Semi-abelian gauge theories, non-invertible symmetry & string tension beyond N-ality.

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MOTIVATION

- There are a handful of QFTs in 3d & 4d where confinement and mass gap generation can be understood analytically.

- Polyakov model on $\mathbb{R}^3$. (75)
  Seiberg - Witten on $\mathbb{R}^4$ (94)
  QCD(adj) on $\mathbb{R}^3 \times S^1$
  deformed Yang-Mills on $\mathbb{R}^3 \times S^1$ (08, w/ Yaffe, & w/ Shifman)

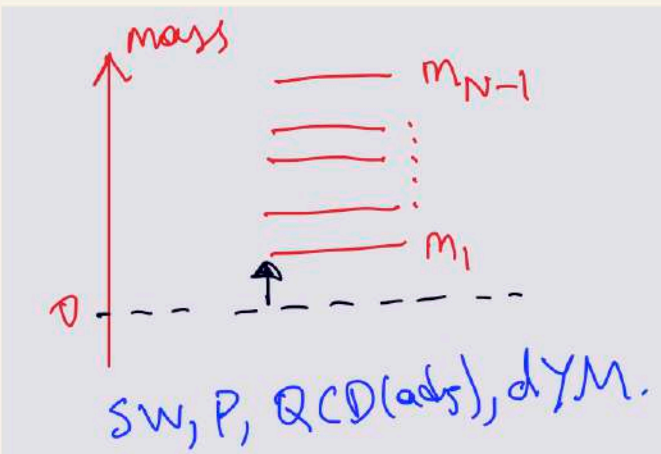
- In all of the above; $SU(N) \longrightarrow U(1)^{N-1}$ due to either an adjoint Higgs vev (algebra valued) or gauge holonomy. (group valued)
  "Dynamical abelianization"

- In all; S_N = Weyl group of $SU(N)$ is Higgsed.
  $\hookrightarrow$ permutation group.

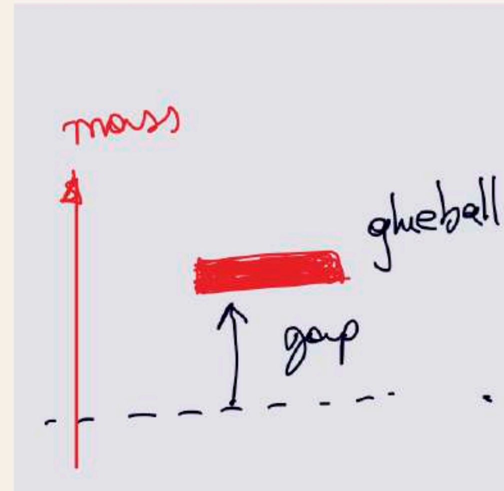
- Higgsing of S_N pervades the physics of these systems & renders them quite distinct from pure YM.

- Spectrum of fundamental string tensions. N-1 types in SW & Polyakov. (see Douglas, Shenker 95) instead of one!

- Spectrum of "glueball" masses. N-1 distinct dual photon masses in all abelianizing theories.



while in YM.

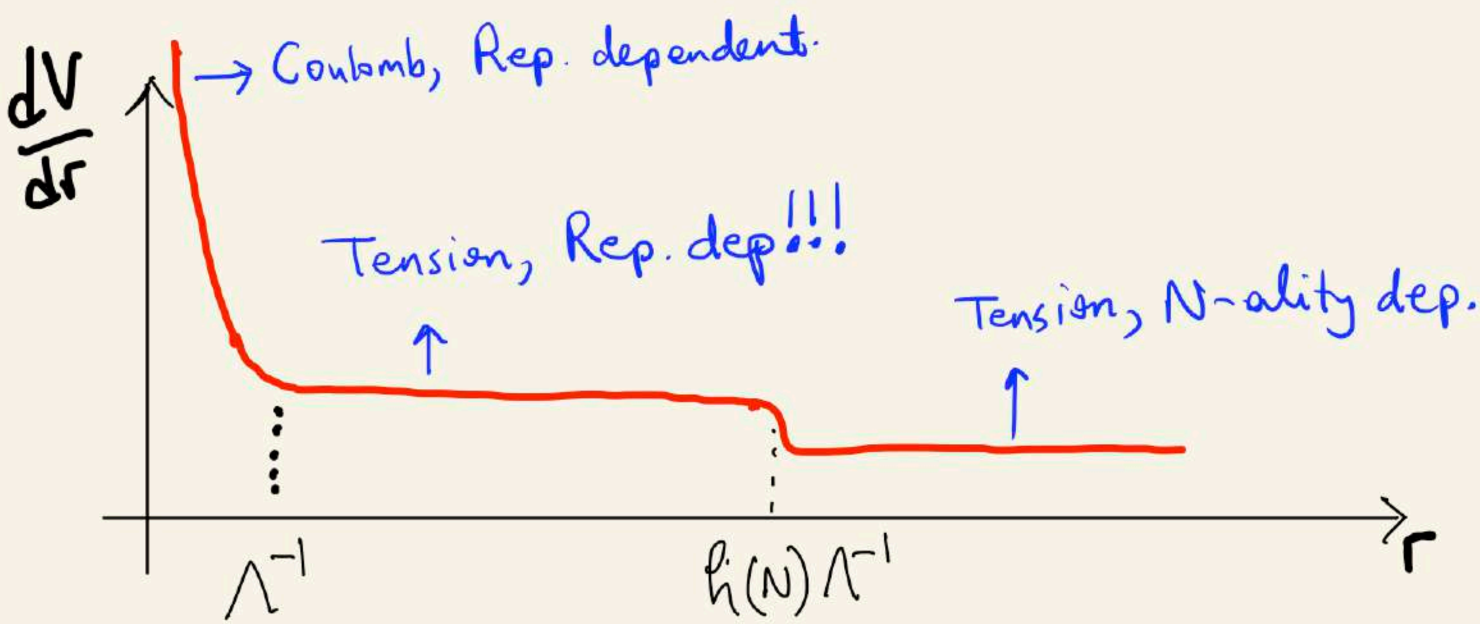


- Splitting to N-1 $\Longrightarrow$ Imprint of broken S_N.

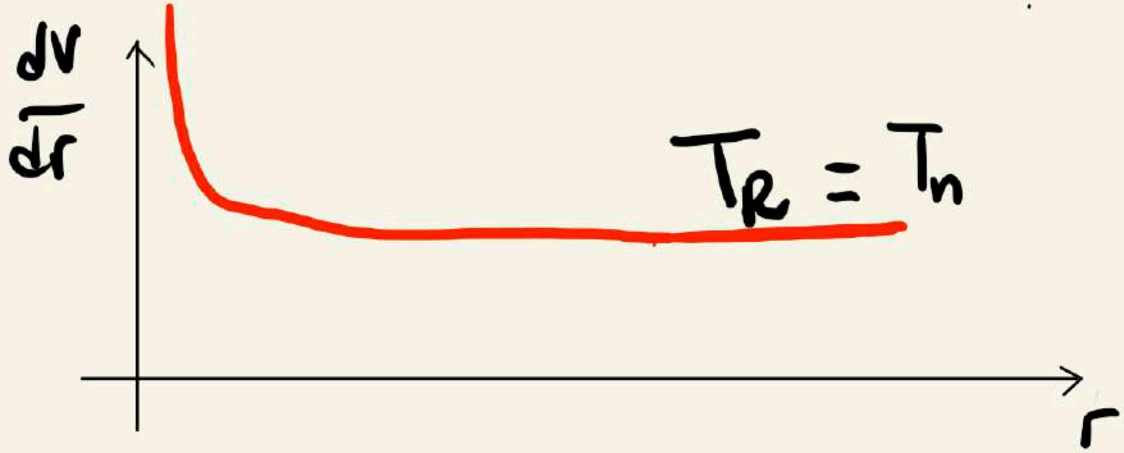
- Tensions dictated by charges $\omega \in \Gamma_w$ (weight lattice) & not N-ality! (sometimes presented as deficiency. Not so, it is a feature.)

