

ONLINE APPENDIX FOR
“DEBT CONTRACTS WITH PARTIAL COMMITMENT”
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OMITTED PROOFS

LEMMA 5: *Function \hat{V} is concave with slope between -1 and 0 .*

PROOF: The fact that $\hat{V}(w)$ is decreasing in w follows from the fact that an increase in w is costly for the lender. When w increases by one unit, the lender can always increase the borrower’s consumption levels by one unit for all s , in which case his value decreases by one unit. He can potentially do better by changing investment and/or continuation values. In addition, if w is such that the promise-keeping constraint does not bind, then $\hat{V}'(w) = 0$. Concavity is a consequence of the possibility of lotteries. \square

PROOF OF CLAIM 1: (i) Autarkic strategies are for the lender to offer zero investment and zero consumption in each period, and for the borrower to reject any offer. Clearly, if $\pi = 0$, this is a subgame perfect equilibrium because if the lender expects the borrower never to repay, his best response is never to invest, and vice versa.

(ii) Let $\pi > 0$, and suppose to the contrary that autarky is an equilibrium. Then it is an equilibrium that delivers the lowest punishment on the lender. Thus the corresponding solution to problem (6)–(10) satisfies $\hat{V}(w) = -w$, and investment is zero in each period. To get a contradiction, I show that whenever w is large enough, the lender can generate a payoff strictly above $-w$.

Since $\hat{V}(w) = -w$, constraint (7) binds for all w , and the lowest payoff to the borrower is $\hat{w}_0 = 0$. Substituting from (7) into the objective function and using the fact that (9) always binds (see the proof of Proposition 1), the maximization problem of the lender can be written as $\max_{K, \{c_s, w_{1s}\}} E[-K + f(K, s) + \beta\pi\hat{V}(w_{1s}) + \beta(1 - \pi)\hat{V}(0)] + \min\{\beta E\pi w_{1s} - w, 0\}$ subject to $c_s + \beta\pi w_{1s} \geq f(K, s)$ for all s . Since $\pi > 0$, the borrower’s participation constraints can be satisfied by setting w_{1s} large enough. Moreover, if w is large enough, $\min\{\beta E\pi w_{1s} - w, 0\} = \beta E\pi w_{1s} - w$. Using $\hat{V}(w) + w = 0$ for all w , the lender’s problem becomes $\max_K -K + Ef(K, s) - w$. The solution is clearly K^* . The lender’s payoff is $-K^* + Ef(K^*, s) - w = (1 - \beta)S^* - w > -w$, a contradiction.

Next, I will show that $v = \hat{V}(0) > 0$. Suppose that $\hat{V}(0) = 0$, and also suppose that the promise-keeping constraint does not bind. Then it is optimal

to set $c_s = 0$ for all s . Expressing w_{1s} from (8), the lender's maximization problem becomes $\max_K E[-K + f(K, s) + \beta\pi\hat{V}(\max\{\hat{w}, [f(K, s)/\beta - (1 - \pi)\hat{w}]/\pi\}) + \beta(1 - \pi)\hat{V}(0)]$. Since $\hat{V}(0) = 0$, $w + \hat{V}(w) > 0$ for w large enough, and \hat{V} is concave, it follows that $E[f(K, s) + \beta\pi\hat{V}(\max\{\hat{w}, [f(K, s)/\beta - (1 - \pi)\hat{w}]/\pi\})]$ is strictly increasing in K . Since $\lim_{K \rightarrow 0} f_K(K, s) = +\infty$ for all $s > 0$, the payoff to the lender is strictly increasing in K for small enough K , and the optimal choice of investment, denoted by \tilde{K} , is strictly positive. Moreover, the lender can obtain a payoff of zero by choosing $K = 0$. Thus $\hat{V}(0) > 0$.

It remains to show that at $w = 0$ the promise-keeping constraint is indeed slack. Using (8), the left-hand side of the promise-keeping constraint is at least $Ef(\tilde{K}, s)$, which strictly exceeds 0. \square

PROOF OF CLAIM 3: The proof is similar to the proof of existence in Kovrijnykh and Szentes (2007) (Proposition 4 in Appendix B).

