INTERNATIONAL ECONOMIC REVIEW
Vol. 58, No. 2, May 2017

SCREENING AS A UNIFIED THEORY OF DELINQUENCY, RENEGOTIATION, AND BANKRUPTCY*

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We propose a parsimonious model with adverse selection where delinquency, renegotiation, and bankruptcy all occur in equilibrium as a result of a simple screening mechanism. A borrower has private information about her endowment, and a lender uses random contracts to screen different types of borrowers. In equilibrium, some borrowers choose not to repay and thus become delinquent. The lender renegotiates with some delinquent borrowers. In the absence of renegotiation, delinquency leads to bankruptcy. Applied to mortgage restructuring, our mechanism generates amplification of house-price shocks through foreclosure spillovers. We also show that government intervention aimed at limiting foreclosures may have unintended consequences.

1. INTRODUCTION

Default in consumer credit markets is not a simple binary event, but rather has multiple stages and possible outcomes. The first "stage" is delinquency, which is defined as being overdue on loan payments for a specified period of time (usually at least 60 days). Some, but not all, delinquent borrowers end up in bankruptcy. Lenders sometimes renegotiate with delinquent borrowers to prevent bankruptcy and achieve debt settlement.

We propose a very simple model where a single key friction gives rise to these three stages of default described above. The friction is adverse selection—a borrower has private information about her endowment. The optimal contract that screens borrowers with different endowment levels endogenously generates the three phenomena, that is, delinquency, renegotiation, and bankruptcy.

To illustrate our basic mechanism in the simplest possible framework, we start by considering a one-period model, where the borrower is indebted to a single lender. We focus on the case where the borrower's endowment, unknown to the lender, can take one of two values, high or low. The lender offers repayment options to the borrower and seeks to maximize the expected repayment. As an alternative to making the offered repayments (or repaying the full amount of owed debt), the borrower can file for bankruptcy.

We first consider a situation where debt is so high that it does not restrict the lender's optimal choice of repayments. We will refer to this scenario as "debt overhang." Faced with adverse selection, the lender has two basic options when restricted to offering deterministic contracts.

*Manuscript received November 2013; revised April 2015.

[1] We thank Andrei Kovrijnykh for getting us started on this project. We have benefited from discussions with Hector Chade, Satyajit Chatterjee, Carlos Garriga, Kristopher Gerardi, Robert Hunt, Nobuhiro Kiyotaki, Ben Lester, Alejandro Manelli, Ben Moll, Salvador Navarro, Ezra Oberfield, Guillermo Ordonez, Andrei Savochkin, Jacob Short, Ali Shourideh, Alexei Tchistyi, and Galina Vereshchagina. We are also grateful for comments by seminar participants at ASU, Atlanta Fed, Bank of Canada, Cardiff Business School, Collegio Carlo Alberto, DIW Berlin, Goethe University of Frankfurt, McMaster University, Princeton University, Ryerson University, UBC, Universidad Carlos III, University of Iowa, University of Konstanz, UNC-Chapel Hill, University of Rochester, University of Winnipeg, University of Windsor, Wharton, and conference participants at SED 2013, SAET 2013, Tepper-LAEF Macro Finance 2013, Vienna Macro 2013, Canadian Macro Study Group 2013, and BEROC 2014. Parts of the work for this article were carried out when Livshits was visiting Collegio Carlo Alberto and CASEE; Livshits thanks them for their kind hospitality. Please address correspondence to: Natalia Kovrijnykh, Department of Economics, W. P. Carey School of Business, Arizona State University, P.O. Box 879801, Tempe, AZ 85287-9801. E-mail: *Natalia.Kovrijnykh@asu.edu*.

First, by asking for repayment that does not exceed the low-income borrower's willingness to pay, the lender can guarantee repayment from both types of borrowers. Second, by asking for a greater repayment, the lender can extract more from the high-income borrowers, but loses the low-income borrowers to bankruptcy.

