1. Matrix Algebra and Linear Economic Models

References

Ch. 1 – 3 (Turkington); Ch. 4 – 5.2 (Klein).

[1] Motivation

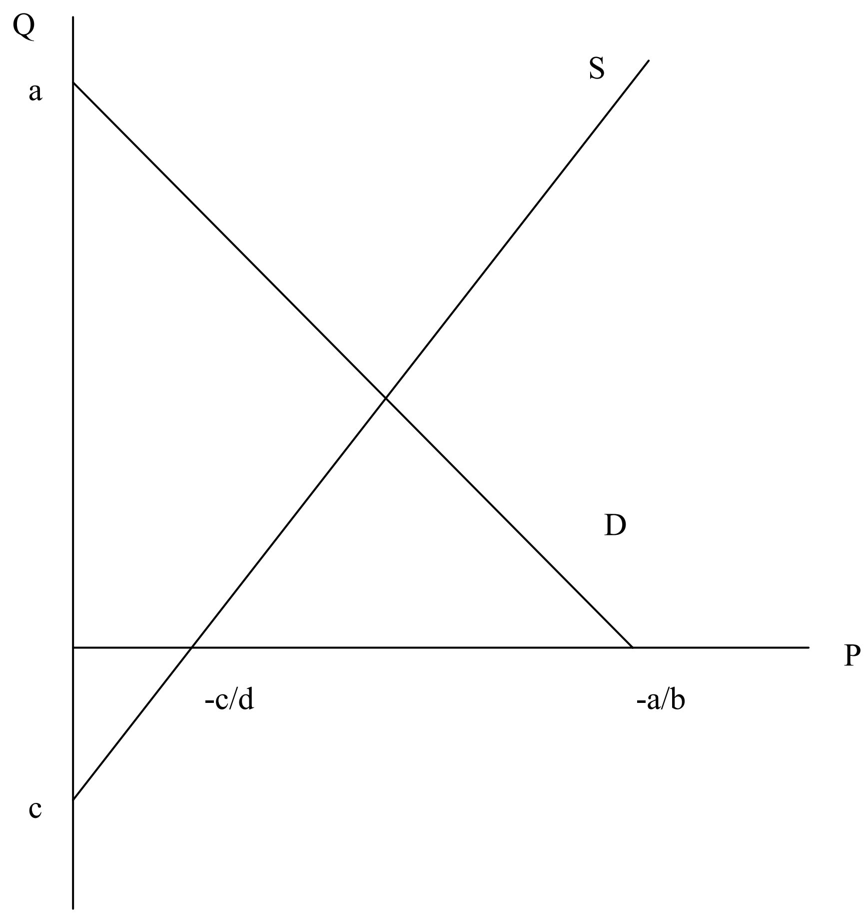
<u>One market equilibrium Model</u>

- Assume perfectly competitive market:

  Both buyers and sellers are price-takers.

- Demand: $Q_d = a + bP$, $a > 0$, and $b < 0$.

- Supply: $Q_s = c + dP$, $c \leq 0$ and $d > 0$.



- Production occurs only if $-c/d < -a/b$:

  $-c/d\times(db) > -a/b\times(db) \rightarrow -cb > -ad \rightarrow 0 > cb - ad$

  $\rightarrow bc - ad < 0$.

- Equilibrium Condition: $Q_d = Q_s$.
- Equilibrium quantity and price: $\overline{Q}$ and $\overline{P}$.
- $\overline{Q}$ and $\overline{P}$ satisfies:

  $$\overline{Q} = a + b\overline{P}; \quad (1)$$

  $$\overline{Q} = c + d\overline{P} \quad (2)$$

- (1) → (2):

  $$a + b\overline{P} = c + d\overline{P} \rightarrow \overline{P} = \frac{c-a}{b-d} > 0 \quad (3)$$

- (3) → (1):

  $$\overline{Q} = a + b\frac{c-a}{b-d} = \frac{a(b-d)}{b-d} + \frac{b(c-a)}{b-d} = \frac{bc-ad}{b-d} \quad (4)$$

- Two lessons here:
  - An economic model should assign proper signs on coefficients.
  - When demand and supply are linear, the equilibrium price and quantity are nothing but the solutions of two linear equations.

