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Debt Contracts with Partial Commitment†

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This paper analyzes a dynamic lending relationship where the borrower cannot be forced to make repayments, and the lender offers long-term contracts that are imperfectly enforced and repeatedly renegotiated. No commitment and full commitment by the lender are special cases of this model where the probability of enforcement equals zero and one, respectively. I show that an increase in the degree of enforcement can lower social welfare. Furthermore, properties of equilibrium investment dynamics with partial commitment drastically differ from those with full and no commitment. In particular, investment is positively related to cash flow, consistent with empirical findings. (*JEL* D82, D86, G21)

Consider a situation where an agent, the borrower, can operate a stochastic production technology, but does not have access to capital that is needed to operate the technology. Another agent, the lender, has capital, but cannot operate the technology. The two parties enter into a repeated relationship where they jointly generate and share surplus. In such a relationship, lack of commitment can cause inefficiencies: if the lender expects the borrower not to repay in the future, then he might not invest in the borrower's technology. Similarly, if the borrower expects the lender not to invest in the future, then she might not repay.

Most studies on dynamic lending with limited commitment by the borrower make one of two extreme assumptions about the lender's ability to commit: either the lender has full commitment power or cannot commit at all.[1] Looking at these extreme cases of commitment, the literature has concluded that more commitment is better. (Since an allocation that can be implemented without commitment can also be implemented with full commitment, social welfare with full commitment is at least as high as without commitment.)

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[1] Full commitment means that at the beginning of a relationship the lender is able to commit to investment decisions for all future periods and all possible histories. In the no-commitment case, the lender chooses investment in each period to maximize his present discounted utility; that is, all agreements must be self-enforcing.

Although the two extremes are natural benchmarks, they are not necessarily very informative about reality: in practice, lending is better described as a situation where there is some but not full commitment to contracts. Indeed, lending relationships are governed by formal contracts, but the terms of these contracts are sometimes renegotiated.

For example, Roberts and Sufi (2009a) use a sample of 1,000 private credit agreements between financial institutions and publicly listed US firms from 1996 to 2005, and find that loan contracts are frequently renegotiated. Unconditionally, 75 percent of contracts are renegotiated before the stated maturity date, and this figure increases to over 90 and 96 percent if the stated maturity is in excess of one and three years, respectively. Moreover, on average loans get renegotiated after about half of the stated maturity has elapsed, with the majority of loans having a duration that is between 25 and 50 percent that of the stated maturity. Likewise, in another sample Roberts (2012) finds that a typical loan is renegotiated every eight months, or four times during the life of the contract.

According to Roberts and Sufi, the primary determinants of renegotiation and its outcomes are: (i) the accrual of new information about the credit quality and outside options (e.g., borrowers' assets, financial leverage, collateral, credit risk, investment opportunities, alternative sources of financing), as well as (ii) fluctuations in credit and equity market conditions and lenders' financial health.

In cases when borrowers' characteristics deteriorate due to the above reasons, renegotiations are linked to covenant violations (which either have occurred or are anticipated).[2] Roberts and Sufi (2009a) show that in cases when loan renegotiations result in an unfavorable outcome for a borrower (as in this paper), 21.4 percent of borrowers report a covenant violation in the year preceding the renegotiation.[3] Covenant violations in private debt contracts are only one example of what causes renegotiation, and there are other situations that one can envision where a lender has the option of not fulfilling contract terms.

Given that renegotiations are so common, the objective of this paper is to understand their economic function and effects. In this paper, I abstract from what causes renegotiations. In my model they emerge exogenously because parties cannot fully commit to contracts, or, in other words, because contracts are not perfectly enforced.[4]

I analyze how the degree of commitment affects allocations and social welfare. I find that if a borrower cannot commit to repay a lender, then having more commitment by the lender can be worse for social welfare than having less commitment, in particular, than having no commitment at all.[5] In addition, the long-run investment dynamics with partial commitment dramatically differ from the dynamics in the extreme cases of full and no commitment. Specifically, investment fluctuates over time and is positively related to cash flow, i.e., the output of the project.

[2] While provisions in loan contracts give creditors the right to accelerate outstanding amounts in response to a covenant violation, in most cases they choose to waive the violation and renegotiate contract terms instead—see, e.g., Beneish and Press (1995) and Chen and Wei (1993). See also Chava and Roberts (2008) and Roberts and Sufi (2009b) who argue that covenant violations are critical to providing creditors the ability to change contract terms.

[3] The authors note that in anticipation of a covenant violation, the parties may renegotiate the contract to prevent a violation from occurring. Therefore renegotiations might not be accompanied by actual covenant violations.

[4] Since one can think of contract enforcement as a commitment device, I will use the words "commitment" and "enforcement" interchangeably.

[5] It will be clear that the lack of commitment by the borrower is essential: if the borrower could commit, her credible promise to repay would give the lender incentives to invest the socially optimal amount in each period, regardless of his own ability to commit.

To establish these results, I consider the following model. In the initial period, the lender offers a contract to the borrower that specifies investments and repayments for all future periods and all possible contingencies. In each period the borrower can default on the contract by walking away with the current output. In addition, in each period before the lender makes the investment, there is a possibility that the contract is voided. In this case the lender is free to choose any level of investment in that period and to offer a new contract. No commitment and full commitment by the lender are special cases of this model where the probability of enforcement is equal to zero and one, respectively.

This simple way of modeling renegotiation affords a tractable recursive formulation of the model.[6] It is not designed to incorporate realistic details of how renegotiations actually happen; instead, my goal is to simply capture the most basic features of renegotiation and trace out its consequences. Thus, the model sidesteps the deeper reasons why provisions such as covenants exist; in reality they presumably help mitigate incentive conflicts between borrowers and lenders. But modeling these conflicts and the details of how various loan provisions facilitate renegotiation would greatly complicate the model. So I opt for a simpler approach of not specifying the exact triggers for renegotiation and suppose that it occurs stochastically.

I derive the main properties of the optimal contract between the lender and the borrower. To see how the probability of enforcement affects welfare, recall that without commitment all agreements must be self-enforcing. In particular, the lender's payoff in the optimal contract must exceed his payoff from the most profitable deviation in all future states. This deviation payoff is the lowest equilibrium payoff that the borrower can inflict on the lender. When the probability of enforcement is greater than zero, such constraints are only imposed in states where a contract is not enforced. Thus, as the probability of enforcement increases, the measure of these states becomes smaller, which leads to a Pareto improvement.

However, the welfare analysis is more complex than that, since the probability of enforcement also impacts the equilibrium punishments. In fact, just as commitment increases the lender's payoff in an optimal equilibrium, it increases his payoff from the most profitable deviation. A deviating lender makes use of his ability to promise investment to the borrower, and he is able to generate higher profits when his promises are more likely to be enforced. This makes it harder to discipline the lender in the optimal equilibrium, which leads to a reduction in welfare.

More commitment can increase or decrease welfare, depending on which of the two effects dominates. While social welfare does not decrease with more commitment for all model parameters, I prove that this is true for an interval of discount factors. Specifically, I first show that for any probability of enforcement strictly below one, there exists a threshold of the discount factor such that, in any optimal equilibrium, the first-best outcome is achieved over time if and only if the discount factor is above this threshold.[7] I then prove that this threshold is strictly increasing in

[6] Note that the concept of renegotiation employed in this paper is very different, e.g., from renegotiation-proofness—an equilibrium refinement where equilibrium payoffs are required to stay on the Pareto frontier in any subgame, even after a deviation.

[7] This result is an extension of the result established in Thomas and Worrall (1994) for the no-commitment case. It means that part of the Pareto frontier coincides with the first-best frontier. Starting from any point on the frontier, the payoffs reach this part of the frontier with probability one.

the probability of enforcement. That is, if enforcement is more likely (but less than perfect), then the agents need to value the future more in order to achieve the first best over time. Moreover, suppose that the discount factor is such that the first best can be sustained with a certain level of enforcement but not with a higher level of it. Then the entire Pareto frontier with this higher level of enforcement is below the Pareto frontier with the lower level of enforcement.

Numerical computations indicate that social welfare as a function of the probability of enforcement is U-shaped, and only starts to increase with enforcement for relatively high levels of enforcement. In other words, unless the contracts are enforced with a high enough probability, it is better not to enforce them at all.

Even though the above result is established in a specific model, the underlying intuition applies more generally. As long as agents cannot fully commit to future decisions, e.g., when contractual enforcement is imperfect, relationships have to rely on self-enforcement. Having a formal enforcement mechanism can make self-enforcement harder to sustain, because it helps agents to generate surplus not only on but also off the equilibrium path. In other words, to have a commitment device is not necessarily welfare-improving, because agents cannot commit not to use it off the equilibrium path.

This simple time-inconsistency argument applies to other economic environments with limited commitment. There is, however, an additional interest in considering partial commitment in the lending framework: it crucially affects predictions about economically relevant issues such as investment inefficiencies and firm dynamics.

Specifically, in models with full or no commitment, investment increases over time to its steady state level. Hence the long-run investment is constant and uncorrelated with stochastic income.[8] In my model, when the discount factor is below the aforementioned threshold, equilibrium investment fluctuates between underinvestment and the first-best level of investment. Moreover, equilibrium investment is positively related to output. This result of cash-flow sensitivity of investment is consistent with empirical findings in the corporate finance literature. Thus, even though with full or no commitment equilibrium investment properties are very similar, they turn out to be drastically different in the intermediate case of partial commitment.

Finally, although the interaction modeled here is that between an investor and a firm, the same model can be applied to many other settings of interest as well. Examples include an input supplier and a producer; a manufacturer who must rely on a local retailer to share profits from selling a product; or a multinational corporation that invests directly abroad and faces the risk that the host country will appropriate profits, as in Thomas and Worrall (1994).