The lender may be able to do better if he extracts different repayments from different types of borrowers. However, he cannot separate the two types of borrowers by offering a menu of deterministic contracts, as every borrower would choose the lowest repayment. But borrowers with different income levels have different willingness to pay to avoid bankruptcy. The lender can utilize this feature and separate the two types of borrowers by using lotteries over repayments and bankruptcy.

We show that the optimal screening mechanism involves the lender offering a menu of random contracts that consists of a deterministic repayment and a lottery, aimed at the high- and low-income borrowers respectively. The lottery for the low-income type is over a repayment that is lower than the deterministic one and a very high repayment that exceeds the willingness to pay of both types. In this optimal mechanism, the high-income borrowers make a higher repayment, whereas the low-income borrowers decline that repayment and are then offered a better deal with some probability, but are forced into bankruptcy with the complementary probability.

Next, we consider a scenario when debt does restrict the lender's offers, that is, there is no debt overhang. We show that when the debt level is below the low-income borrowers' willingness to pay, all borrowers repay their debt in full. However, when debt exceeds this threshold (but there is still no debt overhang) and the fraction of the high-income borrowers is high enough, the optimal mechanism involves screening via random contracts.

One of the central points of the article is that such a mechanism has a natural economic interpretation and delivers the three phenomena—delinquency, renegotiation, and bankruptcy—described above. Indeed, offering the aforementioned menu is equivalent to making the following sequential offers: First, the lender offers a high repayment that only the high-income borrowers accept. As long as there is no debt overhang, this high repayment exactly equals the face value of debt. We interpret the borrowers who have agreed to make the high payment as having repaid the loan, whereas the borrowers who refuse to make it as becoming *delinquent*. Next, the lender offers a lower repayment to a fraction of the delinquent borrowers. We interpret the event of offering the lower repayment as *renegotiation*. The delinquent borrowers with whom the lender does not renegotiate declare *bankruptcy*.[2]

Renegotiation allows the lender to extract some repayment from the low-income borrowers who reject the high repayment. However, the possibility of renegotiation makes delinquency more attractive and thus limits the amount that can be extracted from the high-income borrowers. It is for this reason that the lender does not renegotiate with all delinquent borrowers. Thus, our article also addresses the question of why we see some renegotiation in the consumer credit market but not all bankruptcies are avoided.

Building on our characterization of the one-period interaction between the borrower and the lender given a debt level, we then study a two-period model that endogenizes the debt level in the first period and has our basic mechanism at work in the second period. In the first period, multiple lenders compete in loan terms that they offer to the borrower; they break even, and thus the borrowed amount is chosen so as to maximize the borrower's ex ante utility.

Comparing our framework with the option of bankruptcy to the model where the borrower has no outside option in the second period brings an interesting insight. Notice that borrowers with different income levels have different attitudes toward risk. For example, with decreasing absolute risk aversion (DARA), a borrower with higher income is more tolerant toward risk than a borrower with lower income. Thus, even in the absence of the bankruptcy option, it might be possible to screen different types of borrowers by offering them different lotteries

[2] An implicit assumption needed for the sequential offers to be equivalent to the (simultaneous) menu offer is that the lender can commit not to renegotiate with all delinquent borrowers, for otherwise the high-income borrowers will never agree to make the initial high repayment.

over repayments. However, we show that this is never optimal under DARA. That is, in the absence of the outside option (the loan size is optimally chosen such that), all borrowers make the same repayment in the second period. That is, the presence of the bankruptcy option is *essential* for screening and for the optimal contract to involve delinquency, renegotiation, and bankruptcy.

Our model puts us in a unique position to analyze effects of a government intervention in consumer debt restructuring. One example of such an intervention is a mortgage modification program aimed at limiting foreclosures. Indeed, this intervention is triggered by *delinquency* and offers debt restructuring (i.e., involves *renegotiation*) to avoid *bankruptcy* (foreclosure). Not only does our model capture all these stages of default, but, most importantly, it allows us to explicitly analyze the response of private lenders to the government intervention. We show that a government program that fails to take into account private debt restructuring may have the opposite effect from the one intended—instead of limiting the number of foreclosures, it may actually increase it. We also demonstrate how a seemingly irrelevant intervention can successfully prevent *all* defaults. Our analysis therefore illustrates that it is crucial for a policymaker designing such a program to take into account how private debt restructuring works, or else the program may backfire. If the intervention is anticipated, it is reflected in the equilibrium interest rates and thus affects the amount borrowed in the first period. As a result, the anticipated intervention can reduce (increase) welfare ex ante even though it reduces (increases) foreclosures ex post.