CURIOUS FACT ABOUT STRING TENSION IN YM.

- In YM or QCD with adjoint matter, we are usually told that tensions are dictated by N-ality. This is true.
- However, the real story is actually more interesting.

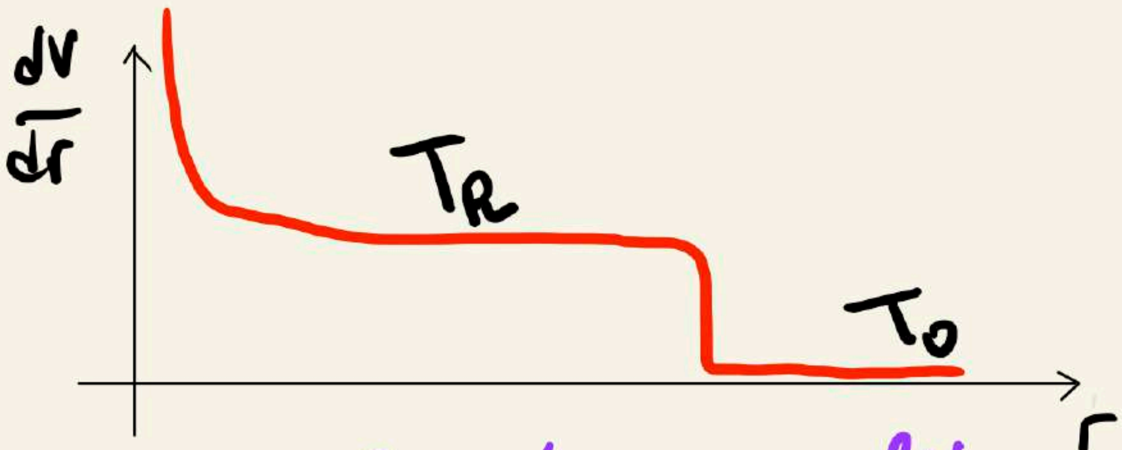


- based on lattice: Bali (93), de Forcrand, Philipsen (99), Wipf et.al.(10), Greensite (03), ...
  (not widely appreciated)

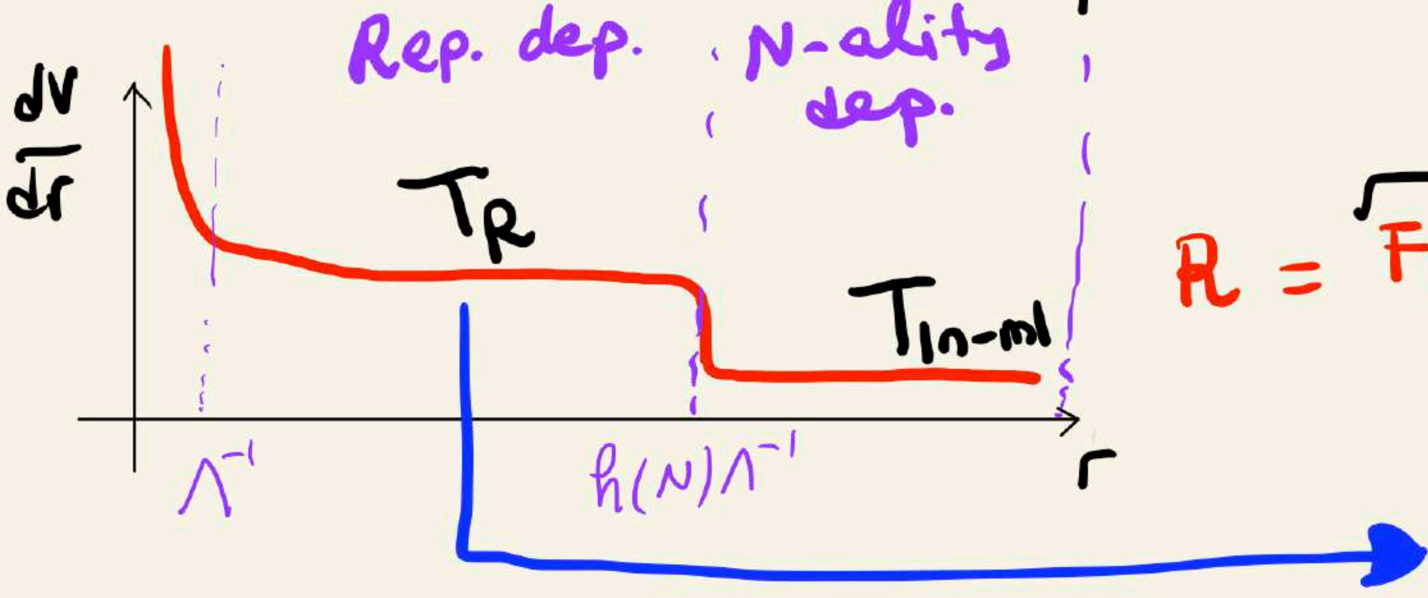


$$R = F, F\otimes F, \ldots, \underbrace{F\otimes \ldots \otimes F}_{n}$$

$$n \leq \left\lfloor \frac{N}{2} \right\rfloor$$



$$R = \mathrm{ad}_J, \mathrm{ad}_J \otimes \mathrm{ad}_J, \ldots$$



$$R = \overbrace{F\otimes \ldots F}^{n} \otimes \overbrace{\bar{F} \otimes \ldots \bar{F}}^{m}$$

WHY?

DEFINITION OF SEMI-ABELIAN THEORY.

- We construct two interesting gauge theories.
- $U(1)^{N-1}$ theory with **global** non-abelian discrete sym. S_N
- $\underbrace{U(1)^{N-1} \rtimes S_N}_{\text{Semi-direct product.}}$ **semi-abelian g.t.** obtained by **gauging S_N**. (coupling to S_N TQFT)

- Abelianizing theories
  $SU(N) \to U(1)^{N-1}$
  S_N: **Higgsed**

  Semi-abelian
  $U(1)^{N-1} \rtimes S_N$
  S_N: **not Higgsed**

  Non-abelian
  $SU(N)$ theories.
  S_N: **not Higgsed**

- Many properties of semi-abelian theory are much closer to pure YM than abelianizing theories.

3d LGT

- Wilson formulation
- Villain formulation

} of 3d $U(1)^{N-1}$ Lattice g.t. (LGT)

- Equivalent at weak coupling.
  - Villain more convenient for exact dualities.
  - Wilson better for gauging S_N.
- We work with 3d LGT; very concrete and simple illustration of some general ideas.

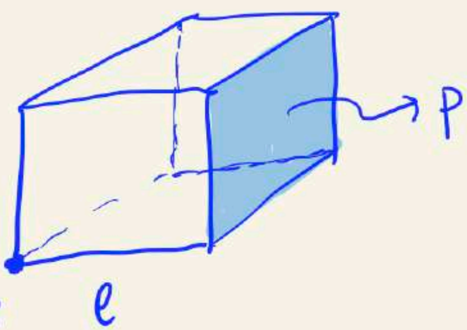
dual lattice & notation.

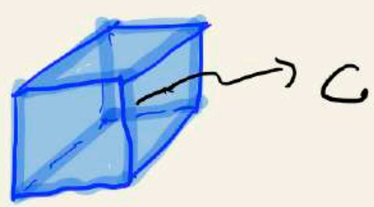
Λ_3 : Original lattice

$C^{(r)}$: r-cells, $r = 0,1,2,3$

site, link, plaquette, cube.