The rest of the paper is organized as follows. The next section reviews the related literature. Section II introduces the environment. Section III formulates the problem recursively. Section IV derives properties of the optimal contract, including equilibrium investment dynamics, while Section V establishes the effects of the probability

[8] In the case of full commitment this steady-state level is the first best; without commitment it is the first best if agents are patient enough—see Thomas and Worrall (1994) and Albuquerque and Hopenhayn (2004). In Albuquerque and Hopenhayn (2004) investment does fluctuate, but only because the productivity shock is observed before investment is made. The first-best investment, the level of which depends on the shock, is made in each period in the long run.

of enforcement on welfare. Section VI discusses robustness of the results to the model assumptions, as well as some possible extensions. Section VII concludes.

I. Related Literature

In the vast literature on optimal contracts with limited commitment, the paper that is most closely related to mine in terms of model structure is Thomas and Worrall (1994). Their model is a special case of my model when the enforcement probability equals zero. A similar environment but with full commitment by the lender is analyzed by Albuquerque and Hopenhayn (2004).[9] The model framework also has similarities with the one analyzed in Kovrijnykh and Szentes (2007), but in that paper there are two lenders, they can only commit to one-period contracts, and Markov equilibria are analyzed.

A paper that derives a result similar to mine is Baker, Gibbons, and Murphy (1994). They study a repeated relationship between a firm and a worker, where the worker's hidden action affects the distribution of output. Implicit contracts use a subjective performance measure—output—while explicit contracts can only use a noisy objective measure—a contractible signal imperfectly correlated with output. The authors find that if the objective measure is sufficiently close to perfect, then no implicit contract is feasible. Also, increasing the signal accuracy increases the lowest discount factor for which the first best can be achieved. These results, although obtained in a very different setting, arise due to a similar mechanism as the one in this paper. However, the authors do not analyze optimal punishments for deviations; instead, they assume that if one party deviates, the other party refuses to cooperate (use implicit contracts) forever after, and the parties only continue to use the objective performance measure. In contrast, in my paper implicit contracts are used in the optimal punishment, and their presence plays a crucial role in its characterization. Moreover, one of the contributions of this paper is precisely in characterizing the optimal punishment.

The result that more commitment can decrease welfare also resembles the result in Ligon, Thomas, and Worrall (2000) that introducing self-insurance by storage in a model of mutual informal insurance can either improve or diminish welfare. In a similar fashion, Krueger and Perri (2011) show that in the absence of complete enforcement, introducing public insurance weakens the enforcement mechanism of private insurance contracts.

My work is also related to studies in the fiscal and monetary policy literature that analyze settings where government promises are not always kept. For example, Debortoli and Nunes (2010) analyze an optimal fiscal policy problem and find that properties of taxes with partial ("loose") commitment by the government are different than in the full- and no-commitment settings. They also show that more commitment leads to higher welfare, in contrast with my results.[10]

[9] Other studies that use the assumption of full commitment by the lender include Atkeson (1991), Bulow and Rogoff (1989b), Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007). No commitment is also assumed, e.g., by Kletzer and Wright (2000), Opp (2012), and Sigouin (2003).

[10] The main reason for this is that the authors analyze Markov perfect equilibria, while I study Pareto optimal equilibria. See also Roberds (1987) and Schaumburg and Tambalotti (2007).

Repeated offers are also modeled, e.g., in Rey and Salanie (1990), where a principal and an agent share a basket of goods in each period, and repeatedly renegotiate contracts.[11] The authors find that short-term contracts implement the full-commitment optimum, in stark contrast to my results. The reason is that their contracts have an overlapping structure: if the agent rejects a new offer, the old contract continues being implemented. This eliminates the principal's incentive problem: at the moment of making an offer, the principal's best alternative is to follow what he has committed to before.

Finally, Koeppl (2007) uses an alternative approach to modeling partial enforcement in a risk sharing model à la Kocherlakota (1996), by assuming that agents can invest into a "punishment technology."[12] Formally, an agent's payoff from deviation (his autarkic value) is multiplied by a parameter, decreasing which is costly. In this model, renegotiation never occurs in equilibrium, which, as I argued above, is an important feature of reality. In addition, Koeppl assumes that autarky is an equilibrium, which means that the agents can commit not to use the enforcement technology if one of them deviates. Thus an increase in the probability of enforcement trivially leads to a *decrease* in the payoff from deviation. And it is precisely the absence of this assumption in my model that is responsible for an increase in the probability of enforcement leading to an *endogenous increase* in the lender's payoff from deviation.

II. The Model

A. *Production and Preferences*

There are two risk-neutral agents, a lender and a borrower. The time horizon is infinite, time is discrete, and the agents discount the future according to the discount factor $\beta \in (0,1)$. The borrower can operate a stochastic technology that transforms capital goods into consumption goods. If the amount of capital investment is K, then output, in terms of consumption good, is $Y = f(K,s)$, where s is the realization of a random shock. The shock is i.i.d. over time, taking values $s \in \mathcal{S} = \{0,1,\ldots,n\}$, with p_s denoting the probability of state s. For all s, the function $f(\cdot,s) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, differentiable, and satisfies $f(0,s) = 0$ and the Inada conditions, $\lim_{K\to 0} f_K(K,s) = +\infty$ for all $s > 0$ and $\lim_{K\to+\infty} f_K(K,s) = 0$ for all s. In addition, for all K, the function $f(K,\cdot) : \mathcal{S} \to \mathbb{R}_+$ is strictly increasing and satisfies $f(K,0) = 0$.

The lender has enough capital to invest in production in every period. In addition, the lender can instantaneously transform one unit of capital good into one unit of consumption good and vice versa. This means the lender is indifferent between the two types of goods.[13] Goods completely depreciate every period. Each agent's goal is to maximize the discounted present value of expected consumption.

[11] See also Bulow and Rogoff (1989a) who study repeated debt rescheduling negotiations via Rubinstein bargaining.

[12] See also Krasa and Villamil (2000) and Popov (2009) who analyze optimal contracts in models with limited commitment, where costly enforcement is a decision variable.

[13] Having two different goods ensures that the borrower cannot invest into the technology herself, which simplifies the problem.

B. *Timing and Contracts*

I model partial commitment and renegotiation in the following way. I assume that similar to the full commitment case, the lender offers a contract to the borrower specifying investments and payments for all future periods and all possible contingencies. In each period the borrower can default on the contract by walking away with the current income. The difference with the full commitment case is that in each period before the lender makes investment, with probability $1 - \pi$ the current contract is voided. In this case the lender is free to choose any level of investment in that period and to offer a new contract.

Formally, let $\mathbf{s}_t^\tau = (s_t, s_{t+1}, \ldots, s_\tau)$ denote the history of productivity shocks starting from date t up to date τ. A typical contract is a pair $(c_\tau(\mathbf{s}_t^\tau), K_{1,\tau+1}(\mathbf{s}_t^\tau))_{\tau=t}^{\infty}$ of sequences of nonnegative functions for the borrower's consumption and investment. More generally, a contract is a probability mixture of these pairs. Nonnegativity of consumption reflects limited liability of the borrower.

The timing at any period is formally described below.

(i) In the beginning of each period $t \geq 1$, a contract is in place. Nature makes a draw x_t of a random variable, which equals one with probability $\pi \in [0, 1]$ and zero with probability $1 - \pi$. Period $t = 0$ starts with no contract in place; otherwise the timing is the same, with the initial realization of the enforcement shock x_0 equal to zero.

(ii) If $x_t = 1$, then the contract is enforced and the lender invests $K_{1,t}$ units of capital prescribed by the contract.[14] If $x_t = 0$, then the old contract is voided and the lender is free to choose any level of investment. Let $K_{0,t}$ denote the lender's optimal choice of investment (possibly a lottery) at that point. (The level $K_{0,t}$ might be different from $K_{1,t}$ because the lender might be tempted to invest a lower amount than he originally promised.)

(iii) If $x_t = 0$, the lender offers a new contract to the borrower. The borrower either accepts or rejects it. If she rejects, the relationship is terminated and the game ends.[15] In this case both the borrower and the lender receive zero payoffs.

(iv) Nature draws s_t, and output $Y_t = f(K_{x_t,t}, s_t)$ is realized and simultaneously observed by the agents. The borrower either pays the prescribed $Y_t - c_t$ units of consumption good to the lender, or the relationship is terminated, in which case the borrower receives Y_t, and the lender receives zero.

[14] If the contract is a lottery, then the outcome of the lottery is first observed.

[15] This assumption means that the lender can commit to terminate the relationship if his contract is rejected. Given the lender's limited ability to commit, this might not be a realistic assumption. Without it, the punishment on the borrower becomes endogenous. Specifically, if the borrower rejects an offer on (off) the equilibrium path, then the worst (best) equilibrium for the borrower must be played from the next period onward. While these endogenous punishments considerably complicate the analysis, the effects of partial commitment that I focus on are still present.

Notice that no commitment and full commitment by the lender are special cases of this model, where the probability of enforcement equals zero and one, respectively. Indeed, if $\pi = 0$, the lender's promises are irrelevant since the actual investment choice will be the one that maximizes the lender's present discounted utility at that time. Therefore there are effectively no formal contracts, and all agreements must be self-enforcing.[16] And if $\pi = 1$, an accepted contract is enforced with probability one in each period and is never renegotiated. In this case there is full commitment by the lender. When $\pi \in (0, 1)$, contractual terms are renegotiated over time, and the higher the π, the longer the expected duration of a contract.[17]

One way to interpret the above approach to modeling partial commitment is to think of a judiciary system that engages in rationing via a simple randomization device. When defaulted upon, borrowers go to court and see their requests rejected with probability $1 - \pi$. In other words, courts decide not to hear cases with probability $1 - \pi$. When the court refuses to hear the case, the borrower sees her claim reduced to a certain level. This level ensures the lender a payoff determined as the maximum punishment that the borrower can levy on him by means of equilibrium continuations, as described in the next section.

I analyze the set of Pareto optimal subgame perfect equilibria in the described game. I will refer to this set as the *optimal contract*.[18] Before I turn to the recursive formulation of the problem, I first characterize the benchmark case where commitment problems are absent.

