will lead to non-local correlations that never equilibrate,  
 $N^\mu \rightarrow N^\mu + U \partial^\mu U$

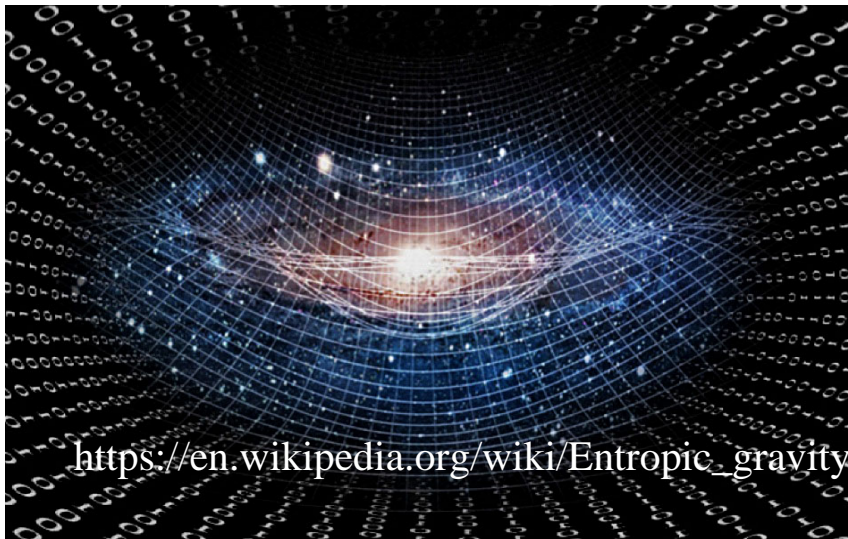
GT, 1810.12468

Wild speculations



General relativity/Theory of everything T.Jacobson,gr-qc/9504004

$$dS \propto d\mathcal{A} \quad , \quad dQ = TdS \Rightarrow G_{\mu\nu} \propto T_{\mu\nu}$$



T.Jacobson, gr-qc/9504004

T.Padmanabhan 0911.5004

E.Verlinde, 1001.0785



T.Jacobson, gr-qc/9504004

T.Padmanabhan 0911.5004

E.Verlinde, 1001.0785

## Started the field of “entropic gravity”

- gravity is emergent and spacetime is a thermalized state
- “Quantum dynamics” is actually fluctuating equilibrium state
- Difficulty of quantizing gravity makes it an interesting idea

## Combining Crooks theorem with relativistic field theory

$$S = \int d\mathcal{A} + \text{Tr} [\rho \ln \rho]$$

Dynamics of the geometry given by  $\exp [\Delta S] = P(W)/P(-W)$   
 $P(W)$  given by the density matrix ,  $P(W) = \text{Tr}[W \cdot \hat{\rho}]$

$$\hat{\rho} = \frac{1}{Z} \int \mathcal{D}\phi \langle \phi | \Psi \rangle \langle \Psi | \phi \rangle$$

Could lead to way to update density matrix. "detailed balance"  
fluctuation/dissipation  $\Rightarrow$  general covariance (GT, 1501.00435 )

*Fluctuation* :  $i \rightarrow \{ijk\dots\}$  , *Dissipation* :  $\{ijk\dots\} \rightarrow i$

Ensemble generally covariant/Foliation independent

"Every tenured physics professor should write at least one paper on a theory of everything."

What if the universe is governed by Crooks?



## Seriously... some conclusions



- Fluctuations force us to go beyond transport and perturbation theory
- Zubarev hydrodynamics and Crooks fluctuation theorem naturally provide us with a way!
- Lots to do but lots of potential!