$\{s, \ell, p, c\}$, $s = x$



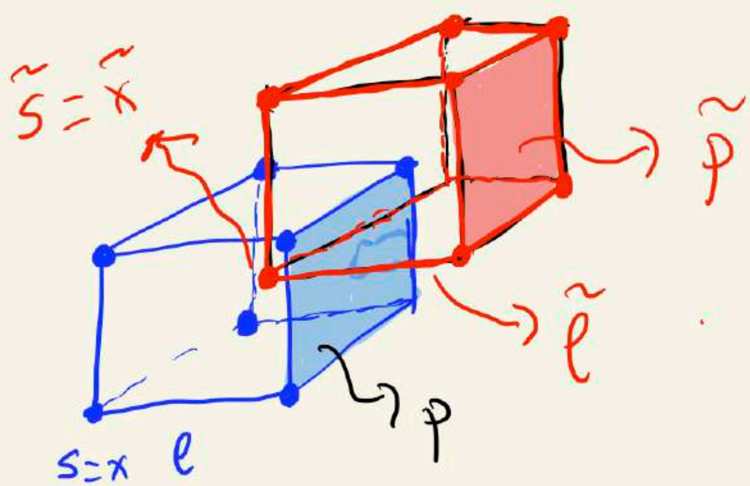


$\tilde{\Lambda}_3$: Dual lattice.

$\tilde{C}^{(r)}$

$$\boxed{*C^{(r)} = \tilde{C}^{(d-r)}}$$

$$*s = \tilde{c}$$
$$*\ell = \tilde{p}$$
$$*p = \tilde{\ell}$$
$$*c = \tilde{s} = \tilde{x} = \overbrace{x + \tfrac{1}{2}(\hat{1}+\hat{2}+\hat{3})}^{\text{shifted by half.}}$$



Wilson formulation

$$S_W = \beta \sum_p \sum_{i=1}^{N} (1-\cos f_p^i) - i \sum_\ell v_\ell \sum_{i=1}^{N} a_\ell^i \ ;$$

$\hookrightarrow$ Lagrange multiplier.

$$f_p^i = (da^i)_p \ ; \quad a_\ell^i \in [0, 2\pi)$$

produce constraint $\sum_{i=1}^{N} a_\ell^i = 0$.

- Constraint tells us that only $N-1$ photons are physical.
- Global S_N permutation symmetry manifest.

$$(a_\ell^1 \ \cdots \ a_\ell^N) \xrightarrow{P \in S_N.} (a_\ell^{P(1)}, \ \cdots, a_\ell^{P(N)})$$

- This is the first example of a pure gauge theory that is equipped with non-abelian global symmetry. (up to my knowledge.)

Villain formulation.

$$S = \frac{1}{4\pi e^2} \sum_p \left(F_p + 2\pi n_p\right)^2, \qquad F_p = (dA)_p$$

A_ℓ valued in $\mathbb{R}^{N-1}$

Γ_r : Root lattice of $SU(N)$.

n_p valued in $\Gamma_r \subset \mathbb{R}^{N-1}$

0- form gauge invariance: $A_\ell \to A_\ell + (d\lambda)_\ell \qquad \lambda_s \in \mathbb{R}^{N-1}$

1- form gauge inv. $A_\ell \to A_\ell + 2\pi \beta_\ell$; $n_p \to n_p - (d\beta)_p$, $\beta_\ell \in \Gamma_r$.

- Since $\boxed{\mathbb{R}^{N-1}/2\pi\Gamma_r \simeq U(1)^{N-1}}$; this also defines a $U(1)^{N-1}$ LGT.

$$\boxed{Z = \sum_{\{n_p \in \Gamma_r\}} \int_{\mathbb{R}^{N-1}} [dA_\ell]\, e^{-S}}$$

- Action in Villain form; can be exactly dualized.

Global symmetries (formal)

- Improved recent understanding of p-form symmetries.

Symmetry : $\stackrel{\text{def}}{=}$ existence of topological generators $U[M_{d-p-1}]$ (Gaiotto, Kapustin, Seiberg, Willet '14)

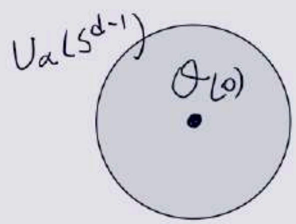
- Textbook case ($p=0$)

$$Q = \int_{\text{space}} J_0 \implies \text{consider } Q = \int_{M^{d-1}} *J \equiv Q(M^{d-1})$$

charge on any $(d-1)$ manifold M^{d-1}.

$$U_\alpha(M^{d-1}) = e^{i\alpha Q(M^{d-1})}$$

α: Group label.



$$= e^{i q_{\mathcal{O}} \alpha}\, \mathcal{O}(0)$$

- $U_\alpha(M^{d-1})$ implements symmetry action on charged operators.

$p=1$ 1-form center sym.

- Center sym. is a 1-form symmetry (acting on lines) generated by co-dim 2 defect.

$$W_R(C) \longmapsto e^{\frac{2\pi i}{N}|R|}\, W_R(C)$$

- $|R|$ = N-ality.

Nice explanation of N-ality rule.

Global symmetries

- 1-form symmetry: $A_e \to A_e + \theta_e$; $\theta_e \in \mathbb{R}^{N-1}$ & $(d\theta)_p = 0$.
- Discrete non-abelian 0-form symmetry:

$$A_e \to \pi A_e \quad ; \quad n_p \longrightarrow \pi\, n_p$$

Π : $O(N-1)$ transformations that preserve root lattice Γ_r.

Weyl group of $SU(N)$, S_N & $\mathbb{Z}_2$ charge conj.