The First Best.—The first-best level of investment maximizes $-K + Ef(K, s)$. The unique solution, K^*, is defined by the following first-order condition: $1 = Ef_K(K^*, s)$. Let S^* denote the first-best social surplus, that is, $S^* = [-K^* + Ef(K^*, s)]/(1 - \beta)$.

III. Recursive Formulation

A. *The Optimal Contract*

To solve for the optimal contract, one maximizes the value to one party subject to delivering at least a certain value to the other party. Following Spear and Srivastava (1987), one can rewrite the sequence problem corresponding to the optimal contract in recursive form, with the promised value to one of the agents as a state variable, and continuation values to this agent as control variables. Essentially, the promised value summarizes the previous history of the play. Let $V(w)$ denote the value to the lender when the value to the borrower is at least w.

An agent has incentives to follow a strategy as long as the payoff from doing so exceeds a payoff from deviating. A Pareto optimal equilibrium is sustained by using

[16] Such agreements are usually referred to as relational or implicit contracts, as opposed to formal or explicit contracts, which parties can commit to.

[17] Modeling renegotiation as exogenous and stochastic is certainly a simplification that completely abstracts from how and why renegotiations occur in reality. I discuss possible extensions that add some features of realism into the way renegotiation is modeled in Section VI.

[18] This is not to be confused with the (formal, or explicit) contracts defined above. Regarding the equilibrium concept, since there are multiple equilibria in this game, there is, of course, no a priori reason to believe that a Pareto optimal equilibrium would be played. While this is certainly a weakness of this technique, that is the standard way relational contracting is approached.

optimal punishments for deviations. The optimal punishment on an agent is a subgame perfect equilibrium that yields her the lowest possible payoff.

In this model, the incentive problem on the side of the borrower is quite simple. The borrower's choice is whether to accept or reject the lender's offer, and whether to make payments prescribed by a contract. She will prefer to accept a contract and to make payments given an output realization Y so long as doing so delivers her at least payoffs from walking away, which equal zero and Y in the two cases, respectively.

Analyzing punishments for deviations by the lender is more complicated. In each period when a contract is voided (or in period $t = 0$, when there is no contract in place), the lender chooses how much to invest and then which new offer to make. Let v denote the value to the lender from the most profitable deviation when making investment, followed by making a new offer. Then the recursive problem corresponding to the optimal contract can be written as follows:[19]

$$V(w) = \max_{K,\{c_s, w_{1s}, w_{0s}\}_{s\in\mathcal{S}}} E[-K + f(K,s) - c_s + \beta(\pi V(w_{1s}) + (1-\pi)V(w_{0s}))] \tag{1}$$

subject to

$$E[c_s + \beta(\pi w_{1s} + (1-\pi)w_{0s})] \geq w, \tag{2}$$

$$c_s + \beta(\pi w_{1s} + (1-\pi)w_{0s}) \geq f(K,s) \quad \text{for all} \quad s \in \mathcal{S}, \tag{3}$$

$$V(w_{0s}) \geq v \quad \text{for all} \quad s \in \mathcal{S}, \tag{4}$$

$$c_s \geq 0,\ w_{1s} \geq 0,\ w_{0s} \geq 0 \quad \text{for all} \quad s \in \mathcal{S}. \tag{5}$$

The control variables are investment K, the borrower's consumption levels c_s, and her continuation values w_{xs}, conditional on the current period's productivity shock realization s and the next period's enforcement shock realization x, given that neither agent has deviated.[20]

Notice that given w_{xs}, the continuation payoffs to the lender are given by the function V. That is, in the optimal contract, the agents' continuation payoffs must lie on the Pareto frontier. The standard argument for this result is that the optimal contract cannot be Pareto dominated after any history: if it were, it would be possible to replace the part of the contract that was dominated, increase payoffs of both parties, and at the same time relax constraints in all previous periods.

Constraint (2) in the above problem is the promise-keeping constraint that guarantees that the equilibrium value to the borrower is at least w. Constraint (3) is the participation constraint of the borrower. Constraint (4) is the lender's incentive constraint; it guarantees that if the contract is not enforced, the lender will prefer making the equilibrium level of investment followed by making the equilibrium offer, to investing some other amount and making any other offer. Constraints in (5) ensure

[19] Since I allow use of lotteries, the expectations in this problem are taken over the randomness of the choice variables as well as over the current period's productivity shock s.

[20] To make a parallel with the previous notation for investment that had subscript 1 or 0, if the promised value to the borrower in period t is w, then the investment level made in period $t+1$ is $K_{x_{t+1},t+1} = K(w_{x_{t+1},s_t}(w))$.

nonnegativity of consumption, and that the borrower receives at least her outside option in the next period.

Let $\underline{w} \equiv \sup\{w | V(w) = V(0)\} = \sup\{w | w \in \arg\max V(w)\}$ be the lowest w at which (2) binds. Then the actual equilibrium payoff to the borrower when her promised value is w equals $\max\{\underline{w}, w\}$. The set of equilibrium payoffs $\{(w, V(w)) | w \geq \underline{w}\}$ constitutes the Pareto frontier, where each point on the frontier corresponds to an optimal equilibrium. (In the initial period, w must also be such that $V(w) \geq v$.)

B. *The Optimal Punishment*

What distinguishes this model from standard models considered in the literature is that the optimal punishment on the lender is nontrivial and is rather complex to derive. One of the contributions of this paper is in providing its characterization using recursive methods. In what follows, I formulate the Bellman equation that corresponds to the worst equilibrium for the lender, and describe some important properties of the resulting value function.

PROPOSITION 1 (Optimal Punishment): *The optimal punishment on the lender is $v = \hat{V}(0)$, where $\hat{V}(w)$ is the lowest subgame perfect equilibrium payoff to the lender given that the borrower's payoff is at least w. The function $\hat{V}$ solves the following recursive problem:*

$$(6) \quad \hat{V}(w) = \max_{K, \{c_s, w_{1s}, w_{0s}\}_{s \in \mathcal{S}}} E[-K + f(K,s) - c_s + \beta(\pi \hat{V}(w_{1s}) + (1 - \pi)\hat{V}(w_{0s}))]$$

subject to

$$(7) \quad E[c_s + \beta(\pi w_{1s} + (1 - \pi) w_{0s})] \geq w,$$

$$(8) \quad c_s + \beta(\pi w_{1s} + (1 - \pi) w_{0s}) \geq f(K,s) \quad \text{for all} \quad s \in \mathcal{S},$$

$$(9) \quad \hat{V}(w_{0s}) \geq \hat{V}(0) \quad \text{for all} \quad s \in \mathcal{S},$$

$$(10) \quad c_s \geq 0, \; w_{1s} \geq 0, \; w_{0s} \geq 0 \quad \text{for all} \quad s \in \mathcal{S}.$$

Notice how the optimal punishment is constructed. Problem (6)–(10) can be obtained from problem (1)–(5) (after substituting $\hat{V}(0)$ for v in the latter) by replacing V with $\hat{V}$ everywhere. Just as in the optimal contract the lender receives the highest possible payoff given the value to the borrower after any history, in the punishment equilibrium he receives the lowest possible payoff. The argument is essentially the same as in the case of the optimal contract: if this was not the case, it would be possible to replace the continuation equilibrium and deliver a lower initial value to the lender.

Another way to see the difference between the punishment equilibrium and the optimal contract is to note that in order to minimize the lender's payoff today, he should not receive a payoff higher than $\hat{V}(0)$ after any future history.[21] But just as

[21] In particular, constraint (9) will always bind as $w_{0s} \geq 0$ and $\hat{V}$ is a decreasing function.

in the optimal contract, the lender can receive a lower payoff in cases when he can commit to it, i.e., when future enforcement shock realizations equal one.

Thus, the recursive representation of the optimal punishment on the lender is also described by a "frontier," except it gives the lowest, and not the highest payoff to the lender given the payoff to the borrower. After every history, the agents' payoffs stay on this "punishment frontier," and the value to the lender returns to the highest point on it whenever a contract is not enforced.

The following claim shows that unless the probability of enforcement is zero, autarky is not an equilibrium, and hence cannot serve as the optimal punishment on the lender.

CLAIM 1:

(*i*) *If* $\pi = 0$, *then autarky is an equilibrium, and* $v = 0$.

(*ii*) *If* $\pi > 0$, *then autarky is not an equilibrium, and* $v > 0$.

If the lender cannot commit, his promises are not credible. Therefore, if the borrower expects the lender never to invest, her best response is never to repay, and vice versa. Hence when $\pi = 0$, autarky is an equilibrium. But as long as $\pi > 0$, autarky is not an equilibrium, and the worst equilibrium for the lender generates strictly positive welfare. Indeed, for autarky to be an equilibrium, the borrower's equilibrium strategy must be to appropriate any output. However, if the lender offers a contract that specifies payments in exchange for a credible promise of future investments, the borrower will find it profitable to accept it. The possibility that the lender's promise will be carried through motivates the borrower to make these payments. This in turn provides the lender with incentives to invest, because he expects to recover his costs through future payments.

Existence.—Standard dynamic programming techniques cannot be used to show existence and uniqueness of the fixed point of the operator corresponding to problem (6)–(10). The reason is that the value function $\hat{V}$ enters the constraint set. As a result, Blackwell's sufficient conditions fail to hold. Claim 3 in the Appendix establishes the existence of the fixed point using Schauder's fixed-point theorem.

As should be clear from the above problem, the payoff to the lender from the optimal punishment is endogenous and is affected by parameters of the model, in particular, the probability of enforcement π. This observation is key to the results of this paper, and is formalized in the next lemma.

LEMMA 1: $\hat{V}(w)$ *is strictly increasing in* π *for all* w.

Lemma 1 claims that a deviating lender is better off if enforcement is more likely. Intuitively, even in the worst equilibrium the lender can use the enforcement technology to his advantage. The higher the probability of delivering a promise, the higher the payoff that a deviating lender can achieve. This implies that as the probability of enforcement increases, it becomes harder to punish the lender in the optimal contract.