Another interesting insight from our analysis is that the endogenous debt restructuring generates amplification of house-price shocks when foreclosures negatively affect neighborhood house prices. In our model, a negative shock to house prices leads to a lower probability of renegotiation and thus a higher foreclosure rate. In the presence of the spillovers, this in turn further lowers the house prices, leading to even more foreclosures, and so on.

In the application to mortgages and foreclosures, our model also matches some key empirical regularities. First, foreclosure rates increase with negative home equity. More interestingly, our model reproduces the "double-trigger" phenomenon that foreclosures are associated not just with negative home equity but also with a negative shock to the homeowner's income.

The rest of the article is organized as follows: The next subsection reviews the related literature. Section 2 sets up the one-period model. Section 3 characterizes the optimal one-period contract and discusses how the screening mechanism captures the three stages of default. Section 4 extends the model to two periods where the debt level is determined endogenously. Section 5 analyzes the applications to mortgage restructuring. Section 6 concludes.

1.1. *Related Literature.* Theoretical studies of default in consumer credit markets have largely focused on bankruptcy and abstracted from delinquency, and especially renegotiation—see, for example, Chatterjee et al. (2007), Livshits et al. (2007), and many others. Notable exceptions are the papers by Chatterjee (2010), Adelino et al. (2013), Benjamin and Mateos-Planas (2012), and Athreya et al. (2012). Although Chatterjee (2010) makes a distinction between delinquency and bankruptcy, he does not allow for renegotiation.[3] Adelino et al. (2013), on the other hand, study renegotiation, but treat delinquency as exogenous. They document that renegotiations of delinquent mortgages are infrequent.[4] In explaining this phenomenon, the authors point out that mortgage restructuring may not be ex post profitable for the lenders as it foregoes to possibility of "self-cures"—delinquent mortgages being repaid in full. In contrast, in our model, renegotiation is always profitable ex post (i.e., after the borrower becomes delinquent), but generates an ex ante distortion by affecting the incentive of the high-income borrowers to make the high repayment instead of choosing delinquency. Thus, we view our explanation for

[3] The distinction between bankruptcy and "informal bankruptcy" is also present in Dawsey and Ausubel (2004) and Dawsey et al. (2009), but the informal bankruptcy is thought of as a terminal state, much like bankruptcy, instead of as a transitional stage that delinquency captures.

[4] Agarwal et al. (2011) also point out that lenders restructure merely a small fraction of delinquent mortgages.

why lenders do not renegotiate more frequently as complementary to that offered by Adelino et al. (2013).

Benjamin and Mateos-Planas (2012) and Athreya et al. (2012) propose quantitative models with symmetric information and incomplete markets where all three stages of default are present. However, the mechanics of their models are very different from ours. In Benjamin and Mateos-Planas (2012), renegotiation occurs with an exogenously given probability, but the possibility of renegotiation leads to an endogenous distinction between delinquency and bankruptcy. In Athreya et al. (2012), delinquency also triggers debt restructuring, but deterministically so.[5] In contrast, in our model, the probability of renegotiation following delinquency is determined endogenously, and, as will be clear from Subsection 5.2, the endogeneity of renegotiation is crucial for policy analysis.

Another related paper is Hopenhayn and Werning (2008), who study a dynamic lending model where the borrower has private information about her outside option. The optimal contract in their framework also features default occurring in equilibrium with positive probability. However, their model does not distinguish between delinquency and default (which is akin to bankruptcy in our setup) and thus does not allow for the possibility of renegotiation.