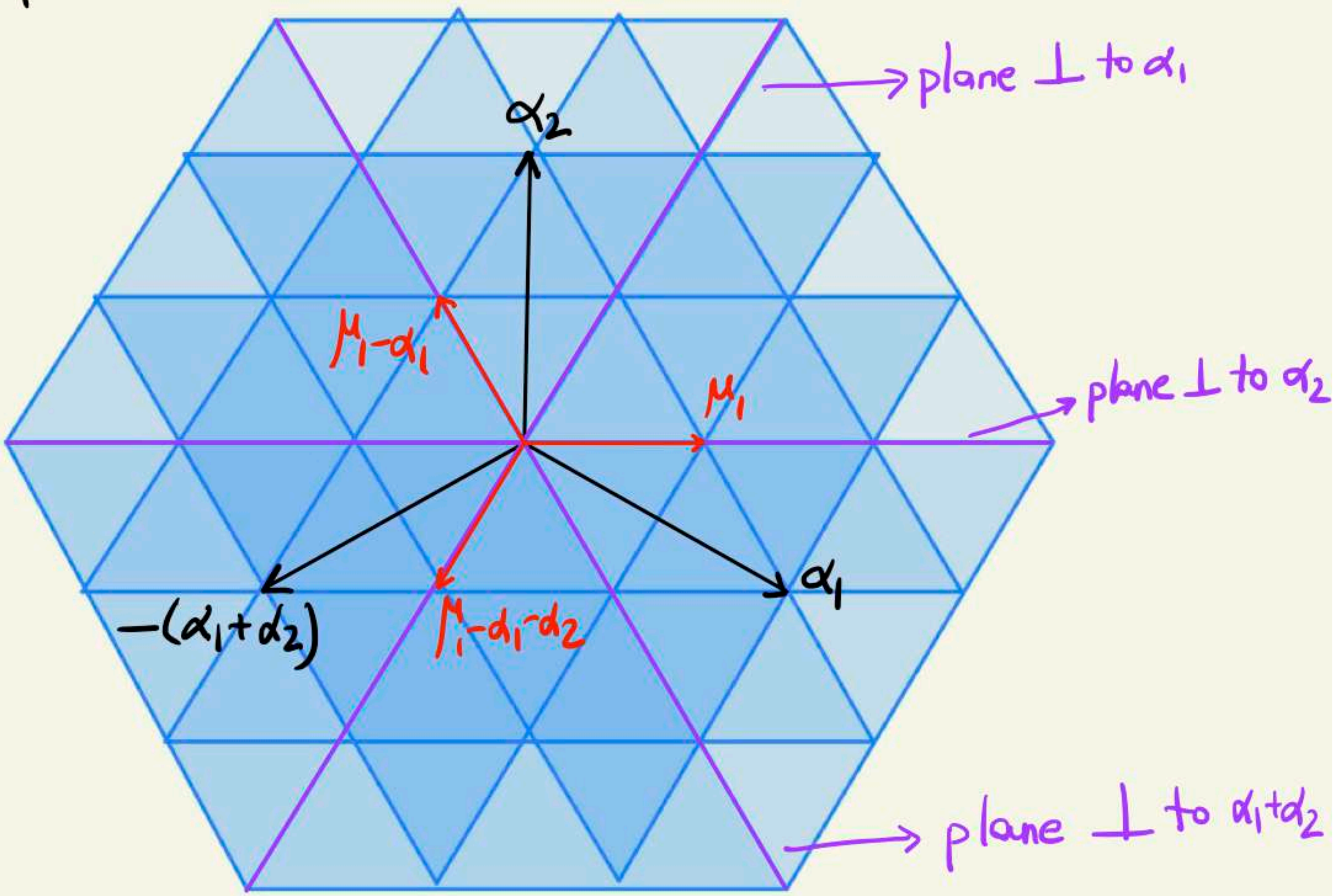
$$A_e \longrightarrow A_e - \alpha(\alpha \cdot A_e) \qquad A_e \to -A_e$$

$$n_p \longrightarrow n_p - \alpha(\alpha \cdot n_p) \qquad n_p \to -n_p\,.$$

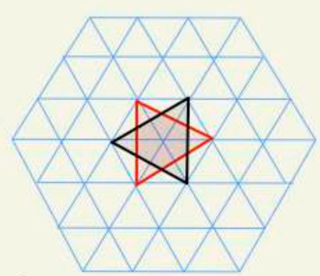
$$G^{[0]} = \begin{cases} S_N \rtimes \mathbb{Z}_2 & N \geq 3 \\ \mathbb{Z}_2 & N = 2 \end{cases} \qquad G^{[1]} = \left\{ U(1)^{N-1} \right.$$

- Weyl reflections: $S_\alpha(v) = v - \alpha(\alpha \cdot v)$

S_3 permutation group. (S_N sym of $N-1$ symplex.)



- $\mathbb{Z}_2$ charge conjugation:



$G^{[0]}: S_3 \rtimes \mathbb{Z}_2$

Observables.

- Wilson lines. $W_\omega(C) = e^{i\int_C \omega \cdot A}$

Invariance under 1-form gauge transformation $\Longrightarrow$ $\omega \in \underline{\Gamma_W}$ weight lattice.

- Transformation under global symmetries.

$$G^{[0]}: \quad W_\omega(C) \longmapsto W_{\pi^{-1}\omega}(C).$$

$$G^{[1]}: \quad W_\omega(C) \longmapsto W_\omega(C)\, e^{i\int_C \omega\cdot\theta}$$

- Contractible loops are invariant under $G^{[1]}$.

Non-contractible loops transform.

Villain $\Longrightarrow$ Γ_w- ferromagnet $\Longrightarrow$ Magnetic Coulomb gas.

Z^{N-1} ferro. (Gopfert, Mack)

- Poisson resummation.

$$\sum_{n_p \in \Gamma_r} e^{-\frac{1}{4\pi e^2}(F_p + 2\pi n_p)^2} = \sum_{k_p \in \Gamma_w} e^{-\pi e^2 k_p^2 + i k_p \cdot F_p}$$

- Integrate out A_ℓ exactly. $\Rightarrow$ $(d^\dagger k)_\ell = 0.$ $\Rightarrow$ $(\star k)_\ell = (dm)_\ell$ where $m_{\tilde{x}}$ is Γ_w-valued scalar on the dual lattice $\tilde{\Lambda}_3$.

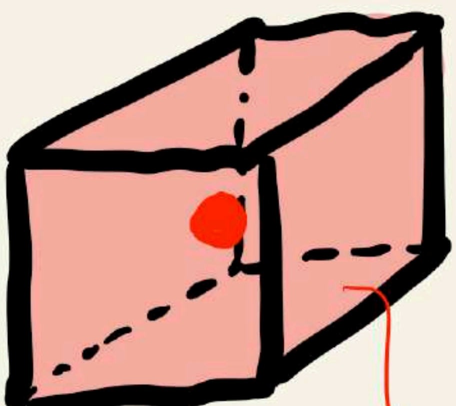
- $$\boxed{Z = \sum_{\{m_{\tilde{x}} \in \Gamma_w\}} e^{-\pi e^2 \sum_{\tilde{\ell}} (dm)^2_{\tilde{\ell}}}}$$

Γ_w- ferro. Exact dual of Villain.

- Let us replace $m(\tilde{x})$ with a continuous field. Using another Poisson resummation;

$$\sum_{m(\tilde{x})\in\Gamma_w} \delta(\sigma(\tilde{x}) - 2\pi m(\tilde{x})) = \sum_{q(\tilde{x})\in\Gamma_r} e^{i\, q(\tilde{x})\cdot\sigma(\tilde{x})}$$

$$Z = \int [d\sigma(\tilde{x})] \sum_{\{q(\tilde{x})\in\Gamma_r\}} e^{-\frac{e^2}{4\pi}\sum_{\tilde{x}}(\partial_\mu \sigma(\tilde{x}))^2 + i\sum_{\tilde{x}} q(\tilde{x})\sigma(\tilde{x})}$$



$$q(\tilde{x}) = \oint_{\text{faces, center } \tilde{x}} n_p \quad \in \Gamma_r$$

$\hookrightarrow$ Magnetic charge.

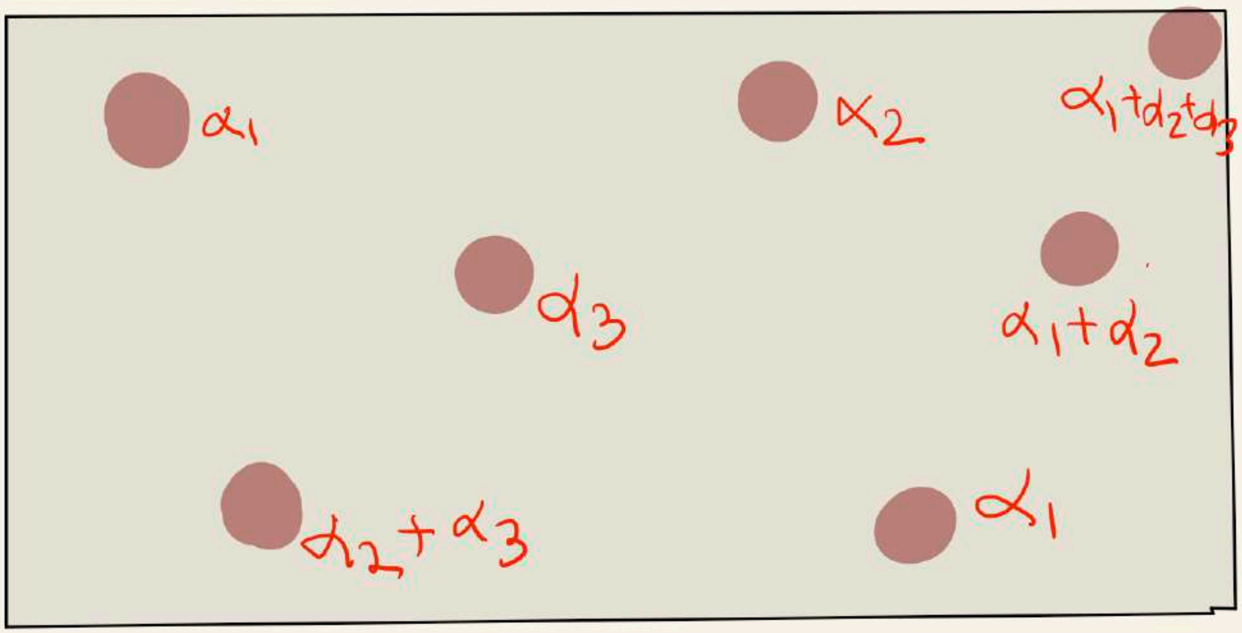
Gauss surface.