The above result implies that an increase in the probability of enforcement has an ambiguous effect on welfare. One simple way to see it is to notice that the higher the π, the smaller the measure of states on which the lender's incentive constraint (4) is imposed. This has a positive effect on welfare. On the other hand, the right-hand side of (4) is larger for larger π, which tends to reduce welfare.[22]

I will use Lemma 1 to further analyze the effects of an increase in the probability of enforcement on welfare in the optimal contract in Section V.

IV. Properties of the Optimal Contract and Investment Dynamics

Before I turn to the characterization of the optimal contract, notice that the Pareto frontier corresponding to the optimal contract is not the only fixed point of the operator corresponding to problem (1)–(5). In particular, $\hat{V}$ is the other fixed point. What can be shown is that an iterative mapping starting from the first-best frontier converges to the fixed point that corresponds to the optimal contract. (See Claim 4 in the Appendix.)

The next lemma describes basic properties of the value function V.

LEMMA 2: *The function V is concave and has slope between* -1 *and* 0.

The Pareto frontier is downward sloping because an increase in the borrower's promised value is costly to the lender. The cost from a one-unit increase in w never exceeds one because the lender can always respond to it by increasing the borrower's consumption in all states by one unit. The lender might be able to do better by changing investment and/or continuation values instead. The possibility of lotteries ensures the concavity of V.

Next, I discuss the role of the discount factor in the properties of the optimal contract. If the relationship between the lender and the borrower only lasted for one period, the lender would never invest since any resulting output would be appropriated by the borrower. The fact that the relationship is repeated provides the lender with incentives to invest because he expects to be compensated in the future. Similarly, the borrower has incentives to repay because she is motivated by future investments and hence future profits. Therefore one should expect that the degree to which the agents value the future must be crucial for how close the welfare in the optimal contract is to the first best.

Claim 2 shows that if the agents sufficiently value the future, there is an equilibrium in which the first-best level of investment is made in every period. This means that part of the Pareto frontier coincides with the first-best frontier. Starting from any point on the frontier, optimal equilibrium payoffs eventually reach this part of the frontier.

CLAIM 2: *For any $\pi \in [0,1)$ there exists $\beta_\pi \in (0,1)$ such that the first-best level of investment can be sustained in a Pareto optimal equilibrium in every period if and*

[22] Things are more subtle because an increase in π also affects the left-hand side of (4) by changing the value function V.

only if $\beta \in [\beta_\pi, 1)$. *If* $\pi = 1$, *then the first best can be sustained in a Pareto optimal equilibrium for any* $\beta \in (0, 1)$.

One of the main results of the paper, presented in Section V, is that the threshold level β_π is strictly increasing in π. In order to prove this result, I first need to characterize the optimal choices of consumption, investment, and continuation values.

The following lemma describes an optimal choice of continuation values.[23]

LEMMA 3: *The following choice of* w_{1s} *and* w_{0s} *is optimal for all s:*

(*i*) $w_{1s}(w) = w$ *if (3) does not bind, and* $w_{1s}(w) > w$ *otherwise.*

(*ii*) $w_{0s}(w) = \min\{w_{1s}(w), w_v\}$.

Since V is concave, it is optimal to set w_{xs} equal to w whenever possible. For s such that the borrower would be better off reneging on the contract rather than receiving w, w_{xs} is set such that (3) is satisfied. The choice of w_{0s} is the same as w_{1s} so long as constraint (4) does not bind, and equals w_v otherwise.

Thus Lemma 3 implies the following dynamics of the promised value. The borrower's promised value increases over time while a contract is enforced, strictly so whenever her participation constraint binds. The strict increase happens when the realized output is high enough, and is done to prevent the borrower from walking away with it. As the value to the borrower continues to grow, it becomes more and more likely that the lender's incentive constraint will bind and he will want to renegotiate the contract upon no enforcement (i.e., to offer a new contract different from the old one). At that point, the promised value drops to w_v.

Next, I turn to the characterization of the optimal choice of investment. In what follows, it will be convenient to distinguish between the following two cases:

Case 1: $\pi = 1$, or $\pi < 1$ and $\beta \geq \beta_\pi$. This case occurs if (4) never binds.

Case 2: $\pi < 1$ and $\beta < \beta_\pi$. This case occurs if (4) binds for some (high enough) w.

As Proposition 2 shows, properties of the optimal investment, as well as the corresponding equilibrium dynamics, have important differences in these two cases.

Recall that the lowest promised value at which the borrower's promise-keeping constraint binds is denoted by $\underline{w}$. For $w \leq \underline{w}$, the optimal investment will be constant and at its minimum. Also, define

$$(11) \quad \overline{w} = \begin{cases} \inf\{w | V'(w) = -1\} & \text{if } \pi > 0, \text{ or } \pi = 0 \text{ and } V'(w_v) = -1, \\ w_v & \text{otherwise.} \end{cases}$$

The value $\overline{w}$ will correspond to the promised value to the borrower above which the optimal investment is constant and is at its maximum.

[23] An optimal choice of consumption levels is described in the Appendix.

PROPOSITION 2 (Optimal Investment):

(i) *Investment level $K(w)$ is increasing in w, strictly increasing on $(\underline{w}, \overline{w})$.*

(ii) *The lowest and highest investment levels satisfy $K(\underline{w}) < K^*$ for all π* and *$K(\overline{w}) = K^*$ for $\pi > 0$. The highest investment made if a contract is not enforced is $K(w_v) = K^*$ in Case 1 and $K(w_v) < K^*$ in Case 2.*

(iii) *In Case 1, equilibrium investment is increasing over time attaining maximum level $\overline{K} = K^*$ with probability one. In Case 2, if $\pi = 0$ then investment is increasing over time, attaining maximum level $\overline{K} < K^*$ with probability one; if $\pi \in (0, 1)$, then along any optimal equilibrium path investment fluctuates between underinvestment and the first-best level of investment.*

Part (i) of the proposition states that in the optimal contract investment is increasing in the promised value to the borrower. This value is positively related to the borrower's income since the higher the borrower's income, the higher the value she can secure in equilibrium. Therefore, investment is also positively related to income.

Part (ii) shows that underinvestment can occur in equilibrium. This is caused by the borrower's inability to commit. Since the lender cannot appropriate the full amount of output, his marginal value of output is strictly smaller than one. Therefore, when the lender does not have to deliver a high value to the borrower, he underinvests.

If the promised value to the borrower is sufficiently high, then her participation constraints become slack, and the first-best level of investment is made. When a contract is not enforced, such a high value to the borrower might not be sustainable if the discount factor is below the threshold β_π, in which case there is underinvestment.

Part (iii) combines the predictions of parts (i), (ii), and Lemma 3 and describes properties of equilibrium investment dynamics. For $\pi \in (0, 1)$ fluctuations occur in Case 2 because every time a contract is not enforced, the borrower's continuation value drops to w_v, and investment drops below the first-best level. Eventually the continuation value rises to (or above) $\overline{w}$, and the first-best investment is made, but only until the enforcement fails again.

Figure 1 illustrates the results found in Proposition 2 in a numerical example. In all numerical examples shown in this paper I use the following parameter values: the number of productivity shocks is $n + 1 = 1{,}500$, $p_s = 1/(n + 1)$ for all s, and the production function is $f(K, s) = 2(s/n)K^{0.85}$. In addition, in the example illustrated in Figure 1, $\pi = 0$ and $\pi = 0.8$ for the left and right panels, respectively. The discount factor β is set to $1/1.1$, which satisfies $\beta_0 < \beta < \beta_{0.8}$. Thus the left and right panels correspond to Cases 1 and 2, respectively. In addition, since $\beta > \beta_0$, the optimal contract with $\pi = 0$ is the same as with $\pi = 1$.

When $\pi = 0$ or $\pi = 1$, investment is increasing over time, eventually attaining the first-best level. In this case long-run investment is at a steady state level and is uncorrelated with output, which is stochastic. In sharp contrast, when $\pi = 0.8$, the optimal contract exhibits fluctuations between underinvestment and the first-best level of investment. In this case even in the long run, investment is positively correlated with output. This result is consistent with sensitivity of investment to cash flow

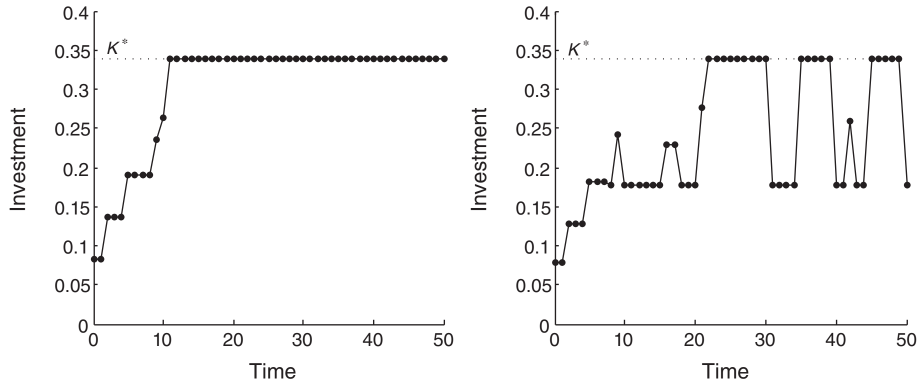

FIGURE 1. INVESTMENT TIME SERIES

Notes: The discount factor is $\beta = 1/1.1 \in (\beta_0, \beta_{0.8})$. Left panel: $\pi = 0$ (or $\pi = 1$, since $\beta > \beta_0$); Right panel: $\pi = 0.8$.

(which is also equivalent to liquidity in this setup) found in the empirical corporate finance literature. (See Hubbard 1998 for a survey.)