The reason for delinquency in our model is distinct from the consumption-smoothing motive in Herkenhoff (2012a), where delinquency ("informal default") is basically a temporary reprieve for borrowers with negative income shocks while they wait for their incomes to recover, at which point they make a full repayment of the rolled-over debt.[6] Since in our model, uncertainty is realized in a single period (and thus there are no future income shocks), delinquency never results in a full repayment. Although "self-curing" delinquencies are clearly present in the data,[7] this article does not attempt to explain them and focuses on a complementary mechanism instead.

Unlike in the consumer debt literature, analysis of renegotiation has played a central role in the sovereign debt literature—see the seminal work by Bulow and Rogoff (1989) and more recent contributions by Kovrijnykh and Szentes (2007), Benjamin and Wright (2009), Yue (2010), Arellano and Bai (2013), and others. Our work differs from this strand of literature in a number of ways. One distinction is that the key friction in our article is private information about the borrower's income, which arguably is more relevant in consumer debt than sovereign debt context. Also, unlike the sovereign default papers, our model allows us to study an "extensive margin" of renegotiation, as the fraction of borrowers with whom the lender renegotiates is determined endogenously. This in turn allows us to analyze the effect of an intervention operating along this extensive margin.

From the modeling standpoint, our article is closely related to papers by Maskin and Riley (1984), Matthews (1983), and Miller et al. (2006). Maskin and Riley (1984) study a problem of designing an auction that maximizes the expected revenue of a seller of an indivisible good facing risk-averse bidders with unknown preferences. They show that making buyers bear risk relaxes incentive constraints. In addition, they find that the probability of winning the auction (obtaining the good) and the amount paid in the case of winning increase with a buyer's valuation. Our result is similar in that, in our screening contract, a low-income borrower makes a lower repayment, and with a lower probability, than a high-income borrower. Matthews (1983) studies a problem similar to the one analyzed by Maskin and Riley (1984), but also analyzes the case where there is an unlimited supply of indivisible units sold. This case is closer to our setup, where it is possible for the lender to obtain repayments (which is analogous to selling a good) from multiple borrowers. Matthews (1983) finds that the optimal selling scheme gives some buyers only a probability of obtaining the good. Finally, Miller et al. (2006) also consider a setup similar to that in the other two papers, but have the seller making sequential price offers.

[5] Coexistence of bankruptcy and delinquency in Athreya et al. (2012) arises from an exogenously imposed additional cost of delinquency, namely, income garnishment.

[6] A similar mechanism is also present in Chatterjee (2010).

[7] Herkenhoff and Ohanian (2012) offer a nice summary of empirical facts regarding transitions of mortgages into and out of delinquencies.

They show that the optimal selling scheme involves the seller making an offer that, if rejected, is followed by a subsequent, more attractive offer, but only with some probability. This selling scheme is similar to the sequential interpretation of the one-period optimal contract in our model. Although our mechanism shares features common with those in the aforementioned papers, the application is quite distinct. When applied to consumer credit, screening through randomization offers a novel, unified theory of delinquency, renegotiation and bankruptcy. Furthermore, it generates a number of interesting insights into mortgage debt restructuring.

Finally, our analysis of the government intervention in debt restructuring contributes to the literature on the effects of the most notable such intervention in recent years—the Home Affordable Mortgage Program (HAMP). This program is aimed at restructuring troubled mortgages and preventing foreclosures and has been in place in the United States since 2009. Agarwal et al. (2012) offer a comprehensive empirical analysis of the effects of this program. The authors highlight the importance of accounting for changes in private restructuring in evaluating the effects of the program. Our theoretical model allows us to explicitly analyze the private sector's response to an intervention and to illustrate that it can lead to unexpected, and possibly undesired, consequences. These insights are complementary to the existing studies pointing out possible shortcomings of HAMP. Most notably, Mulligan (2009, 2010) points out severe distortions imposed by the means-testing aspect of the program that induces an excessively high effective income tax rate. Specifically, since the restructured payments depend directly on the borrower's income, HAMP creates a strong incentive for the borrower to earn less. We treat income of borrowers as exogenous, thus ignoring such distortions. Instead, we highlight the distortions imposed by such a government program on the private sector debt renegotiation.