Equilibrium model of two markets

- Assumptions:
  - Two goods (coffee and tea).
  - Both markets are perfectly competitive.
  - Two goods are substitutable (not complementary).
  - Each producer can produce only one good (short-run).

Market 1:

- $Q_{d1} = 10 - 2P_1 + P_2$;
- $Q_{s1} = -2 + 3P_1$;
- $Q_{d1} = Q_{s1}$

Market 2

- $Q_{d2} = 15 + P_1 - P_2$;
- $Q_{s2} = -1 + 2P_2$.
- $Q_{d2} = Q_{s2}$

Question:

- Why is the coefficient of P_2 in the demand for Good 1 positive?
- Why is the coefficient of P_1 in the demand for Good 2 positive?
- Why is no P_2 in the supply of Good 2?

- At equilibrium:

1) $\bar{Q}_1 = 10 - 2\bar{P}_1 + \bar{P}_2$;

2) $\bar{Q}_1 = -2 + 3\bar{P}_1$

3) $\bar{Q}_2 = 15 + \bar{P}_1 - \bar{P}_2$;

4) $\bar{Q}_2 = -1 + 2\bar{P}_2$.

- 2) → 1):

5) $5\bar{P}_1 - \bar{P}_2 = 12$.

3) → 4):

6) $\bar{P}_1 = 3\bar{P}_2 - 16$.

- 6) → 5):

$5 \times (3\bar{P}_2 - 16) - \bar{P}_2 = 12 \rightarrow$

7) $\bar{P}_2 = 92/14$

- 7) → 6):

$\bar{P}_1 = 3 \times 92/14 - 16 = 52/14$

- You can solve for $\bar{Q}_1$ and $\bar{Q}_2$.

- Lesson:
  - As the number of equations increases, it becomes harder to solve a system of linear equations.

- Question:
  - How can we find the equilibrium prices and quantities for multiple market models?

- Use Matrices:
  - 1´) $1\times\bar{Q}_1 + 0\times\bar{Q}_2 + 2\bar{P}_1 + (-1)\times\bar{P}_2 = 10$;

    2´) $1\times\bar{Q}_1 + 0\times\bar{Q}_2 - 3\times\bar{P}_1 + 0\times\bar{P}_2 = -2$;

    3´) $0\times\bar{Q}_1 + 1\times\bar{Q}_2 + (-1)\times\bar{P}_1 + 1\times\bar{P}_2 = 15$;

    4´) $0\times\bar{Q}_1 + 1\times\bar{Q}_2 + 0\times\bar{P}_1 + (-2)\times\bar{P}_2 = -1$.

- $$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \bar{P}_1 \\ \bar{P}_2 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 15 \\ -1 \end{pmatrix}$$

- $$\begin{pmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \bar{P}_1 \\ \bar{P}_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ -2 \\ 15 \\ -1 \end{pmatrix}.$$

[2] Matrix and Operations

Definition: **Matrix**

A matrix, A, is a rectangular array of real numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n},$$

where i indexes **row** and j indexes **column**.

- A is called an m×n matrix (m = # of rows; n = # of columns).
- $A_{m\times n}$ or $[a_{ij}]_{m\times n}$ denote an m×n matrix.

EX: $\begin{bmatrix} 1 & 1 & 3 \\ 5 & -1 & 0 \end{bmatrix}_{2\times 3}; [2 \quad 1 \quad 3]_{1\times 3}, [4]_{1\times 1}$

- A 1×1 matrix is called “scalar”.

Definition: **Square Matrix**

If m = n for an m×n matrix A, A is called a square matrix.