Many existing papers explain cash-flow sensitivity of investment using models of asymmetric information. This paper shows that asymmetric information is not needed to obtain this prediction, and suggests imperfect enforcement and renegotiation as an alternative explanation.[24] Close bank-firm relationships, as the one modeled in this paper, are often suggested to mitigate information problems and hence the associated liquidity problems—see, e.g., Hoshi, Kashyap, and Scharfstein (1991). However, Fohlin (1998) finds that relationship banking provides no consistent lessening of firms' liquidity sensitivity. This finding indicates that asymmetric information might not always be the source of liquidity problems, which is consistent with the predictions of my model that even without it liquidity sensitivity of investment is still present. I further discuss the result of cash-flow sensitivity of investment and how it is affected by possible modifications of the model in Section VI.

Notice that a drop in investment occurs in my model only when a contract is not enforced. As I argued in the introduction, renegotiations in loan contracts are sometimes triggered by covenant violations. The prediction that covenant violations reduce the availability of credit is consistent with empirical findings by Chava and Roberts (2008) and Roberts and Sufi (2009a, b).[25]

Finally, in both cases presented on Figure 1, along the equilibrium path under-investment is more severe initially. The liquidity problem is relaxed over time, but might not be completely eliminated. This is consistent with empirical findings that borrowing constraints are more important for younger firms, and that firm growth is negatively related to firm age. (See, e.g., Evans 1987.)

[24] Another possible explanation is suggested, e.g., by Opp (2012, online Appendix), who finds that long-run fluctuations in investment can occur in the no-commitment case if the lender and the borrower have different discount factors.

[25] Roberts and Sufi (2009a) also document that renegotiation that leads to a reduction in the amount of credit following a decline in cash flow is more likely to occur in loan contracts with a pricing grid on cash flow (i.e., when the interest rate depends on cash flow), as in my paper.

Since in cases $\pi = 0$ and $\pi = 1$ properties of investment dynamics are very similar, one might expect analogous properties in the intermediate case of partial commitment as well. Interestingly, my results show that this is not true in general, and the properties of equilibrium investment with partial commitment are drastically different than in the two extreme cases.

V. Welfare and the Probability of Enforcement

This section formally states the result that more enforcement can decrease social welfare. Compare, for instance, cases $\pi = 0$ and $\pi \in (0, 1)$. Casual intuition might suggest that since with $\pi \in (0, 1)$ the lender can always offer a contract that specifies investment levels that would be his optimal choices ex-post, he can replicate the same allocation as without commitment. This would be the case if optimal punishments for deviations in the two cases were the same. However, by Claim 1, with $\pi = 0$ the optimal punishment on the lender is autarky, whereas with $\pi \in (0, 1)$ the deviating lender generates profits. That is, in the environment with partial commitment it is harder to punish the lender compared to the environment without commitment.

More generally, for any π the positive effect of an increase in π comes from the fact that a higher investment is made with a higher probability. The negative effect is that a higher probability of enforcement increases the value to the lender from the most profitable deviation (see Lemma 1).

An alternative way of seeing the two competing effects is that an increase in π reduces the measure of states on which the lender's incentive constraint (4) is imposed, which increases welfare. However, at the same time an increase in π tightens constraint (4) by increasing its right-hand side, which reduces welfare.[26]

Depending on which of the two effects dominates, social welfare in the optimal contract can increase or decrease with π. In general, it is hard to determine the sign of the net effect analytically. Proposition 3 determines the range of the discount factors for which the negative effect dominates. Below, subscripts refer to the dependence of the value function V on the parameters π and β.

PROPOSITION 3:

(i) *The threshold level β_π is strictly increasing in π for all $\pi \in [0, 1]$.*

(ii) *Consider any π and π' such that $\pi' > \pi$, and let $\beta \in [\beta_\pi, \beta_{\pi'})$, so that the first best can be sustained in an optimal equilibrium with π but not with π'. Then $V_{\pi',\beta}(w) < V_{\pi,\beta}(w)$ for all w.*[27]

Part (i) of Proposition 3 says that for a higher probability of enforcement, a higher discount factor is needed in order to achieve the first best in the long run. By Claim 2, if the agents sufficiently value the future, it is possible to split the total

[26] Of course, this result depends on how imperfect enforcement is being modeled. For example, if upon the failure of enforcement the agents' payoffs are changed to some exogenous values, this reasoning does not apply. But if the enforcement failure is tied to endogenously modeled renegotiation, as in my model, its outcome will also be endogenous.

[27] The proof also shows that if $\beta \geq \beta_{\pi'}$ then $V_{\pi,\beta}(w) = V_{\pi',\beta}(w)$ for all w.

surplus between them in such a way that their incentive constraints are slack in each period. By Lemma 1, the higher the π, the greater the value to the lender from deviating. Therefore a higher threshold of the discount factor is needed in order to keep the incentive constraints slack and achieve the first best.

Part (i) implies that if $\beta \in [\beta_\pi, \beta_{\pi'})$, then the maximum social welfare that can be achieved in the optimal contract is higher with π than with π': $\max_w w + V_{\pi';\beta}(w) < S^* = \max_w w + V_{\pi;\beta}(w)$. Part (ii) provides an even stronger result: for any value to the borrower, the value to the lender is higher with π than with π'. That is, increasing the probability of enforcement from π to π' shifts the Pareto frontier downward.

The argument behind the proof is as follows. As I mentioned earlier, the positive effect of an increase in π comes from the fact that a higher level of investment is enforced with a higher probability. This is due to the fact that the borrower receives a higher promised value with a higher probability, and investment is increasing in the value to the borrower. However, since with π and $\beta \in [\beta_\pi, \beta_{\pi'})$ the lender's incentive constraint (4) never binds, by Lemma 3 the optimal continuation values are the same whether the contract is enforced or not: $w_{0s}(w) = w_{1s}(w)$ for all s and w. Thus, if the value to the deviating lender could be held fixed, the social surplus achieved with $\pi' > \pi$ would be the same as with π. In other words, there is no positive effect of having a higher probability of enforcement. There is, however, the negative effect because the value to the lender from the most profitable deviation with π' strictly exceeds that with π.

Notice that Proposition 3 does not imply that $V_{\pi,\beta}$ is decreasing in π for all π. In fact, we know that $V_{0,\beta}(w) \leq V_{1,\beta}(w)$ for all w, and as π approaches 1, we expect the frontier to approach the full-commitment frontier. Indeed, let $\pi < 1$ be arbitrarily close to 1, and $\beta < \beta_\pi$. Then every time the contract is not enforced, there is underinvestment, but this event occurs with an arbitrarily small probability. Thus, even though the social surplus is strictly below the full-commitment one, it becomes arbitrarily close to it as π approaches 1.

This reasoning together with Proposition 3 suggests that $V_{\pi,\beta}$ is nonmonotone in π. This conclusion is supported by numerical computations illustrated on Figure 2. The left panel plots the highest social welfare that can be achieved in the optimal contract as a fraction of the first-best social surplus, $(\max_w w + V_{\pi,\beta}(w))/S^*$, against the probability of enforcement π. The solid and dashed lines correspond to $\beta = 1/1.1 > \beta_0$ and $\beta = 1/1.15 < \beta_0$, respectively. The right panel depicts Pareto frontiers for $\pi = 0.4$, $\pi = 0.8$, and $\pi = 1$ (the dotted, dashed, and solid lines, respectively), with $\beta = 1/1.15$.

The curves on the left panel are U-shaped, and only for π sufficiently high the social welfare starts to increase with more enforcement. If $\beta \geq \beta_0$, not only is there no benefit of increasing enforcement, but in fact it can be a bad idea. And if $\beta < \beta_0$, then an increase in the probability of enforcement only becomes beneficial if the resulting enforcement is sufficiently strong, close to full enforcement.

The right panel of Figure 2 shows that not only the maximum welfare but the whole Pareto frontier changes nonmonotonically with enforcement: it first shifts downward (compare the frontiers for $\pi = 0.4$ and $\pi = 0.8$) and then upward (compare the frontiers for $\pi = 0.8$ and $\pi = 1$).

These results suggest an important conclusion: when contracts are enforced only imperfectly, contractual enforcement may interfere with self-enforcement. As a result, stronger contractual enforcement may reduce welfare.

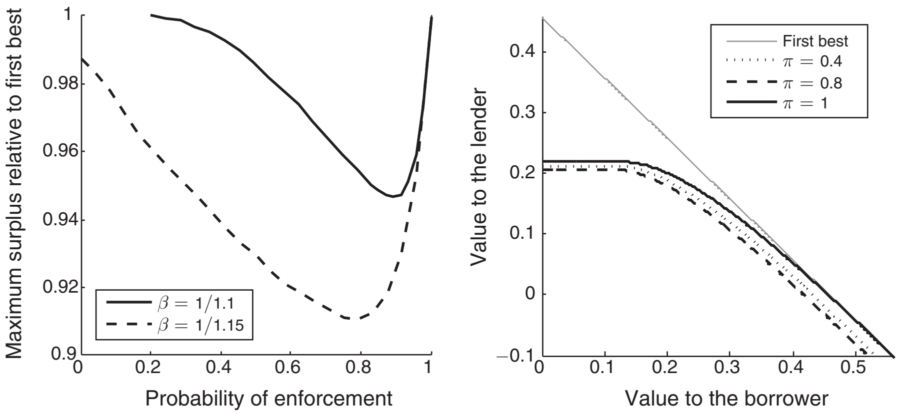

FIGURE 2. WELFARE AND THE PROBABILITY OF ENFORCEMENT

Notes: Left panel: The maximum total surplus as a fraction of the first-best surplus against π for different values of β. Right panel: Pareto frontiers for different values of π ; $\beta = 1/1.15$.

VI. Robustness of the Results and Extensions

In this section I discuss robustness of my main results to the model assumptions and suggest some possible extensions of the model.

