Factorial. Before we can continue on to our next counting technique, we will need to learn a new idea and notation. The idea is called the factorial, it has the notation \( ! \). It is easiest to understand the idea by looking at the pattern.

\[
\begin{align*}
0! &= 1 \\
1! &= 1 \\
2! &= 2 \cdot 1 \\
3! &= 3 \cdot 2 \cdot 1 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\
\end{align*}
\]

The idea can be expressed in general as:

\[
 n! = n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1.
\]

We could stop the expansion process at any point and indicate the remainder of the factorial in terms of a lower factorial.

\[
\begin{align*}
5! &= 5 \cdot 4! \\
    &= 5 \cdot 4 \cdot 3! \\
    &= 5 \cdot 4 \cdot 3 \cdot 2! \\
    &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1! \\
    &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1
\end{align*}
\]

The ability to expand any factorial partially can be an aid in simplifying expressions involving more than one factorial.

Example 1. Compute, by expanding and simplifying,

\[
\frac{12!}{9!}.
\]

Solution:

\[
\begin{align*}
\frac{12!}{9!} &= \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!} \quad \text{(by expanding)} \\
&= \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!} \quad \text{(cancelling like terms)} \\
&= 12 \cdot 11 \cdot 10 = 1320
\end{align*}
\]

Next we will consider the difference in the following tasks and learn how to count them. The first task is to take 3 books from a pile of 8 distinct books and line them up. The second task is to select 3 of the 8 distinct books.
The first thing to notice is that we can distinguish each object from any other and that we cannot replace any book that has been used, this is called without replacement. Both of our tasks have this feature. Next, we note that the first task is to line up the books, thus order makes a difference; \{\text{math, art, english}\} would look different from \{\text{art, math, english}\}. So for task one, ORDER MATTERS. In the second task, we simply choose a group of 3 with no arranging, so for task two, ORDER DOES NOT MATTER.

This leads us to the definitions for these situations. A permutation is an arrangement of objects. A combination is a collection of objects.

Next, we will learn how to count the number of permutations and combinations.

The number of permutations (arrangements) without replacement of \(r\) objects from a group of \(n\) distinct objects is denoted \(P(n, r)\) or \(nPr\) and is calculated with the formula:

\[
P(n, r) = \frac{n!}{(n-r)!}.
\]

The number of combinations (groups) without replacement of \(r\) objects chosen from a group of \(n\) distinct objects is denoted \(C(n, r)\) or \(nCr\) and is calculated with the formula:

\[
C(n, r) = \frac{n!}{r!(n-r)!}.
\]

Example 2. Consider the set \{a, b, 5\}.

(1) How many permutations of 2 of the objects are possible?

Solution: The first solution is to simply list out the permutations:

\[ab, a5, ba, b5, 5a, 5b\]

to see that there are a total of 6. The second solution is to use the formula where \(r\) is 2 (the number of objects being arranged) and \(n\) is 3 (the number of objects from which we are selecting those to arrange). This gives the result:

\[
P(3, 2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{6}{1} = 6.
\]

(2) How many groups of two of the objects are there?

Solution: Again, our first solution will be to list the possibilities:

\[\{a, b\}, \{a, 5\}, \{b, 5\}\]

to see that there are only three ways to choose two of the objects. The second solution is to use the formula with \(r = 2\) (the number of objects to be chosen) and \(n = 3\) (the number of objects from which we are choosing). This gives the result:

\[
C(3, 2) = \frac{3!}{2!(3-2)!} = \frac{3!}{2!1!} = \frac{6}{2 \cdot 1} = \frac{6}{2} = 3.
\]

Example 3. A museum has 12 paintings. They have space to display 6 of them in the current exhibit. How many ways are there for the museum to choose and arrange the 6 paintings?
Solution: Since we are asked how many ways there are to arrange the painting, then the order is important and we are dealing with a permutation. We want to arrange 6 of the 12 painting. Thus in the formula \( P(n, r) = \frac{n!}{(n-r)!} \), \( n = 12 \) and \( r = 6 \). We then calculate \( P(12, 6) \) to find that there are 665,280 ways for the museum to arrange six of the twelve paintings.

**Example 4.** Katie is taking an exam containing eight questions. She is required to answer five of the questions. How many ways are there for Katie to choose the five questions?

Solution: Since it does not matter which order Katie answers the questions, this is a combination problem. Thus in the formula \( C(n, r) = \frac{n!}{r!(n-r)!} \), \( n = 8 \) and \( r = 5 \). We then calculate \( C(8, 5) \) to find that there are 56 ways for Katie to pick out the five questions to answer.

**Example 5.** An animal shelter has 30 dogs and 20 cats. They are going to be holding an adoption event where there will be room for 15 dogs and 10 cats. How many ways are there for the shelter to choose the dogs and the cats for the adoption event?

Solution: To solve this problem, we will need to use multiple counting methods. Since there are two types of animals that we will be choosing, the question is a fundamental counting principle question at its base. We will need to figure out how many ways there are to choose the 15 dogs and then how many ways there are to choose the 10 cats.

\[
\text{ways to choose dogs} \times \text{ways to choose cats}
\]

We now just need to figure out what the numbers are that go into each box. We will start with the ways to choose the dogs. Since the order that we choose the 15 dogs out of the 30 dogs does not matter, this is a combination questions. The same is true for choosing the cats. Thus we can fill in the boxes with the appropriate combinations for the dogs and the cats.

\[
\text{ways to choose dogs} \times \text{ways to choose cats}
\]

When we do the final calculation

\[
155,117,520 \times 184,756 = 28,658,892,530,000
\]

ways to pick the animals for the event.