EX: $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ -1 & 4 & 1 \end{bmatrix}_{3\times 3}$

Definition: **Transpose**

Let A be an m×n matrix. The transpose of A is denoted by A^t (or A'), which is a n×m matrix; and it is obtained by the following procedure.

- 1st column of A → 1st row of A^t,
- 2nd column of A → 2nd row of A^t… etc.

EX: $A = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 2 & 2 \end{bmatrix} \rightarrow A^t = \begin{bmatrix} 2 & 6 \\ 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Note: $(A^t)^t = A$.

Definition: **Symmetric Matrix**

Let A be a square matrix. A is called symmetric if and only if (iff) A = A^t.

EX: $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 5 \\ 1 & 5 & 7 \end{bmatrix} = A^t$; $B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = B^t$.

Definition: **Adding Matrices**

Let A and B be m×n matrices. (A + B) is obtained by adding corresponding entries of A and B.

EX: $\begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}; \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -2 & -3 \end{bmatrix}.$

Definition:

Let $A = [a_{ij}]$ be an m×n matrix and c be a scalar (real number). Then, cA is obtained by multiplying all the entries of A by c: $cA = [ca_{ij}]$.

EX: $6\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 18 & 30 \end{bmatrix}.$

Definition: **Vector**

Any m×1 matrix is called a column vector. Any 1×n matrix is called a row vector. Vectors are normally denoted by lower cases (e.g., x, y, a, b).

EX: $x = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}; z = \begin{pmatrix} 6 & 2 & 3 \end{pmatrix}.$

Note:

An m×n matrix can be viewed as a collection of m row vectors or n column vectors.

EX: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{pmatrix} = \begin{pmatrix} a_{\bullet 1} & a_{\bullet 2} & \cdots & a_{\bullet n} \end{pmatrix},$

where,

$$a_{i\bullet} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}; a_{\bullet j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Definition: **Multiplication of Vector**

Suppose *a* and *b* are 1×p and p×1 vectors, respectively:

$$a = (a_1, \ldots, a_p);\ b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}.$$

Then, $ab = a_1 b_1 + a_2 b_2 + \ldots + a_p b_p = \Sigma_{i=1}^{p} a_i b_i$.

EX: $a = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix};\ b = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \rightarrow ab = 1 \times 4 + 2 \times 1 + 3 \times 2 = 12$.

Definition: **Multiplication of Matrices**

Let A and B are m×p and p×n matrices, respectively. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mp} \end{bmatrix} = \begin{pmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{pmatrix};\ B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \ldots & b_{pn} \end{bmatrix} = \begin{pmatrix} b_{\bullet 1} & b_{\bullet 2} & \ldots & b_{\bullet n} \end{pmatrix}$$

Then,

$$AB = \begin{bmatrix} a_{1\bullet} b_{\bullet 1} & a_{1\bullet} b_{\bullet 2} & \ldots & a_{1\bullet} b_{\bullet n} \\ a_{2\bullet} b_{\bullet 1} & a_{2\bullet} b_{\bullet 2} & \ldots & a_{2\bullet} b_{\bullet n} \\ \vdots & \vdots & & \vdots \\ a_{m\bullet} b_{\bullet 1} & a_{m\bullet} b_{\bullet 2} & \ldots & a_{m\bullet} b_{\bullet n} \end{bmatrix}_{m \times n}$$

EX 1: $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}; B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$

$$\to AB = \begin{pmatrix} 1\times 4 + 3\times 2 + 5\times 1 & 1\times 3 + 3\times 1 + 5\times 0 \\ 2\times 4 + 4\times 2 + 6\times 1 & 2\times 3 + 4\times 1 + 6\times 0 \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 22 & 10 \end{pmatrix}.$$

EX 2: System of m linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$\to Ax = b$, where $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

EX 3:

$$x_1 + 2x_2 = 1;$$
$$x_2 = 0$$

$$\to \begin{matrix} 1\times x_1 + 2\times x_2 = 1 \\ 0\times x_1 + 1\times x_2 = 0 \end{matrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Some Caution in Matrix Operations:

- AB may not be equal to BA (Commutative Law does not hold.)