First, notice that $0 \leq \hat{V}(w) + w \leq S^* < \infty$. Define $\hat{S}(w) = \hat{V}(w) + w$. From Lemma 5, \hat{S} is concave, with slope between 0 and 1.

I will define the set of possible candidates for \hat{S} . Consider the following set of functions: $\Gamma = \{\hat{S}_g | \hat{S}_g \in \mathbb{C}[0, \infty), \hat{S}_g \geq 0, \hat{S}_g \text{ is concave and bounded, and for } \delta > 0, [\hat{S}_g(w + \delta) - \hat{S}_g(w)]/\delta \in [0, 1]\}$. Observe that Γ with the sup norm is a convex compact set.

Next, I will define a fixed-point operator on Γ . For all $\hat{S}_g \in \Gamma$, let $\hat{V}_g(w) = \hat{S}_g(w) - w$ for all w . Define the operator $\hat{T} : \Gamma \rightarrow \Gamma$ by $\hat{T}\hat{S}_g = \text{cav}(\max\{\hat{T}_0\hat{S}_g, 0\})$, where cav denotes the concavification of a function, and the operator \hat{T}_0 is defined as follows:

$$\begin{aligned} \hat{T}_0\hat{S}_g(w) &= w + \max_{K, \{c_s, w_{0s}, w_{1s}\}_{s \in \mathcal{S}}} -K + E[f(K, s) - c_s + \beta(\pi\hat{V}_g(w_{1s}) \\ &\quad + (1 - \pi)\hat{V}_g(w_{0s}))] \\ \text{s.t.} \quad &E[c_s + \beta(\pi w_{1s} + (1 - \pi)w_{0s})] \geq w, \\ &c_s + \beta(\pi w_{1s} + (1 - \pi)w_{0s}) \geq f(K, s) \text{ for all } s \in \mathcal{S}, \\ &\hat{V}_g(w_{0s}) \geq \hat{V}_g(0) \text{ for all } s \in \mathcal{S}, \\ &c_s \geq 0, w_{1s} \geq 0, w_{0s} \geq 0 \text{ for all } s \in \mathcal{S}. \end{aligned}$$

To see that \hat{T} indeed maps into Γ , note that the fact that $\hat{T}\hat{S}_g$ is continuous with slope between zero and one follows from Lemma 5. The concavity of $\hat{T}\hat{S}_g$ and $\hat{T}\hat{S}_g \geq 0$ follow immediately from construction. Next, I will show

that (a) the operator \hat{T} has a fixed point, and (b) there is a bijection between fixed points and punishment equilibria.

(a): I apply Schauder's fixed-point theorem to the operator \hat{T} . Notice that Γ is a convex compact set. The operator \hat{T} is continuous with respect to the sup norm because the operator \hat{T}_0 is continuous.

(b): If $\hat{S}_g = \hat{S}$, where $\hat{V}(w) = \hat{S}(w) - w$ is the value to the lender in a punishment equilibrium, then \hat{S}_g is obviously a fixed point of \hat{T} . If \hat{S}_g is a fixed point of \hat{T} , then it follows from the proof of Proposition 9.2 in Stokey and Lucas (1989) that $\hat{V}(w) = \hat{S}_g(w) - w$ corresponds to a punishment equilibrium. \square