Coulomb gas representation.

- Integrating out $\sigma(\tilde{x})$ exactly; we reach to Coulomb gas representation.

$$\boxed{Z = \sum_{\{q(\tilde{x}) \in \Gamma_r\}} e^{-\frac{\pi}{e^2} \sum_{\tilde{x}, \tilde{x}'} v(\tilde{x} - \tilde{x}')\, q(\tilde{x}) \cdot q(\tilde{x}')}}$$

- Euclidean Vacuum: Proliferation of monopoles, but very different from Polyakov model & deformed YM due to unbroken S_N.



Effective Field Theory.

- $$\mathcal{L} = -\frac{e^2}{4\pi} \sum_{\tilde{x}} \sigma(\tilde{x}) \Delta \sigma(\tilde{x}) + 2 e^{-I} \sum_{\tilde{x}} \sum_{\alpha \in \Phi} \cos(\alpha \cdot \sigma(\tilde{x}))$$

Φ : all $N^2 - N$ roots of $SU(N)$ algebra. $\alpha \in Adj(SU(N))$

All on the same footing due to unbroken S_N.

- In Polyakov model where $SU(N) \to U(1)^{N-1}$, actions are hierarchical.

Hierarchical (Polyakov)

$\alpha_1, \alpha_2, \cdots, \alpha_{N-1}$: e^{-S_0}

$\alpha_1+\alpha_2, \alpha_2+\alpha_3, \cdots \alpha_{N-1}+\alpha_N$: e^{-2S_0}

$\alpha_1+\alpha_2+\alpha_3, \cdots$: e^{-3S_0}

$\vdots$

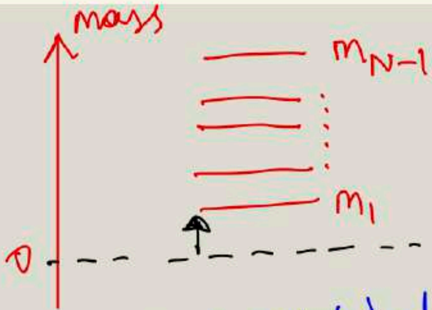
$\alpha_1 + \cdots + \alpha_{N-1}$: $e^{-(N-1)S_0}$

Egalitarian (Semi-abelian)

All N^2-N roots $\alpha \in \Phi$ have the same action. $Adj(SU(N))$

Dramatic differences

- In Polyakov; there are $N-1$ dual photon mass & $N-1$ fundamental string tension. (same in SW $\mathcal{N}=2$)

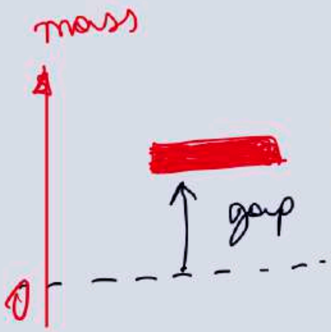


SW, P, QCD(adj), dYM.

& $T_{\mu_1} \neq T_{\mu_1-\alpha_1} \neq T_{\mu_1-\alpha_1-\alpha_2} \neq \cdots T_{\mu_1-\alpha_1-\cdots-\alpha_{N-1}}$

- Imprints of broken (Higgsed) S_N.

- In $U(1)^{N-1}$ LGT with S_N.



$N-1$ fold deg. & $T_{\mu_1} = T_{\mu_1-\alpha_1} = \cdots\cdots = T_{\mu_1-\alpha_1-\alpha_{N-1}}$

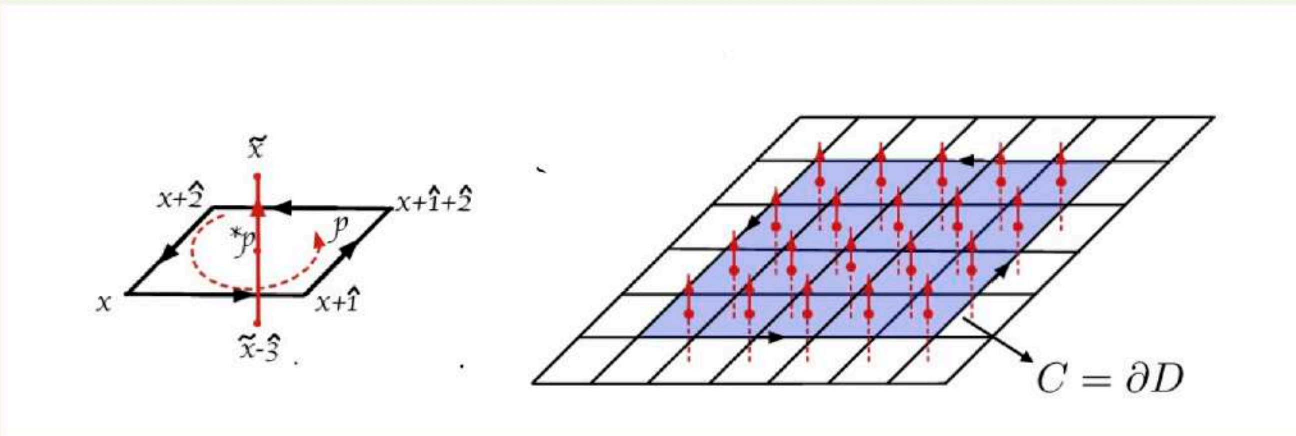
- Imprints of unbroken S_N.

Wilson loops & string tensions.

- Let $\omega \in \Gamma_W$ electric charge & $C = \partial D$ contractible loop.

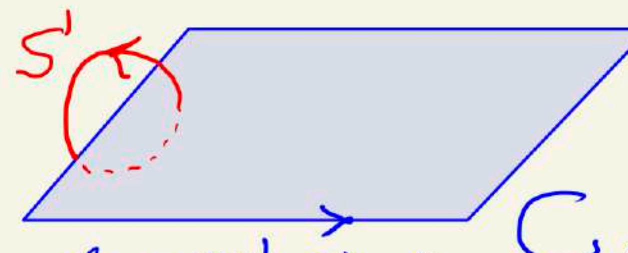
$$W_\omega(C) = e^{i\oint \omega . A} \rightarrow \text{Line integral.}$$

$$= \underbrace{e^{i\int_D \omega . F}}_{\text{surface int}} = \underbrace{e^{i\sum_{p\in\Lambda_3} [D]_{*p}(\omega F_p)}}_{\text{volume integral.}}$$



$[D]$: Poincaré dual, bump one form function equal to $\begin{cases} 1 & \text{on } {*p} \\ 0 & \text{otherwise.} \end{cases}$

- $W_\omega(C)$: Defect operator in dual formulation. Delete C & restrict path integral to config. $\oint_{S^1} d\sigma = 2\pi\omega \in 2\pi\Gamma_W$.