  EX: $A=\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \rightarrow AB \neq BA$.

- But the distributive law holds:
  - A(B+C) = AB + AC, if AB and AC are computable.
  - (B+C)A = BA + CA, if BA and CA are computable.

- $AB = AC$ does not mean $B = C$.

  EX: $A=\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$, $C=\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \rightarrow AB = AC$.

- $AD = 0$ (zero matrix) does not mean that $A = 0$ or $D = 0$.

  EX: $A=\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$, $D=\begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix} \rightarrow AD=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Theorem:

$(AB)^t = B^t A^t$.

$(A+B)^t = A^t + B^t$.

Some Special Matrices

(1) **Identity Matrix**

Let I_n be an n×n square matrix. I_n is called an identity matrix if all of the diagonal entries are ones and all of the off-diagonals are zeros.

EX: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Note:

$$A_{m\times n} I_n = A_{m\times n}; I_m A_{m\times n} = A_{m\times n}.$$

(2) **Zero (Null) Matrix**

Let $A = [a_{ij}]_{m\times n}$. If $a_{ij} = 0$ for all i and j, A is called a zero matrix.

EX: $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ A is a zero matrix, but not B.

(3) **Scalar Matrix**

For any scalar λ, $\lambda I_n = \begin{pmatrix} \lambda & 0 & \ldots & 0 \\ 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda \end{pmatrix}$ is called a scalar matrix.

(4) **Diagonal Matrix**

$$diag(\lambda_1, \lambda_2, ..., \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ : & : & & : \\ 0 & 0 & ... & \lambda_n \end{pmatrix}.$$

(5) **Triangular Matrix**

A square matrix $A_{n\times n} = [a_{ij}]$ is called an upper (lower) triangular if $a_{ij} = 0$ for all i < j (i > j).

EX: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ (upper); $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$ (lower)

$C = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 6 & 4 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ (not triangular)

(6) **Idempotent Matrix**

A matrix $A_{n\times n}$ is said to be idempotent iff $AA = A$.

EX: $\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}; \begin{pmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{pmatrix}.$

[3] Inverse and Determinant

Definition: **Inverse**

For $A_{n\times n}$ and $B_{n\times n}$, B is called the inverse of A iff AB = I_n or BA = I_n.

The inverse of A is denoted by A^{-1}.

EX: $A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}; B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \rightarrow AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow B = A^{-1}$.

Theorem:

$$A_{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $ad - bc = 0$, then, A^{-1} does not exist.

Terminology:

If a square matrix A has an inverse, A is said to be "**invertible**" or "**nonsingular**". If A does not have an inverse, A is said to be "singular."

Theorem:

1) For $A_{n\times n}$ and $B_{n\times n}$, if AB = I_n, then, BA = I_n.

2) A^{-1} is unique if it exists.

3) $(AB)^{-1} = B^{-1}A^{-1}$ if both A and B are invertible and conformable.

4) $(A^{-1})^{-1} = A$.

5) $(A^t)^{-1} = (A^{-1})^t$.

Proof:

2) Suppose that AB = AC = I_n. B = BI_n = BAC = I_nC = C.

3) $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$.

4) $A^{-1}A = I_n \rightarrow (A^{-1})^{-1} = A$.

5) $A^t(A^{-1})^t = (A^{-1}A)^t = I_n^t = I_n$. $\rightarrow$ $(A^{-1})^t$ is the inverse of A^t.

EX:

- A system of linear equations is given $A_{m\times n}x_{n\times 1} = b_{m\times 1}$.
- If m = n and A is invertible,

$$A^{-1}A\overline{x} = A^{-1}b \rightarrow I_n\overline{x} = A^{-1}b \rightarrow \overline{x} = A^{-1}b.$$

Question:

How can we find an inverse if n > 2?