One important prediction of the paper is the cash-flow sensitivity of investment. Recall that in the optimal contract (i) investment always rises along contract duration and can only fall as a result of renegotiation, and (ii) the level to which investment falls does not depend on cash flow. The first result is a robust prediction that occurs in all setups that rely on the borrower's limited commitment. However, the second result is not robust to slight changes in the timing of the model. In particular, if the timing is changed as I describe below, then following renegotiation investment changes to a level that is increasing in cash flow. In this modified model, it will no longer be the case that investment will only react positively to positive productivity shocks, but will never decline after negative shocks. That is, correlation between cash flow and investment will no longer be due to increases in capital only.

To be more specific, consider the following modification of the original setup. In the beginning of a period, productivity and enforcement shocks are realized. If the enforcement shock, x, equals one, parties continue following the old contract. If $x = 0$, then the lender offers a new contract, which consists of new repayment in the current period, new investment, and the schedule of history-contingent repayments and investments from the next period onward. If the borrower rejects the contract, she walks away with the current income. If she accepts it, she makes the repayment specified in the contract, the lender makes investment (he can commit to it for one period), and the period ends.[28]

[28] When $\pi = 1$, this new setup still corresponds to full commitment by the lender. However, when $\pi = 0$, it corresponds to the case when the lender can commit to one-period contracts. That is, no commitment is no longer a special case of this model.

The difference between this setup and the original one is that here the lender offers a new contract at the moment when the borrower has income in her hands. Hence the offered contract, and investment in particular, will depend on it.[29] It can be shown that the higher the income, the higher the investment. Thus the level to which investment drops if enforcement fails will depend on cash flow.

Also, since in reality renegotiations are sometimes triggered by covenant violations, it might be natural to think that shocks s and x might be negatively correlated.[30] (Indeed, as Sufi 2009 documents, changes in cash flow are a strong predictor of covenant violations.[31]) This would create an additional channel of cash-flow sensitivity of investment in my model: lower cash flow would be more likely to result in renegotiation, which would lower investment.

Next, I discuss how adding certain features of realism into the framework would affect the main predictions. Recall the extreme assumption of the model that it is always the lender who makes a take-it-or-leave-it offer to the borrower. In reality, one or the other party can initiate renegotiation (have the bargaining power) depending on the events that trigger it. (See, e.g., Roberts and Sufi 2009a and Roberts 2012.) However, the qualitative results of my paper would still apply if, for example, upon the renegotiation shock nature chose randomly which of the parties makes an offer.[32] (Only in the extreme case where the borrower always makes offers would the results change.) The model can also be further extended by assuming that who makes an offer is correlated with the shock to cash flow. The analysis of this extension would be very similar to the one where shocks s and x are negatively correlated.

Another unrealistic feature of the model is the infinite contract maturity. This assumption allows me to write the model in a fairly simple recursive form. In contrast, it is very cumbersome to write down and analyze a recursive model with N-period contracts for an arbitrary N. Solving such a model can be hard even numerically unless N and the number of productivity shocks are very small, as the number of possible next period's states, and thus the number of control variables, become very large. The model becomes even more complicated if one wants to capture renegotiation prior to the stated maturity. However, at least in the relatively simple case of one-period contracts, the main analytical and numerical results of this paper still apply.[33]

[29] Technically, the values v and w_v will depend on income.

[30] Alternatively, one could assume that the distribution of x directly depends on output rather than on s. Formally, one can specify the probability of enforcement as $p(y) = \min\{\pi g(y), 1\}$, where y is either s or Y. The function g is strictly increasing and $g(0) = 1$, so that $\pi = 0$ and $\pi = 1$ still correspond to no- and full-commitment cases. If $y = Y$, the probability of enforcement becomes endogenous; the marginal benefit of investment increases, as a higher level of investment will increase the lender's expected ability to commit.

[31] Also, Roberts and Sufi (2009a) find that despite the fact that loan agreements can be made contingent on cash flow, deviation in cash flow is a strong predictor of renegotiation. They argue that noncontractible outcomes that are correlated with cash flow make it hard for lenders and borrowers to specify all contingencies in the original contract. That is, while the parties understand that they might want to change contract terms in response to deviation in cash flow, they cannot always identify how it should be changed, given other nonverifiable information that will be available when future cash flow is realized.

[32] What happens when the borrower makes an offer is fairly straightforward to analyze, assuming that upon rejection of an offer relationship is terminated. If the value to the lender under the original contract was positive, it will drop to zero; if it was negative, then the borrower will not want to renegotiate the old contract. As a result, investment will either rise—to the level $K(w')$, where w' is such that $V(w') = 0$}—or remain the same.

[33] I have studied a setup where after income is realized, the lender makes an offer that specifies repayment and the next period's investment, where the latter is enforced with probability π. In that model, $\pi = 0$ corresponds to no commitment and $\pi = 1$ corresponds to perfect enforcement of one-period contracts. Among the results, the threshold β_π is strictly increasing in π for $\pi \in [0, 1]$. In numerical computations welfare is nonmonotone in π, and is (weakly) lower with any $\pi \in (0, 1]$ than with $\pi = 0$. Investment is again positively related to cash flow.

The model can also be easily generalized by allowing the borrower's outside option upon termination of the relationship to be different from zero. This extension leads to an additional result: social welfare may rise or fall with the borrower's outside option. (In numerical computations, social welfare as a function of the outside option has an inverse U-shape.) The reason is that even though a lower outside option makes it easier to punish the borrower, it also makes it harder to punish the lender: a deviating lender is able to generate higher profits if he faces a borrower with a lower outside option. That is, improving the borrower's incentives endogenously aggravates the lender's incentive problem. This result reinforces the main message of the paper that mitigating incentive problems through enforcement does not necessarily improve social welfare.

The borrower's outside option can be interpreted as a measure of competition on the credit market. With full and no commitment competition is unambiguously welfare decreasing. Indeed, since the optimal punishments on the lender—which equal $-\infty$ and 0 in the two cases—are independent of the borrower's outside option, increasing it necessarily lowers social welfare. In contrast, in the environment with partial commitment competition can be beneficial as it mitigates the lender's incentive problem. This result is consistent with empirical predictions that bank competition can have both positive and negative economic effects. (See, e.g., Cetorelli and Gambera 2001, and reviews of the literature by Boot 2000 and Cetorelli 2001.)

Finally, as I mentioned in the introduction, renegotiations are also often tied to changes in borrowers' outside options. Thus another possible extension would be to assume that the outside option is stochastic and the probability of the lender offering a new contract is decreasing in its realization.

VII. Conclusions

The main question addressed in this paper is how commitment in a lender-borrower relationship affects the agents' incentives, allocations, and social welfare. The majority of papers on lending with limited commitment focuses on the borrower's incentives. This paper shows that a realistic assumption of the possibility of writing formal contracts that are renegotiated over time creates a nontrivial incentive problem on the side of the lender.

I characterize the optimal contract and the optimal punishment on the lender using recursive methods, and derive the main properties of the optimal contract, in particular, comparative statics with respect to the probability of contract enforcement. I find that an increase in the probability of enforcement can decrease welfare. The reason is that while the enforcement mechanism helps generate surplus on the equilibrium path, it also helps the lender generate profit off the equilibrium path, making it harder to discipline him in equilibrium.

Furthermore, investment dynamics are distinctly different with partial commitment and with full or no commitment. With full or no commitment, investment is increasing over time, eventually attaining a steady state level. In contrast, with partial commitment, investment fluctuates between underinvestment and the first-best level of investment, and is positively related to cash flow even in the long run.

Finally, it is important to emphasize that while the paper models the occurrence of renegotiation in a very simple, exogenous way, in reality its determinants (in particular, inclusion of covenants in loan contracts) may be endogenous. Thus endogenizing the occurrence of renegotiation in a tractable manner is an important issue that calls for further research.

Appendix

PROOF OF PROPOSITION 1:

Let $\hat{V}(w)$ be the lowest subgame perfect equilibrium payoff to the lender if the borrower's payoff is at least w. Then the deviating lender can secure himself $\hat{V}(0)$, since he has to deliver to the borrower at least zero—her payoff if the relationship is terminated. Thus, $v = \hat{V}(0)$.

Finding out what continuation equilibrium payoffs the agents receive in the punishment equilibrium at any subsequent period after any history is equivalent to looking at the following problem:

$$\hat{V}(w) = \min_{\{V_{1s}(\cdot), V_{0s}(\cdot)\}_{s\in\mathcal{S}}} \max_{K, \{c_s, w_{1s}, w_{0s}\}_{s\in\mathcal{S}}} E[-K + f(K,s) - c_s + \beta(\pi V_{1s}(w_{1s}) + (1-\pi)V_{0s}(w_{0s}))]$$

$$\text{s.t.} \quad E[c_s + \beta(\pi w_{1s} + (1-\pi)w_{0s})] \geq w,$$

$$c_s + \beta(\pi w_{1s} + (1-\pi)w_{0s}) \geq f(K,s) \quad \text{for all} \quad s \in \mathcal{S},$$

$$V_{0s}(w_{0s}) \geq \hat{V}(0) \quad \text{for all} \quad s \in \mathcal{S},$$

$$c_s \geq 0, \ w_{1s} \geq 0, \ w_{0s} \geq 0 \quad \text{for all} \quad s \in \mathcal{S},$$

where $V_{xs}(w_{xs})$, $x \in \{0,1\}$, $s \in \mathcal{S}$, is a subgame perfect equilibrium payoff to the lender given that the corresponding payoff to the borrower is at least w_{xs}.

First, notice that for any w, setting $V_{xs}(w_{xs})$ as low as possible for all x and s minimizes the above objective function and tightens the constraints. Thus $V_{xs} = \hat{V}$ for all x and s. Hence the problem of finding the optimal punishment on the lender can be written recursively, as a functional equation in $\hat{V}$.

The lender's incentive constraint becomes $\hat{V}(w_{0s}) \geq \hat{V}(0)$. Clearly, $\hat{V}(w)$ is (weakly) decreasing in w, as an increase in w tightens the promise-keeping constraint in the above problem. This together with $w_{0s} \geq 0$ implies that $\hat{V}(w_{0s}) = \hat{V}(0)$. Moreover, w_{0s} equals $\underline{\hat{w}} \equiv \sup\{w | \hat{V}(w) = \hat{V}(0)\}$ for all s, the lowest w at which the promise-keeping constraint binds.