2. THE ONE-PERIOD MODEL

We begin by studying a simple one-period environment. There is one lender and one borrower.[8] The borrower is risk averse and derives utility from consumption according to the utility function u: $[0, +\infty) \to \bar{\mathbb{R}}$. The function u is continuous, strictly increasing and strictly concave and satisfies the Inada condition, $\lim_{c\to 0} u'(c) = +\infty$. The borrower has endowment (income) I that can be low or high, $I \in \{I_L, I_H\}$, where $0 < I_L < I_H$. The income is known to the borrower, but is unobservable to the lender. The prior belief of the lender that the income is high is denoted by γ, where $0 < \gamma < 1$. Alternatively, γ can be interpreted as the fraction of high-income borrowers.

The borrower has a debt obligation to the lender. We denote the level of debt that the borrower owes to the lender by D. For now, we take D as given. Focusing on the one-period setup with a given level of debt highlights the simplicity of the mechanism we propose. We endogenize the level of debt in a two-period model in Section 4.

The borrower can always fulfill her obligation to the lender by simply repaying D. But the lender can also offer her alternative payment arrangements, as we will explain later. As an alternative to making repayments to the lender, the borrower has an option of declaring bankruptcy. If the borrower declares bankruptcy, she receives utility $v(I)$, whereas the lender receives nothing. The function $v(I)$ is nondecreasing in I, and $u(0) < v(I) < u(I)$ for all I. Furthermore, to keep the exposition simple, we will assume that $I - u^{-1}(v(I))$ is strictly increasing in I, which will guarantee that the borrower with the high endowment has a strictly higher willingness to pay to avoid bankruptcy than the borrower with the low endowment.

Standard examples of the function v (that satisfy all of the above assumptions) are $v(I) = \theta$, where the value of bankruptcy is independent of the borrower's income, $v(I) = u(I(1 - \theta))$ with $\theta \in (0, 1)$, where the borrower loses a fraction of her income in the event of bankruptcy, and $v(I) = u(I) - \theta$, where $\theta > 0$ is the "stigma" cost of bankruptcy.[9]

[8] We can alternatively assume that there are many borrowers.

[9] An earlier version of this article established that our analysis can be equivalently applied to the problem where I is observable to everyone, but the borrower's bankruptcy cost (such as the parameter θ in the above examples) is only known to the borrower.

The lender is risk neutral and maximizes the expected repayment that he extracts from the borrower. We assume that the lender makes a take-it-or-leave-it offer to the borrower. An offer consists of a menu of contracts, where each contract—which can be deterministic or random—specifies how much the borrower should repay to the lender. A deterministic contract is simply an amount R that the borrower is asked to repay; a random contract is a lottery over repayments. The borrower chooses one contract from the offered menu or rejects all contracts. In the latter case (or if the borrower does not make the repayment specified in the contract he chose) she has to declare bankruptcy.

3. OPTIMAL ONE-PERIOD CONTRACTS

In this section, we describe the optimal solution to the lender's problem for a given level of debt D. Before considering possible contracts that the lender can offer, it is useful to define R_j—the largest amount that a borrower with income I_j is willing to pay to avoid bankruptcy. This payment solves

$$u(I_j - R_j) = v(I_j), \tag{1}$$

where $j \in \{L, H\}$. Notice that, because of our assumption that $I - u^{-1}(v(I))$ is increasing in I, the "willingness to repay" of the low-income borrowers is lower than that of the high-income borrowers: $R_L < R_H$. We will refer to the borrower with the high (low) income as the "high type" ("low type").

It is convenient to start by analyzing the case of debt overhang, defined as the situation where the debt is so large that it does not restrict the contracts that the lender may offer. After characterizing the optimal contracts in this simpler case, we will return to the original problem where repayments may be constrained by the amount of debt.

3.1. *One-Period Contracts: The Debt Overhang Case.*

3.1.1. *Deterministic contracts.* Suppose first that the lender is restricted to offering a single deterministic contract. Depending on the level of the demanded repayment, denoted by R, three situations may arise. If $R \leq R_L$, then both types of borrowers will accept the contract. If $R \in (R_L, R_H]$, then only the high type will accept the contract, whereas the low type will prefer to declare bankruptcy. Finally, if $R > R_H$, no borrower will accept the contract. Therefore, to maximize the expected repayment, the lender will offer either $R = R_L$ or $R = R_H$. We will refer to the first alternative as "pooling," as it attracts both types of borrowers, and to the second one as "exclusion," as it excludes—that is, forces into bankruptcy—the low-income borrowers.