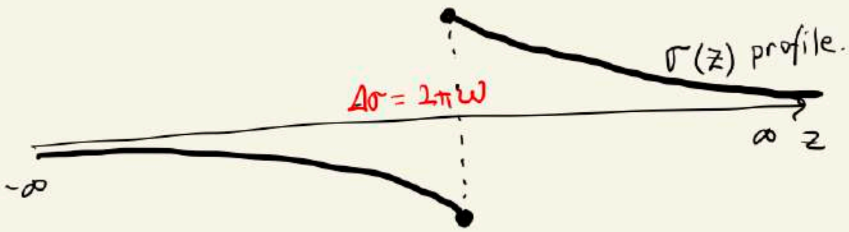


$\text{link}(S^1, C) = 1.$

- $\langle W_\omega(C)\rangle = \int \mathcal{D}\sigma \; e^{-\frac{e^2}{2\pi}\int d^3x \left(\frac{1}{2}|d\sigma - 2\pi\omega[D]|^2 + M^2 \sum_{\alpha\in\Phi_+}(1-\cos(\alpha\cdot\sigma))\right)}$

$$= e^{-T_\omega \, \mathrm{Area}(D)}$$

- Area law of confinement.
- $T_\omega \neq 0$ string tension

- 

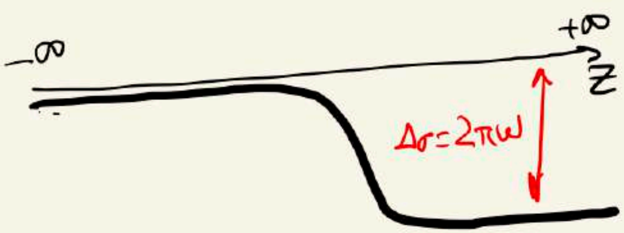
- Action of $\sigma(z) =$

$S = \mathrm{Area}[D] \times T_\omega$

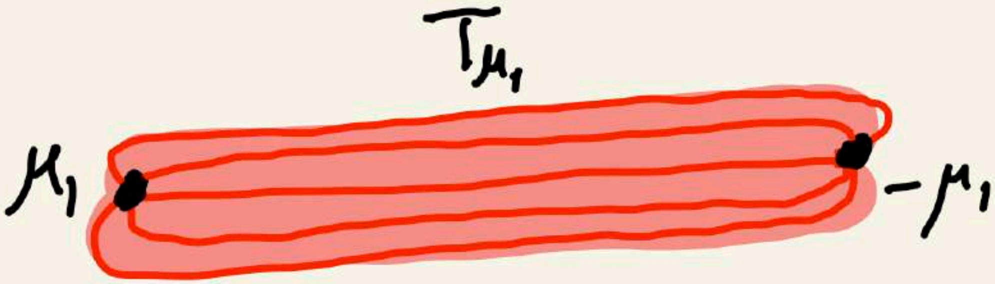
where T_ω is instanton action in reduced QM system.

- $A[D]$ is area.

- Equivalent to instanton calculation in QM



$$T_\omega = \min_{\sigma(z)} \frac{e^2}{2\pi}\int_{-\infty}^{\infty} dz \left(\frac{1}{2}\left(\frac{d\sigma}{dz}\right)^2 + M^2 \sum_{\alpha\in\Phi^+}(1-\cos\alpha\cdot\sigma)\right)$$



- $\{\mu_1, 2\mu_1, \mu_2, \alpha\} \in \Gamma_W$ highest weights of $\{F, S, AS, adj\}$ reps of $SU(N)$.

T_{μ_1} ; $T_{2\mu_1} = 2T_{\mu_1}$, $T_{\mu_2} = \frac{2(N-2)}{N-1} T_{\mu_1}$ $T_\alpha = 2T_{\mu_1}$

- If $\omega_1 \neq \Pi\omega_2$; generically; $T_{\omega_1} \neq T_{\omega_2}$.

- There is no $\omega \in \Gamma_W$ for which T_ω vanishes. This simple fact will be important later.

Gauging S_N is good.

- Clearly, properties of our abelian theory with S_N global symmetry resembles to YM more than dynamically abelianizing theories.
- But global symmetries are different from YM.

$U(1)^{N-1}$ with global S_N
$G^{[0]}: S^N \rtimes \mathbb{Z}_2$ → charge conj.
$G^{[1]}: U(1)^{N-1}$

$SU(N)$ YM
$G^{[0]}: \mathbb{Z}_2$ → charge conj.
$G^{[1]} = \mathbb{Z}_N^{[1]}$

- Let us gauge S_N. and construct $U(1)^{N-1} \rtimes S_N$ theory. Then, we will show that

$U(1)^{N-1} \rtimes S_N$
$G^{[0]}: \mathbb{Z}_2$
$G^{[1]}: \mathbb{Z}_N$

Semi-abelian gauge theory.

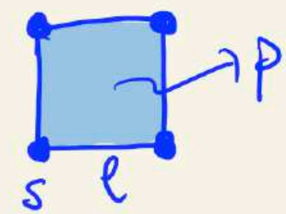
- Start with Wilson formulation and gauge S_N.
- Embed elements of $U(1)^{N-1} \rtimes S_N$ inside $SU(N)$ as

$$P \circ C \in SU(N) \; ; \qquad C = (e^{ia_1}, \ldots e^{ia_N})$$

$P \in S_N$: $N \times N$ matrix rep. of Weyl reflection.

$$(P_1 \cdot C_1)(P_2 \cdot C_2) = \underbrace{P_1 P_2}_{\in S_N} \cdot \underbrace{(P_2^{-1} C_1 P_2 C_2)}_{\in U(1)^{N-1}}$$

- Let $(P_\ell \cdot C_\ell) \in SU(N)$ denote link variable.



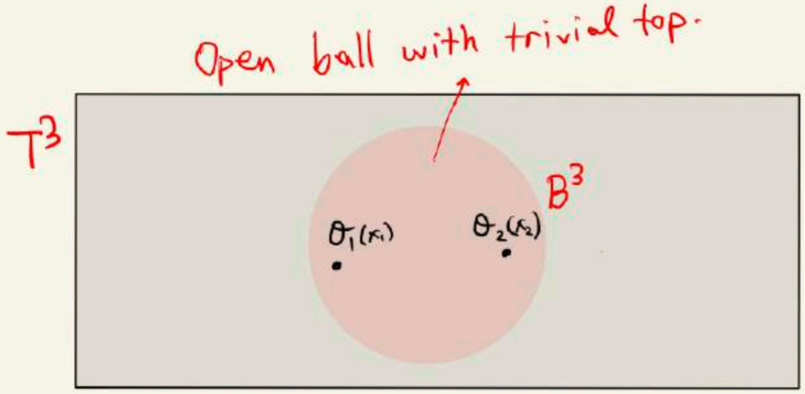
- $$S = \sum_p \beta_1 \left(\mathrm{tr}\Big(1_N - \prod_{(\ell \in \partial p)} (P_\ell \cdot C_\ell)\Big)\right) + \underbrace{\sum_p \beta_2 \, \mathrm{tr}\Big(1_N - \prod_{\ell \in \partial p} P_\ell\Big)}_{S_N}$$

Coupling to S_N TQFT.