PROOF OF CLAIM 1:

See the online Appendix.

CLAIM 3: *A solution to problem* (6)–(10) *exists.*

PROOF OF CLAIM 3:

See the online Appendix.

PROOF OF LEMMA 1:

See the online Appendix.

PROOF OF LEMMA 2:

See the main text.

PROOF OF CLAIM 2:

First suppose that $\pi < 1$. For an equilibrium where investment K^* is made in every period to exist, the following constraints have to hold simultaneously:

$$-R_{s'} + E[f(K^*, s) - R_s]\,\beta/(1-\beta) \geq 0 \quad \text{for all} \quad s' \in \mathcal{S}, \tag{A1}$$

$$[-K^* + ER_s]/(1-\beta) \geq v_{\pi,\beta}, \tag{A2}$$

where $R_s \in [0, f(K^*, s)]$ denotes repayment from the borrower to the lender if the productivity shock realization is s, and the subscripts refer to the dependence of v on the parameters π and β. It must be the case that $R_s < f(K^*, s)$ for some s, for otherwise the borrower never consumes and will thus find it profitable to deviate. Hence $ER_s < Ef(K^*, s)$.

First, I will show that (A1)–(A2) are simultaneously satisfied for $\beta < 1$ high enough. By definition of the first-best surplus, $[-K^* + Ef(K^*, s)]/(1-\beta) \geq v_{\pi,\beta}$. Moreover, since $v_{\pi,\beta}$ is strictly increasing in π by Lemma 1 and S^* is independent of π, the above inequality must be strict:

$$[-K^* + Ef(K^*, s)]/(1-\beta) > v_{\pi,\beta}. \tag{A3}$$

Thus for ER_s sufficiently close to $Ef(K^*, s)$, $[-K^* + ER_s]/(1-\beta) \geq v_{\pi,\beta}$ by (A3), and hence (A2) is satisfied. Since $ER_s < Ef(K^*, s)$, (A1) will hold for $\beta < 1$ sufficiently high.

Next, I will show that if $\beta > 0$ is low enough, then (A1)–(A2) cannot hold simultaneously. From (A2), $[-K^* + ER_s]/(1-\beta) \geq v_{\pi,\beta} \geq 0$, and hence $-ER_s \leq -K^*$. Then from (A1), $R_{s'} \leq E[f(K^*, s) - R_s]\,\beta/(1-\beta) \leq [Ef(K^*, s) - K^*]\beta/(1-\beta)$, which can be made strictly less than K^* for β small enough. Thus for $\beta > 0$ low enough, $R_{s'} < K^*$ for all s', so that $[-K^* + ER_s]/(1-\beta) < 0$, violating (A2).

Let β_π be the lowest β for which (A1)–(A2) are simultaneously satisfied. I will show that they are also satisfied for any $\beta \geq \beta_\pi$. One can write $v_{\pi,\beta}$ as $\sum_{t=0}^{\infty} E_t \beta^t(-\hat{K}_t + f(\hat{K}_t, s_t) - \hat{c}_t)$, where $\hat{K}_t$ and $\hat{c}_t$ are the stochastic processes of investment and consumption levels (corresponding to the optimal choices of the lender) in the punishment equilibrium. Using the Envelope theorem, $\partial v_{\pi,\beta}/\partial\beta = \partial\left[\sum_{t=0}^{\infty} E_t \beta^t(-\hat{K}_t + f(\hat{K}_t, s_t) - \hat{c}_t)\right]/\partial\beta < \partial[(-K^* + Ef(K^*, s))/(1-\beta)]/\partial\beta$. Hence the difference between the left- and right-hand sides of (A3) is strictly increasing in β. Therefore as β increases, one can increase repayments so that (A1)–(A2) continue to hold.

If $\pi = 1$, then there is no restriction on the lender's profits as (A2) is omitted. Thus (A1) can be satisfied for any $\beta \in (0, 1)$ by setting R_s sufficiently low.

Denote by $\partial V(w)$, $V'_-(w)$, and $V'_+(w)$ the superdifferential, left derivative, and right derivative, respectively, of the function V at w. Let λ, $p_s\mu_s$, and $\beta(1-\pi)\gamma$

denote the Lagrange multipliers on constraints (2), (3), and (4), respectively. The first-order condition with respect to c_s, with complementary slackness, is

$$-1 + \lambda + \mu_s \leq 0, \qquad c_s \geq 0. \tag{A4}$$

I will take the first-order conditions with respect to w_{1s} and w_{0s} ignoring constraints in (5), and will show that they indeed never bind. These first-order conditions and the Envelope condition are

$$-(\lambda + \mu_s) \in \partial V(w_{1s}), \tag{A5}$$

$$-(\lambda + \mu_s)/(1 + \gamma) \in \partial V(w_{0s}), \tag{A6}$$

$$-\lambda \in \partial V(w). \tag{A7}$$

The first-order condition with respect to K is

$$1 \in E f_K(K, s)(1 - \mu_s). \tag{A8}$$

Conditions (A4)–(A6) and Lemma 2 imply the following results about the optimal choices of consumption. If $\pi > 0$ then $c_s = 0$ unless $w_{1s} > \overline{w}$, where $\overline{w}$ is defined in (11). Similarly, if $\pi = 0$ and (4) never binds (in which case $w_v > \overline{w}$), then $c_s = 0$ unless $w_{0s} > \overline{w}$. And if $\pi = 0$ and (4) binds for w high enough (in which case $w_v = \overline{w}$), then $c_s = 0$ unless $-1 \in \partial V(w_{0s})(1 + \gamma)$, i.e., $w_{0s} = w_v$ and $V'_{-}(w_v)(1 + \gamma) \leq -1$.

PROOF OF LEMMA 3:

See the online Appendix.

Let $\{s_t\}_{t \geq 0}$ and $\{x_t\}_{t \geq 0}$ (with $x_0 = 0$) be sequences of productivity and enforcement shocks, respectively, and let $w_{(0)} \geq 0$ be the initial condition. Define the sequences $\{w_{(t)}\}_{t \geq 0}$ and $\{K_{(t)}\}_{t \geq 0}$ as follows: $w_{(t+1)} = w_{x_{t+1}, s_t}(w_{(t)})$ and $K_{(t)} = K(w_{(t)})$. The following result follows immediately from Lemma 3:

LEMMA 4:

(i) If $w_{(t)} \leq w_v$, then $w_{(t+1)} \geq w_{(t)}$.

(ii) If $w_{(t)} \geq \min\{w_v, \overline{w}\}$, then $w_{(\tau)} \geq \min\{w_v, \overline{w}\}$ for all $\tau \geq t$.

PROOF OF PROPOSITION 2:

(i) Since for $w \leq \underline{w}$ (2) does not bind, it is optimal to set $c_s = 0$ and $w_{1s} = w_{0s} = w_s$ for all s. Using (3) and Lemma 3, $w_s = \max\{w, f(K, s)/\beta\}$, and $V(w_s) = V(0)$ if $w_s = w (\leq \underline{w})$ and $V(w_s) = V(f(K, s)/\beta)$ if $w_s = f(K, s)/\beta$. Then the optimal choice of K is $\text{argmax}_K E[-K + f(K, s) + \beta \min\{V(0), V(f(K, s)/\beta)\}]$, the same for all $w \leq \underline{w}$.

Let $w \in (\underline{w}, \overline{w})$ so that (2) binds, and suppose that w increases marginally by Δ. In this case, if K is kept unchanged, with the choices of consumption levels described above and the choices of continuation values given in Lemma 3, the left-hand side of (2) increases by strictly less than Δ, and hence (2) would be violated. Therefore $K(w)$ has to be strictly increasing on $(\underline{w}, \overline{w})$.

If $\pi = 0$ and $\overline{w} = w_v$, then $\overline{w}$ is the maximum value to the borrower that can be sustained in equilibrium. Suppose that either (a) $\pi > 0$, or (b) $\pi = 0$ but $\overline{w} < w_v$, and consider $w > \overline{w}$. Then $V'(w_{1s}) = V'(w) = -1$ for all s in case (a), and $V'(w_{0s}) = V'(w) = -1$ for all s in case (b). Then it is optimal to increase all c_s by one unit in response to an increase in w by one unit, and leave investment unchanged. Thus $K(w) = K(\overline{w})$ for $w \geq \overline{w}$.

(ii) From (A8), $K(\underline{w}) \leq K^*$. Suppose that $K(\underline{w}) = K(0) = K^*$. This would imply that at $w = 0$, $\mu_s = 0$ for all s and $V'(0) = -1$, i.e., $\overline{w} = 0$. Then using (3), the left-hand side of (2), which holds with equality at $w = 0$, is at least $Ef(K^*, s)$. Thus $Ef(K^*, s) \leq 0$, a contradiction.

Let $\pi > 0$ and $w > \overline{w}$. Then $V'(w_{1s}) = V'(w) = -1$ and $\mu_s = 0$ for all s. Hence from (A8), $K(w) = K^*$ for $w > \overline{w}$, and by continuity $K(\overline{w}) = K^*$.

Next, consider how $K(w_v)$ compares to K^*. By Lemma 3, once the value w_v is reached, the value to the borrower never goes below w_v. Hence by part (i) of this proposition, once w_v is reached, the level of investment in each period is at least K_v, and equal to it every time a contract is not enforced. In particular, if $\pi = 0$, $K(w_v)$ is invested in each period. The fact that $K(w_v) = K^*$ in Case 1 and $K(w_v) < K^*$ in Case 2 follows from the definitions of β_π and the two cases.