Definition: **Determinant of 2×2 Matrix**

Let $A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then, $|A| \equiv \det(A) = a_{11}a_{22} - a_{12}a_{21}$.

EX: $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \rightarrow \det(A) = 2 \times 4 - 1 \times 3 = 5$.

Definition: **Determinant of 3×3 Matrix**

Let $A_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Write

$$\begin{matrix} a_{11} & a_{12} & a_{13} & : & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & : & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & : & a_{31} & a_{32} & a_{33} \end{matrix}$$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

EX: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 3 & 4 \end{pmatrix} \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 3 & 4 \end{matrix} \rightarrow$ $\det(A) = 1 \bullet 5 \bullet 4 + 2 \bullet 1 \bullet 1 + 3 \bullet 4 \bullet 3$ $- 1 \bullet 1 \bullet 3 - 2 \bullet 4 \bullet 4 - 3 \bullet 5 \bullet 1 = 8$

Definition: **Minor and Cofactor**

Let $A_{n\times n}=[a_{ij}]$. Then, the minor of $a_{ij}(\equiv M_{ij})$ is the (n-1)×(n-1) matrix excluding the i$^{\text{th}}$ row and the j$^{\text{th}}$ column of A. The cofactor of a_{ij} $(\equiv|C_{ij}|)$ is $(-1)^{i+j}\det(M_{ij})$.

Definition: **Determinant of n×n Matrix (Laplace Expansion)**

For $A_{n\times n}$,

$$\det(A)=\Sigma_{j=1}^{n}a_{ij}\,|C_{ij}|=a_{i1}\,|C_{i1}|+a_{i2}\,|C_{i2}|+\ldots+a_{in}\,|C_{in}|,$$

for any choice of i. Also,

$$\det(A)=\Sigma_{i=1}^{n}a_{ij}\,|C_{ij}|=a_{1j}\,|C_{1j}|+a_{2j}\,|C_{2j}|+\ldots+a_{nj}\,|C_{nj}|,$$

for any choice of j.

EX 1: $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 1\\ 1 & 3 & 4\end{pmatrix}$.

- Choose the first row:

$$a_{11}=1:|M_{11}|=\begin{vmatrix}5 & 1\\ 3 & 4\end{vmatrix}=17\rightarrow|C_{11}|=(-1)^{1+1}\,|M_{11}|=17.$$

$$a_{12}=2:|M_{12}|=\begin{vmatrix}4 & 1\\ 1 & 4\end{vmatrix}=15\rightarrow|C_{12}|=(-1)^{1+2}\,|M_{12}|=-15$$

$$a_{13}=3;|C_{13}|=7.$$

$$\det(A)=a_{11}\,|C_{11}|+a_{12}\,|C_{12}|+a_{13}\,|C_{13}|=1\times17+2\times(-15)+3\times7=8.$$

EX 2: $A=\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 1 \\ 3 & 0 & 0 & 0 \\ 4 & 3 & 1 & 4 \end{pmatrix} \rightarrow \det(A)=3(-1)^{3+1}\begin{vmatrix} 2 & 1 & 2 \\ 6 & 1 & 1 \\ 3 & 1 & 4 \end{vmatrix}$.

Theorem:

$\det(I_n)=1$.

Theorem:

If all of the entries in the i[th] row (j[th] column) of $A_{n\times n}$ are zero, then, $\det(A)=0$.

Theorem:

If $A_{n\times n}=[a_{ij}]$ is a triangular matrix, $\det(A)=a_{11}a_{22}\bullet\bullet\bullet\bullet\bullet a_{nn}$.

EX: $A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$; $B=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix} \rightarrow \begin{matrix} \det(A)=1\times 4\times 6=24; \\ \det(B)=1\times 3\times 6=18 \end{matrix}$.

Theorem: **Elementary Row (Column) Operations**

(a) If multiplying a single row (column) of A by a constant k results in B, $\det(B) = k \times \det(A)$.