- $\beta_2 \to \infty$: $\begin{cases} \text{imposes flatness condition on } S_N \text{ gauge field.} \\ S_N \text{ monopoles are energetically forbidden.} \end{cases}$

$$\prod_{e \in \partial p} P_e = 1_N \implies \boxed{S_N \text{ gauge fields become topological.}}$$

- As long as correlators do not involve non-contractible cycles on M_3; local dynamics must be the same as abelian model



- P_e can be fixed to 1 inside B^3.

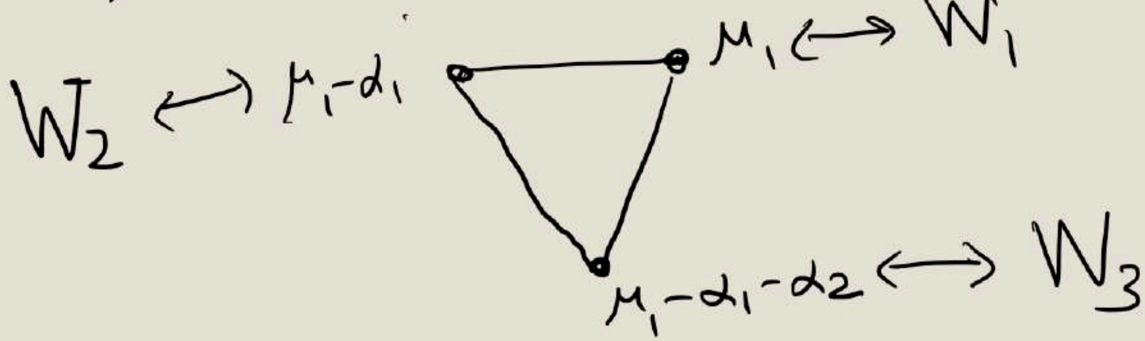
Local dynamics same as abelian model.

$$\langle O_1(x_1)\, W^{S_N}(x_1,x_2)\, O_2(x_2) \rangle_{U(1)^{N-1} \rtimes S_N} = \langle O_1(x_1)\, O_2(x_2) \rangle_{U(1)^{N-1}}$$

- Globally different theories. e.g. correlators involving non-contractible cycles know about S_N TQFT.

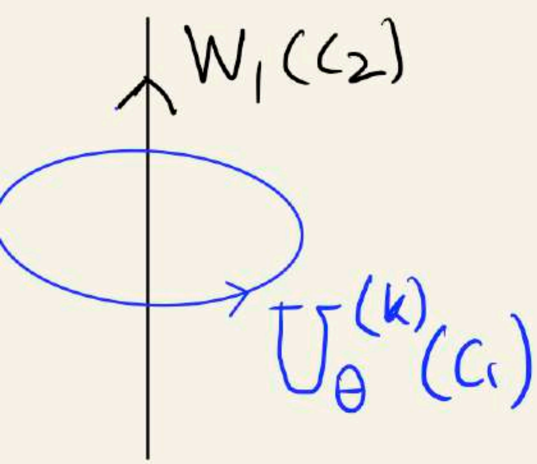
Wilson Loops & symmetry generators in abelian (original) $U(1)^{N-1}$ model.

Wilson loops : $W_k(C) = e^{i\left(\mu_1 - \sum_{J=1}^{k-1} \alpha_J\right)\cdot \int_C A}$ $\quad k = 1, \dots N.$

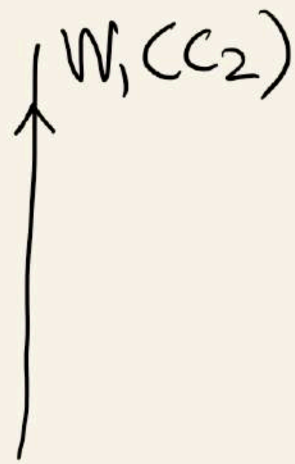


$\left(U(1)^{[1]}\right)^{N-1}$ generators: $U_\theta^{(k)}(C) = e^{i\frac{\theta}{2\pi}\int_C \alpha_k \cdot d\sigma}$ $\quad k = 1, \dots N-1.$

$\sigma \in \mathbb{R}^{N-1}/2\pi\Gamma_w$



$= e^{i\theta\delta_{k1}}$



Eigenvalue eq.

$W_1(C_2)$: Eigen-operator.

$U_\theta(C_1)$: Topological line operator. (generator)

Gauging S_N

- After gauging S_N; generators & Wilson operators are no longer gauge invariant.
- We must symmetrize Wilson loops & topological operators to build gauge invariant objects.

- Wilson loops in $U(1)^{N-1} \rtimes S_N$ theory.

$$W_{fd}(C) = W_1(C) + W_2(C) + \cdots + W_N(C)$$

- $\mathbb{Z}_N$ generators.

$$U_n(C) = \prod_{k=1}^{N-1} U^{(k)}_{\frac{2\pi}{N}kn}(C) = e^{i\frac{n}{N}\int_C (\alpha_1 + 2\alpha_2 + \cdots + (N-1)\alpha_{N-1})\cdot d\sigma}$$

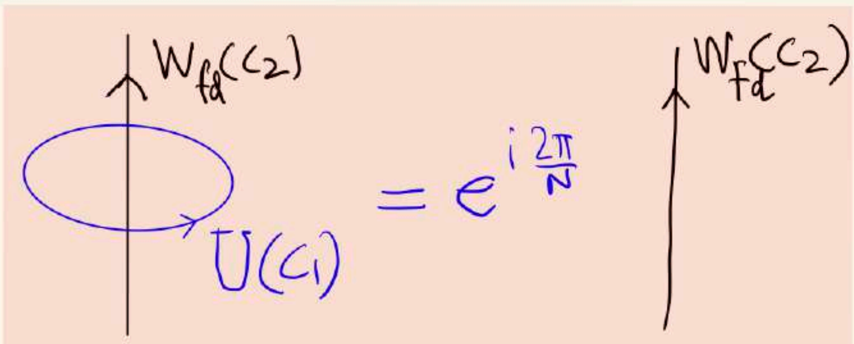
$$U_n(C)\, U_m(C) = U_{n+m \bmod N}(C)$$

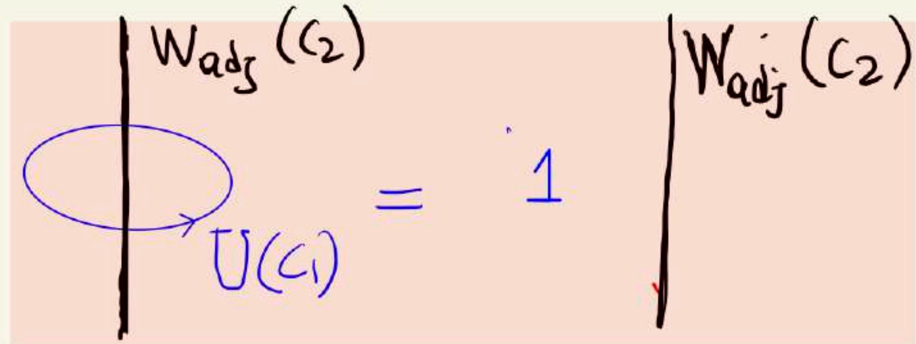
Fusion obeys group multiplication laws.

$\mathbb{Z}_N^{[1]}$ center & failure of N-ality rule.

$$\mathbb{Z}_N^{[1]} : W_{fd} \longmapsto e^{2\pi i/N} W_{fd} . \quad \text{or}$$

$$\langle U(C_1) W_{fd}(C_2) \rangle = e^{\frac{2\pi i}{N} \mathrm{Link}(C_1, C_2)} \langle W_{fd}(C_2) \rangle$$





Q: Does 1-form $\mathbb{Z}_N^{[1]}$ center sym. control string tensions?
No. $T_{\mu_1} \neq 0$, but $T_\alpha \neq 0$ as well (zero N-ality). $T_{2\mu_1} \neq T_{\mu_2}$ (both N-ality 2).