(iii) By part (i) of Lemma 4, $w_{(t)}$ is increasing over time as long as $w_{(t)} \leq w_v$, and thus by part (i) of this proposition so is $K_{(t)}$. In Case 1, $\min\{w_v, \overline{w}\} = \overline{w}$, and hence by part (ii) of Lemma 4, if $w_{(t)} \geq \overline{w}$ then $w_{(\tau)} \geq \overline{w}$ for all $\tau \geq t$. In addition, by part (ii) of this proposition, $K(w) = K^*$ for $w \geq \overline{w}$. Since the borrower must consume in equilibrium, and $c_s = 0$ unless $w_{1s} \geq \overline{w}$ (or $w_{0s} \geq \overline{w}$ if $\pi = 0$), $\overline{K} = K^*$ is attained with probability one.

In Case 2, if $\pi = 0$, then $w_v = \overline{w}$ and $K(\overline{w}) < K^*$ by part (ii) of this proposition. By part (ii) of Lemma 4, if $w_{(t)} \geq \overline{w}$, then $w_{(\tau)} \geq \overline{w}$ for all $\tau \geq t$. By the same argument as above, $\overline{K} < K^*$ is attained with probability 1. Suppose that $\pi \in (0, 1)$. Then $K(w_v) < K^*$ and $K(\overline{w}) = K^*$. Thus along any optimal equilibrium path, investment increases and eventually reaches $K(\overline{w}) = K^*$ along a sequence of $x = 1$ shocks, and falls to $K(w_v) < K^*$ every time $x = 0$, resulting in fluctuations between underinvestment and the first-best level of investment.

CLAIM 4: *Let T be the operator corresponding to problem* (1)–(5), *and let $V^*(w)$ denote the value to the lender in the first best when the value to the borrower is w,* *i.e.*, $V^*(w) = S^* - w$. *The sequence* $T^n(V^*)$ *converges point wise to* V *as* $n \to \infty$, *where* V *is a fixed point of* T *corresponding to the constrained Pareto frontier.*[34]

PROOF OF CLAIM 4:

See the online Appendix.

PROOF OF PROPOSITION 3:

(i) Let $\pi' \in (\pi, 1)$, and suppose that $\beta_{\pi'} \leq \beta_\pi$. Consider constraints (A1)–(A2) for π' and $\beta_{\pi'}$. It has to be the case that the lender's incentive constraint and at least one of the borrower's participation constraints bind. Hold R_s and $\beta_{\pi'}$ constant, and decrease the probability of enforcement from π' to π. By Lemma 1, $v_{\pi',\beta} > v_{\pi,\beta}$. Therefore (A2) becomes slack. But then it is possible to decrease both β and all R_s marginally so that (A1)–(A2) continue to hold. That is, (A1)–(A2) are simultaneously satisfied with π and $\beta < \beta_{\pi'} \leq \beta_\pi$, a contradiction with the fact that β_π is the lowest β for which this is true.

(ii) Define the operator $\tilde{T}_{\pi',\pi,\beta}$ as follows:

$$\tilde{T}_{\pi',\pi,\beta}V(w) = \max_{K,\{c_s, w_{1s}, w_{0s}\}_{s\in\mathcal{S}}} E[-K + f(K,s) - c_s + \beta(\pi' V(w_{1s}) + (1 - \pi')V(w_{0s}))]$$

$$\begin{aligned} \text{s.t.} \quad & E[c_s + \beta(\pi' w_{1s} + (1 - \pi')w_{0s})] \geq w, \\ & c_s + \beta(\pi' w_{1s} + (1 - \pi')w_{0s}) \geq f(K,s) \quad \text{for all} \quad s \in \mathcal{S}, \\ & V(w_{0s}) \geq v_{\pi,\beta} \quad \text{for all} \quad s \in \mathcal{S}, \\ & c_s \geq 0,\ w_{1s} \geq 0,\ w_{0s} \geq 0 \quad \text{for all} \quad s \in \mathcal{S}. \end{aligned}$$

For $\pi = \pi'$, this operator coincides with the operator T corresponding to problem (1)–(5). Let $\tilde{V}_{\pi',\pi,\beta}$ be the fixed point of $\tilde{T}_{\pi',\pi,\beta}$ corresponding to the Pareto frontier, given that the probability of enforcement is π' and the lender's punishment is fixed at $v_{\pi,\beta}$. For any π, $\tilde{V}_{\pi,\pi,\beta} = V_{\pi,\beta}$. Denote the policy functions for continuation values corresponding to the above problem by $w_{xs;\pi',\pi,\beta}$.

Take $\pi' \in (\pi, 1)$, and let $\beta \in [\beta_\pi, 1)$. Then (4) never binds so that $w_{1s;\pi',\pi,\beta}(w) = w_{0s;\pi',\pi,\beta}(w)$ are optimal for all w and all s. Thus $\tilde{T}_{\pi',\pi,\beta}(V_{\pi,\beta}) = V_{\pi,\beta}$ and hence $V_{\pi,\beta} = \tilde{V}_{\pi,\pi,\beta} = \tilde{V}_{\pi',\pi,\beta}$. Next, consider applying the operator $\tilde{T}_{\pi',\pi',\beta}$ to $V_{\pi,\beta}$. By Lemma 1, $v_{\pi',\beta} > v_{\pi,\beta}$. Hence the only difference between the operators $\tilde{T}_{\pi',\pi,\beta}$ and $\tilde{T}_{\pi',\pi',\beta}$ is that in the latter the constraint set is smaller. Therefore $\tilde{T}_{\pi',\pi',\beta}(V_{\pi,\beta}) \leq V_{\pi,\beta} = \tilde{T}_{\pi',\pi,\beta}(V_{\pi,\beta})$. By induction, suppose that $\tilde{T}^n_{\pi',\pi',\beta}(V_{\pi,\beta}) \leq \tilde{T}^{n-1}_{\pi',\pi',\beta}(V_{\pi,\beta})$ for some $n \geq 1$. Then $\tilde{T}^{n+1}_{\pi',\pi',\beta}(V_{\pi,\beta}) \leq \tilde{T}^n_{\pi',\pi',\beta}(V_{\pi,\beta})$. Applying the same argument as in the proof of Claim 4, $V_{\pi',\beta} = \lim_{n\to\infty} \tilde{T}^n_{\pi',\pi',\beta}(V_{\pi,\beta}) \leq V_{\pi,\beta}$.

[34] In other words, $V(w)$ is the value to the lender who solves the optimal sequence problem with the promised value to the borrower equal w.

If $\beta \geq \beta_{\pi'} > \beta_{\pi}$, then in the problems corresponding to both $\tilde{T}_{\pi',\pi',\beta}$ and $\tilde{T}_{\pi',\pi,\beta}$ the lender's incentive constraint is always slack, and hence $V_{\pi',\beta} = V_{\pi,\beta}$. Let $\beta \in [\beta_{\pi}, \beta_{\pi'})$. Then the lender's incentive constraint binds in some states. The rest of the proof shows that in this case $V_{\pi',\beta}(w) < V_{\pi,\beta}(w)$ for all w.

First, since $\beta \in [\beta_{\pi}, \beta_{\pi'})$, $\max_w w + V_{\pi',\beta}(w) < \max_w w + V_{\pi,\beta}(w) = S^*$ by part (i) of this proposition. That is, $V_{\pi',\beta}(w) < V_{\pi,\beta}(w)$ for $w \geq \max\{\overline{w}_{\pi}, \overline{w}_{\pi'}\}$, where $\overline{w}_{\pi}$ is $\overline{w}$ defined by (11) for $V = V_{\pi,\beta}$. Recall that for $\pi = 0$, V is only defined on $[0, w_v]$. In order to be able to compare $V_{\pi,\beta}(w)$ and $V_{\pi',\beta}(w)$ for all w when $\pi = 0$, define $V_{0,\beta}(w) = V_{0,\beta}(w_v) + w_v - w$ for $w > w_v$.

Suppose there exists $w \geq 0$ such that $V_{\pi',\beta}(w) = V_{\pi,\beta}(w)$, and let $w' = \max\{w \mid V_{\pi',\beta}(w) = V_{\pi,\beta}(w)\}$. First I want to show that $w' \leq \overline{w}_{\pi'}$. Suppose that $w' > \overline{w}_{\pi'}$. Then it has to be the case that $\max\{\overline{w}_{\pi}, \overline{w}_{\pi'}\} = \overline{w}_{\pi} > \overline{w}_{\pi'}$ and $w' \in (\overline{w}_{\pi'}, \overline{w}_{\pi})$. At w', $V'_{\pi,\beta+}(w') > -1 = V'_{\pi',\beta}(w')$. Using Lemma 2, this contradicts to $V_{\pi,\beta}(w) = V_{\pi',\beta}(w)$ for $w \leq w'$. Thus $w' \leq \overline{w}_{\pi'}$.

Next, I will show that $V_{\pi',\beta}(w') < V_{\pi,\beta}(w')$, thus arriving to a contradiction with the definition of w'. For all s, $\pi' V_{\pi',\beta}(w_{1s;\pi',\pi',\beta}(w')) + (1-\pi')V_{\pi',\beta}(w_{0s;\pi',\pi',\beta}(w')) \leq \pi' V_{\pi,\beta}(w_{1s;\pi',\pi',\beta}(w')) + (1-\pi')V_{\pi,\beta}(w_{0s;\pi',\pi',\beta}(w'))$, with strict inequality for s such that $w_{xs;\pi',\pi',\beta}(w') > w'$ for at least some x. It must be the case that $w_{xs;\pi',\pi',\beta}(w') > w'$ for some s and x; if not, then since $w' \leq \overline{w}_{\pi'}$, all consumption levels are zero, and the promise-keeping constraint is violated. Hence at w' the lender's objective function in the operator $\tilde{T}_{\pi',\pi',\beta}$ with $V_{\pi',\beta}$ as the continuation value function is strictly lower than that with $V_{\pi,\beta}$, while the constraint set is no larger. Thus $V_{\pi',\beta}(w') = \tilde{T}_{\pi',\pi',\beta}(V_{\pi',\beta})(w' < \tilde{T}_{\pi',\pi',\beta}(V_{\pi,\beta})(w') \leq V_{\pi,\beta}(w')$.

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