(b) If adding a multiple of one row (column) of A to another row (column) results in B, $\det(B) = \det(A)$.

(c) If B results when two rows (columns) of A are interchanged, then, $\det(B) = -\det(A)$.

EX.a:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & somthing & & \end{pmatrix}; B = \begin{pmatrix} 2 & 4 & 6 & 8 \\ & same\ as\ A & & \end{pmatrix} \to \det(B) = 2 \times \det(A).$$

$$A = \begin{pmatrix} 1 & \\ 2 & \\ 3 & something \\ 4 & \end{pmatrix}; B = \begin{pmatrix} 2 & \\ 4 & \\ 6 & same\ as\ A \\ 8 & \end{pmatrix} \to \det(B) = 2 \times \det(A).$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}; B = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

$$\to \det(B) = 2 \times \begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 2 \times 2 \times \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4 \times \det(A).$$

EX.b:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ & \textit{something} & & \end{pmatrix}; B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ & \textit{same as } A & & \end{pmatrix} \rightarrow \det(B) = \det(A).$$

- r2 of B = r2 of A – r1 of A = (-1)×r1 of A + r2 of A.

$$A = \begin{pmatrix} 1 & 2 & \\ 2 & 3 & \\ 3 & 4 & \textit{somthing} \\ 4 & 5 & \end{pmatrix}; B = \begin{pmatrix} 1 & 1 & \\ 2 & 1 & \\ 3 & 1 & \textit{something} \\ 4 & 1 & \end{pmatrix} \rightarrow \det(B) = \det(A).$$

EX.c:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ & \textit{something} & \end{pmatrix}; B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ & \textit{same as } A & \end{pmatrix} \rightarrow \det(B) = -\det(A).$$

EX:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix} \rightarrow \det(A) = -2 \text{ (Check this!)}$$

$$B = \begin{pmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix} \text{ (r1 of B = 4×r1 of A)} \rightarrow \det(B) = 4 \times (-2) = -8.$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{pmatrix} \text{ (r2 of B = r2 of A – 2×r1 of A)}$$

$$\rightarrow \det(C) = -2 .$$

$$D = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \text{ (r1 and r2 of A are interchanged)}$$

$$\rightarrow \det(B) = (-1) \times (-2) = 2 .$$

EX: $C = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 6 & 4 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ (not triangular)

$$\det(C) = (-1) \times \begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 6 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-1) \times (-1) \times \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3.$$

Theorem:

Suppose that two rows (columns) of $A_{n\times n}$ are identical. Then, $\det(A) = 0$.

Proof:

Suppose that r1 = r2. If you subtract r1 from r2, the second row becomes zero. Thus, the determinant of this new matrix should be zero. Since this transformation does not alter the determinant of A, it must be the case that $\det(A) = 0$.

Theorem: **Expansion Using Alien Cofactors**

For $A_{n\times n}$,

$$\Sigma_{j=1}^{n} a_{hj} |C_{ij}| = a_{h1}|C_{i1}| + a_{h2}|C_{i2}| + ... + a_{hn}|C_{in}| = 0,$$

for any choice of $h \neq i$. Also, for any choice of $h \neq j$,

$$\Sigma_{i=1}^{n} a_{ih} |C_{ij}| = a_{1h}|C_{1j}| + a_{2h}|C_{2j}| + ... + a_{nh}|C_{nj}| = 0.$$

Proof:

- Consider a matrix, $B_{n\times n}$, which is identical to A except that the i^th row of A is replaced by the h^th row of A.
- Since the two rows of B are identical, $\det(B) = 0$.
- The Laplace extension of B using the i^th row of B will give us:

$$\Sigma_{j=1}^{n} a_{hj} |C_{ij}| = a_{h1}|C_{i1}| + a_{h2}|C_{i2}| + ... + a_{hn}|C_{in}| = \det(B) = 0.$$

EX: $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 0 & 1 \\ 2 & 4 & 5 & 7 \\ 0 & 2 & 3 & 4 \end{pmatrix}; B = \begin{pmatrix} 3 & 2 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & 4 & 5 & 7 \\ 0 & 2 & 3 & 4 \end{pmatrix} \to \det(B) = 0$.