PROOF OF LEMMA 1: Let $\pi < 1$, and suppose that π increases marginally. Let $\hat{\lambda}$ and $p_s \hat{\mu}_s$ denote the Lagrange multipliers on constraints (7) and (8), respectively. The current period's change in the value to the lender in response to the increase in π is $\Delta(w) = \beta E[\hat{V}(\hat{w}_{1s}) - \hat{V}(\hat{w}_{0s}) + (\hat{\lambda} + \hat{\mu}_s)(\hat{w}_{1s} - \hat{w}_{0s})]$, where \hat{w}_{1s} and \hat{w}_{0s} denote the optimal choices of continuation values. (From the proof of Proposition 1, $\hat{w}_{0s} = \underline{w}$ and $\hat{V}(\hat{w}_{0s}) = \hat{V}(0)$.) Notice that \hat{w}_{1s} maximizes $\beta E[\hat{V}(\hat{w}_{1s}) + (\hat{\lambda} + \hat{\mu}_s)\hat{w}_{1s}]$. Thus $\Delta(w) = \beta E[\hat{V}(\hat{w}_{1s}) + (\hat{\lambda} + \hat{\mu}_s)\hat{w}_{1s}] - \beta E[\hat{V}(\hat{w}_{0s}) + (\hat{\lambda} + \hat{\mu}_s)\hat{w}_{0s}] \geq 0$, with strict inequality if (9) binds so that \hat{w}_{1s} is different from \hat{w}_{0s} . I want to show that when w is high enough so that the borrower consumes, $\Delta(w) > 0$.

Let $\partial\hat{V}(w)$ denote the superdifferential of the function \hat{V} at w . The first-order condition with respect to c_s , with complementary slackness, is $-1 + \hat{\lambda} + \hat{\mu}_s \leq 0$, $c_s \geq 0$. The first-order condition with respect to w_{1s} (ignoring constraints $w_{xs} \geq 0$, which, as I will show, indeed never bind) is $-(\hat{\lambda} + \hat{\mu}_s) \in \partial\hat{V}(\hat{w}_{1s})$, and the Envelope condition is $-\hat{\lambda} \in \partial\hat{V}(w)$. The first-order condition with respect to K is $1 \in Ef_K(\hat{K}, s)(1 - \hat{\mu}_s)$.

First, notice that since \hat{V} is concave, it is optimal for the lender to set w_{1s} equal to w whenever possible. If setting $w_{1s} = w$ violates (8), then $w_{1s} > w$. (The proof is analogous to the proof of Lemma 3.) Thus $w_{1s} \geq w$ for all s , so indeed $w_{1s} \geq 0$ never binds. Also, consumption in state s is zero unless $-1 \in \partial\hat{V}(\hat{w}_{1s})$. Moreover, notice that if $\hat{V}'(w) = -1$, then since $\hat{w}_{1s} \geq w$, the concavity of \hat{V} implies $\hat{V}'(\hat{w}_{1s}) = -1$ for all s . Since $\hat{V}'(\hat{w}_{1s}) = -\hat{\lambda} - \hat{\mu}_s = -1 = \hat{V}'(w) = -\hat{\lambda}$, it follows that $\hat{\mu}_s = 0$ for all s . Then from the first-order condition with respect to K , the optimal investment, denoted by \hat{K} , equals K^* . In addition, at w such that $\hat{V}'(w) = -1$, $\Delta(w) = \beta E[\hat{V}(\hat{w}_{1s}) + \hat{w}_{1s}] - \beta[\hat{V}(0) + \underline{w}]$. Since \hat{V} is concave, $\Delta(w) > 0$ unless \hat{V} is linear with slope -1 for $w \geq \underline{w}$. It remains to show that indeed \hat{V} cannot

have slope -1 for $w \geq \hat{w}$.

There are two scenarios that one needs to rule out: (a) $\hat{w} = 0$ and \hat{V} is linear with slope -1 for $w \geq 0$, and (b) $\hat{w} > 0$, and \hat{V} is constant on $[0, \hat{w}]$ and is linear with slope -1 for $w > \hat{w}$.