- Infinitely many string tensions, instead of just N.

Puzzle: How is the presence of infinitely many string tensions compatible with finite center symmetry?

Non-invertible topological lines & symmetry.

- Need to explain the failure of the N-lity rule.
- $\Longrightarrow$ Non-invertible top. symmetry. Generalization of 1-form $\mathbb{Z}_N^{[1]}$ center symmetry.
- Symmetrizing center generators $U_\theta^{(1)}(C)$ in abelian theory:

$$\boxed{\begin{aligned} T_\theta(C) &= \frac{1}{N!} \sum_{P \in S_N} P\, U_\theta^{(1)}(C)\, P^{-1} \\ &= \frac{1}{N(N-1)} \sum_{\alpha \in \Phi} e^{i \frac{\theta}{2\pi} \int_C \alpha \cdot d\sigma} \end{aligned}}$$

- Satisfies all features of 1-form symmetry, except group multiplication law.

$$\boxed{T_\theta(C)\, T_{\theta'}(C) \neq T_{\theta+\theta'}(C)}$$

Symmetry, but not a group.

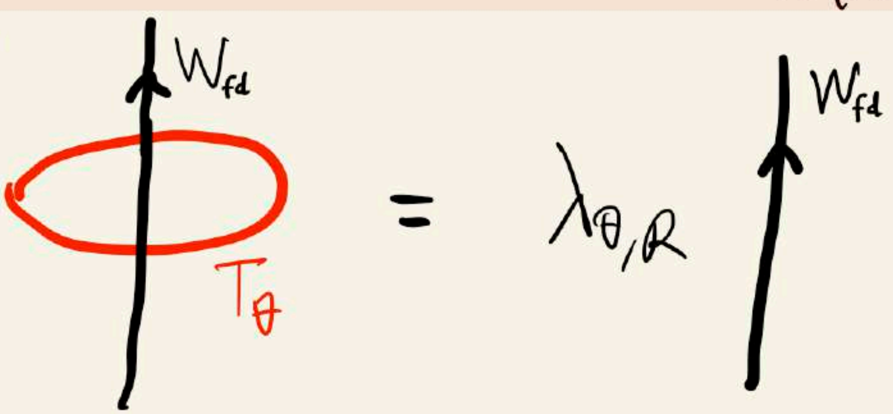
- Consider Wilson loop that corresponds to $\omega \in \Gamma_w$.

  electric charge in weight lattice.

  e.g. $W_R = \frac{1}{N!} \sum_{P \in S_N} P W_\omega P^{-1}$ etc.

- $$T_\theta(C)\, W_R(C') = \lambda_{\theta,R}\, W_R(C')$$

  $$\lambda_{\theta,R} = \frac{1}{N(N-1)} \sum_{\alpha \in \Phi} e^{i\theta\, \alpha\cdot\omega}$$



$$T_\theta(C)\, W_{fd}(C') = \frac{1}{N}\left(N-2+2\cos\theta\right) W_{fd}(C')$$

2d QFTs: cf Bhardwaj, Tachikawa (17), Buican, Gromov (17), Thorngren, Wang (19), ...
Komargodski et. al. (20).

Tension for N-ality zero rep.

- Here is the main point concerning non-invertible sym.

Despite the fact that W_{adj} is trivial under $\mathbb{Z}_N^{[1]}$; $U(C_1)\, W_{adj}(C_2) = \mathbb{1}\, W_{adj}(C_2)$; it obeys

$$T_\theta(C_1)\, W_{adj}(C_2) = \underbrace{\frac{(N-2)(N-3) + 4(N-2)\cos\theta + 2\cos(2\theta)}{N(N-1)}}_{\text{non-trivial. eigenvalue.}}\, W_{adj}(C_2)$$

- String tension beyond N-ality is characterized by non-invertible topological line operators T_θ.

- T_θ is a 1-form sym. But it does not have an inverse. $T_{-\theta} = T_\theta$. Fusion rule does not conform with group law. Fusion category???

Distinguishing two N-ality two reps.

- Consider two N-ality 2-reps, $W_{sym.}$ & $W_{asym.}$
- Eigen-operators of non-invertible generators $T_\theta(C_1)$ are

W_{asym} & $(W_{sym} - W_{asym})$.

$$T_\theta(C_1)\,(W_{sym} - W_{asym})(C_2) = \boxed{\frac{N-2+2\cos 2\theta}{N}}\,(W_{sym} - W_{asym})$$

$$T_\theta(C_1)\,W_{asym}(C_2) = \boxed{\frac{(N-2)(N-3)+2+4(N-2)\cos\theta}{N(N-1)}}\,W_{asym}(C_2)$$

- consistent with $T_{\mu_2} \neq T_{2\mu_1}$.

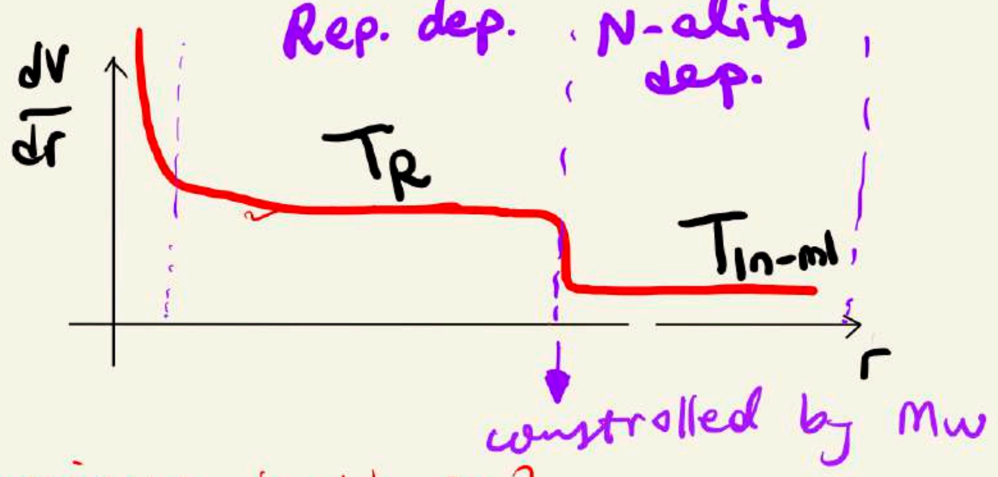
Dynamical electric charges & breaking non-invertible sym.

- Add W-bosons. Since eigenvalue of W_{adj} must be one, (perimeter law or adjoint string breaking);

$$T_\theta(C_1)\, W_{adj}(C_2) = \underbrace{\lambda_{\theta, adj}}_{=1} W_{adj}(C_2)$$

the only solution is for $\theta = 0$, and we lose non-inv. symmetry. (cf. Rudelius, Shao 20, discrete gauge theories.)

- If W is heavy, then;



- suspiciously curious, isn't it?

Conclusions - speculations

- Abelian and semi-abelian dynamics are quite different from other solvable, abelianizing theories: Polyakov, $N=2$ SW, QCD(adj), deformed YM. Provides new insights.

- Non-invertible sym. is very likely also present in pure YM as an approximate sym. We belive that it becomes exact at $N=\infty$ limit and string tensions are not controlled by N-ality.

- Non-inv. sym can very likely provide a meaning to confinement in theories such as QCD(F) and YM with G_2 group etc. It also gives a meaning to confinement of adjoint probes in SU(N) YM.