- For A and B, the cofactors corresponding to the first row entries (say, $|C_{11}|, |C_{12}|, |C_{13}|, |C_{14}|$) are the same.
- $a_{21}|C_{11}| + a_{22}|C_{12}| + a_{23}|C_{13}| + a_{24}|C_{14}| = \det(B) = 0$

Theorem:

For any $A_{n\times n}$ and $B_{n\times n}$, $\det(AB) = \det(A) \times \det(B)$.

EX: $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}; B = \begin{pmatrix} -1 & 3 \\ 5 & 8 \end{pmatrix} \to \det(A) = 1; \det(B) = -23$.

$AB = \begin{pmatrix} 2 & 17 \\ 3 & 14 \end{pmatrix} \to \det(AB) = 28 - 51 = -23 = 1 \times (-23)$.

Note: $\det(A+B) \neq \det(A) + \det(B)$, in general.

Note: $\det(A^{-1}) = 1/\det(A)$.

Theorem:

$A_{n\times n}$ is nonsingular iff $\det(A) \neq 0$.

EX: $A=\begin{pmatrix}1 & 2 & 3\\ 1 & 0 & 1\\ 2 & 4 & 6\end{pmatrix}$ (-2×r1+r3) → $\det(A)=\begin{vmatrix}1 & 2 & 3\\ 1 & 0 & 1\\ 0 & 0 & 0\end{vmatrix}=0$.

A is singular.

<u>Inverse of some special matrices</u>

(1) $$\begin{pmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \dots & \lambda_n\end{pmatrix}^{-1}=\begin{pmatrix}1/\lambda_1 & 0 & \dots & 0\\ 0 & 1/\lambda_2 & \dots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \dots & 1/\lambda_n\end{pmatrix}.$$

(2) $$\begin{pmatrix}A_{p\times p} & 0_{p\times q}\\ 0_{q\times p} & B_{q\times q}\end{pmatrix}^{-1}=\begin{pmatrix}A^{-1} & 0_{p\times q}\\ 0_{q\times p} & B^{-1}\end{pmatrix}.$$

EX: $$\begin{pmatrix}5 & 7 & 0\\ 7 & 10 & 0\\ 0 & 0 & 3\end{pmatrix}^{-1}=\begin{pmatrix}\begin{pmatrix}5 & 7\\ 7 & 10\end{pmatrix} & \begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}0 & 0\end{pmatrix} & 3\end{pmatrix}^{-1}=\begin{pmatrix}\begin{pmatrix}5 & 7\\ 7 & 10\end{pmatrix}^{-1} & \begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}0 & 0\end{pmatrix} & 1/3\end{pmatrix}$$
$$=\begin{pmatrix}10 & -7 & 0\\ -7 & 5 & 0\\ 0 & 0 & 1/3\end{pmatrix}$$

Definition: **Cofactor and Adjoint Matrices**

For $A_{n\times n}$, the cofactor and adjoint matrices are defined as

$$Cof(A)=\begin{pmatrix} |C_{11}| & |C_{12}| & \dots & |C_{1n}| \\ |C_{21}| & |C_{22}| & \dots & |C_{2n}| \\ \vdots & \vdots & & \vdots \\ |C_{n1}| & |C_{n2}| & \dots & |C_{nn}| \end{pmatrix};\ Adj(A)=[Cof(A)]^t.$$

Theorem:

Suppose that $A_{n\times n}$ is invertible, then,

$$A^{-1}=\frac{1}{\det(A)}Adj(A).$$

Proof:

$$A\times adj(A)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \vdots & \vdots & & \vdots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{pmatrix}$$

$$=\begin{pmatrix} \det(A) & 0 & \dots & 0 \\ 0 & \det(A) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \det(A) \end{pmatrix}$$

Note: A^{-1} exists iff $\det(A)\neq 0$.