Consider scenario (a) first. If $\pi > 0$, then $\hat{V}(0) > 0$ by Claim 1. Since $\hat{V}'(w) = -1$ for all $w \geq 0$, the above reasoning implies that $\hat{K} = K^*$ for all $w \geq 0$. From (8) with $w_{0s} = 0$, obtain $w_{1s} \geq [f(K^*, s) - c_s]/(\beta\pi)$. Substituting this into the objective function and using $\hat{V}(w) = \hat{V}(0) - w$, obtain $\hat{V}(0) \leq -K^* + E[f(K^*, s) - c_s] + \beta\pi(\hat{V}(0) - E[f(K^*, s) - c_s]/(\beta\pi)) + \beta(1 - \pi)\hat{V}(0)$. Rearranging terms, this expression becomes $(1 - \beta)\hat{V}(0) \leq -K^* < 0$, a contradiction with $\hat{V}(0) > 0$.

Now consider scenario (b). In this case again $\hat{K} = K^*$ for $w \geq \hat{w}$. Moreover, for $w \in [0, \hat{w})$, the value to the lender is the same as at $w = \hat{w}$, but the promise-keeping constraint (7) holds with strict inequality. Then at $w = 0$, $\hat{V}(0) = \max_K -K + Ef(K, s) + \beta\pi(\hat{V}(0) - E[\max\{\hat{w}, f(K, s)/(\beta\pi) - \hat{w}(1 - \pi)/\pi\}]) + \beta(1 - \pi)\hat{V}(0)$. For some s , $w_{1s} = \max\{\hat{w}, f(K, s)/(\beta\pi) - \hat{w}(1 - \pi)/\pi\} = f(K, s)/(\beta\pi) - \hat{w}(1 - \pi)/\pi > \hat{w}$ (otherwise the borrower never consumes). Then it is optimal for the lender to invest $K < K^*$, and earn profits strictly higher than $\hat{V}(\hat{w})$. Thus $\hat{V}(0) > \hat{V}(\hat{w})$, contradicting the properties of scenario (b).

I have shown that the per-period change in the lender's value is $\Delta(w) \geq 0$, with strict inequality (at least) when w is such that the borrower consumes in that period at least in some states. The total change in the lender's value is the expected present discounted value of changes in the current and all future periods. Since starting from any promised value, the borrower must consume in the future with probability one (for otherwise she would find it profitable to deviate), the total change is strictly positive. \square

PROOF OF LEMMA 3: From (A5) and (A6), so long as $\gamma = 0$, $w_{1s} = w_{0s} = w_s$ for all s . While this is the case, suppose s is such that choosing $w_s = w$ satisfies (3), so that $\mu_s = 0$. Then from (A5) and (A7), this choice is optimal. For s such that choosing $w_s = w$ would violate (3), a higher level of w_s must be chosen. For w high enough, setting $w_{0s} = w_{1s}$ for all s would violate (4), and hence $w_{0s} = w_v$. Notice that since $w \geq 0$, given this optimal choice of continuation values, constraints $w_{xs} \geq 0$ indeed do not bind. \square

PROOF OF CLAIM 4: The proof is by induction. First notice that $T(V^*) \leq V^*$. In addition, suppose that $V_2 \leq V_1$. Then the constraint set in $T(V_1)$ is at least as large as that in $T(V_2)$, and therefore $T(V_2) \leq T(V_1)$. Applying

this argument to $V_2 = T^n(V^*)$ and $V_1 = T^{n-1}(V^*)$, obtain the induction step: if $T^n(V^*) \leq T^{n-1}(V^*)$ then $T^{n+1}(V^*) \leq T^n(V^*)$. Hence $T^n(V^*)$ is a decreasing sequence and therefore must converge point wise to some limit \bar{V} . By continuity, \bar{V} is a fixed point of T . Moreover, by the above induction argument, $V \leq V^*$ implies $T^n(V) \leq T^n(V^*)$ for all $n \geq 1$ and hence $V = \lim_{n \rightarrow \infty} T^n(V) \leq \lim_{n \rightarrow \infty} T^n(V^*) = \bar{V}$. By the definition of V as the constrained Pareto frontier, $V \geq \bar{V}$. Thus $\bar{V} = V$. \square

REFERENCES

Stokey, Nancy, and Robert Lucas, with Edward Prescott. 1989. *Recursive Methods in Economic Dynamics*. Cambridge, MA: Harvard University Press.