EX:

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}.$$

$\det(A) = 64$.

$|C_{11}| = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12; |C_{12}| = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6; |C_{13}| = (-1)^{1+3} \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -16;$

$|C_{21}| = 4; |C_{22}| = 2; |C_{23}| = 16;$

$|C_{31}| = 12; |C_{32}| = -10; |C_{33}| = 16.$

$$A^{-1} = \frac{1}{64} \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}.$$

Note:

There is an alternative way to compute inverses using elementary row (column) operations. See Ch. 1.3 of Turkington.

[4] Cramer's Rule

- Consider a system of equations, $Ax = b$, where $A_{n\times n}$, $x_{n\times 1}$ and $b_{n\times 1}$.
  - n equations and n unknowns:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

  - Can we solve for x?
    - If A is invertible.
    - $\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix} = A^{-1}b$.

- Cramer's Rule:

  Define $A_j = \left(a_{\bullet 1}, a_{\bullet 2}, \ldots, a_{\bullet j-1}, b, a_{\bullet j+1}, \ldots, a_{\bullet n}\right)$. Then,

$$\bar{x}_j = \frac{\det(A_j)}{\det(A)}.$$

Proof:

$$\bar{x} = A^{-1}b = \frac{1}{\det(A)}\begin{pmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ : & : & & : \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ : \\ b_n \end{pmatrix}$$

$$= \frac{1}{\det(A)}\begin{pmatrix} b_1|C_{11}| + b_2|C_{21}| + \dots + b_n|C_{n1}| \\ b_1|C_{12}| + b_2|C_{22}| + \dots + b_n|C_{n2}| \\ : \\ b_1|C_{1n}| + b_2|C_{2n}| + \dots + b_n|C_{nn}| \end{pmatrix} = \frac{1}{\det(A)}\begin{pmatrix} \det(A_1) \\ \det(A_2) \\ : \\ \det(A_3) \end{pmatrix}$$

Why?

- Observe that $|C_{11}|$, $|C_{21}|$, ... , $|C_{n1}|$ are the cofactors of A corresponding to its first column entries.
- $\det(A_1) = b_1|C_{11}| + b_2|C_{21}| + \dots + b_n|C_{n1}|$.

EX: $Q_d = a + bP; Q_s = c + dP; Q_d = Q_s$.

$$\begin{matrix} \bar{Q} = a + b\bar{P} \to \bar{Q} - b\bar{P} = a \\ \bar{Q} = c + d\bar{P} \to \bar{Q} - d\bar{P} = c \end{matrix} \to \begin{pmatrix} 1 & -b \\ 1 & -d \end{pmatrix}\begin{pmatrix} \bar{Q} \\ \bar{P} \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

$$\bar{Q} = \frac{\begin{vmatrix} a & -b \\ c & -d \end{vmatrix}}{\begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix}} = \frac{-ad + bc}{-d + b} = \frac{bc - ad}{b - d}; \bar{P} = \frac{\begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix}}{\begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix}} = \frac{c - a}{-d + b} = \frac{a - c}{b - d}.$$

EX: Solve the following simultaneous equations:

$$x_1 + 2x_2 + 3x_3 = 3$$
$$x_1 + 3x_2 + 5x_3 = 0$$
$$x_1 + 5x_2 + 12x_3 = 6$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$$

$$\rightarrow \overline{x}_1 = \frac{\begin{vmatrix} 3 & 2 & 3 \\ 0 & 3 & 5 \\ 6 & 5 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}} = \frac{39}{3} = 13; \ \overline{x}_2 = \frac{\begin{vmatrix} 1 & 3 & 3 \\ 1 & 0 & 5 \\ 1 & 6 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}} = \frac{-33}{3} = -11;$$

$$\overline{x}_3 = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 1 & 5 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}} = \frac{12}{3} = 4.$$