Which of the two contracts generates higher profits to the lender depends on the parameters of the model, namely, the fraction of the high-income borrowers, γ, and the extent to which R_L and R_H are different from each other. Specifically, the lender prefers exclusion to pooling whenever $\gamma > R_L/R_H$, where the values on the right-hand side are completely pinned down by the primitives of the model (see Equation (1)).

3.1.2. *Random contracts.* Since a deterministic contract specifies only a repayment, it is impossible to offer a menu of deterministic contracts and have different types of borrowers accepting different contracts. However, the lender may be able to achieve this by offering a menu of random contracts. We refer to this case as "screening," as the lender uses lotteries to screen the borrowers of different types. As we only have two types of borrowers, we can, without loss of generality, limit the analysis to just two random contracts.

Since different types of borrowers have different willingness to pay to avoid bankruptcy, assigning a positive probability to bankruptcy allows the lender to screen the borrowers. The following proposition establishes exactly how the lender implements screening.

Proposition 1. *The menu of contracts that maximizes the lender's profits in the debt-overhang case consists of a deterministic repayment $R_S \in [R_L, R_H]$ aimed at the high-income borrower and a lottery aimed at the low-income borrower. The lottery offers R_L with probability $p \in [0, 1]$ and results in bankruptcy (i.e., offers any repayment above R_H) with probability $(1 - p)$.*

Proof. See the Appendix.

The proposition establishes three key points. First, it is never optimal to ask the low-type borrower to repay anything smaller than R_L. Essentially, setting the low type's repayment to R_L maximizes the repayment extracted from the low type and also minimizes the attractiveness of this contract to the high type. Second, the high-income borrower must be offered a deterministic repayment, which we denote by R_S. Intuitively, since the borrower is risk averse and is willing to pay to avoid a lottery, a certainty equivalent of any lottery offered to the high type generates a higher revenue to the lender. Third, to prevent the high type from taking the contract meant for the low type, the latter is a lottery that sends the borrower to bankruptcy with some probability. We denote the lottery offered to the low type by (R_L, p).

Note that the only reason for p to be set strictly below 1 is to keep the high-type borrowers from accepting the contract meant for the low-type borrowers. Indeed, profit maximization requires the deterministic repayment R_S to be such that the high type is just indifferent between the two contracts. That is,

$$u(I_H - R_S) = pu(I_H - R_L) + (1 - p)v(I_H) = pu(I_H - R_L) + (1 - p)u(I_H - R_H), \tag{2}$$

where the second equality follows from (1). Clearly, $R_S < R_H$ as long as $p > 0$, as offering the lottery will prevent extracting the full surplus from the high-income type. Also, $R_S > R_L$ as long as $p < 1$, for otherwise the high-income borrower's incentive constraint is lax, and the lender could increase expected repayment by increasing R_S.

The lender's problem is then simply to choose p to maximize the expected repayment,

$$\max_{p\in[0,1]} \gamma R_S(p) + (1 - \gamma)pR_L, \tag{3}$$

where $R_S(p)$ is given by (2). Notice that choosing $p = 1$ and $p = 0$ corresponds to the pooling and exclusion cases, respectively. Therefore, the lender's problem is fully captured by the maximization problem (3) subject to constraint (2). Equation (2) and strict concavity of the utility function imply that $R_S(p)$ is strictly concave in p. Thus, the objective function in (3) is strictly concave in p, and the problem has a unique solution, which we denote by p^*. The corresponding repayment made by the high-income borrower, $R_S(p^*)$, is denoted by R_S^*. We summarize the above discussion in the following corollary.

Corollary 1. *The repayment scheme that maximizes the lender's profits in the debt-overhang case is to offer a menu consisting of a deterministic repayment R_S^* and a lottery (R_L, p^*), where p^* solves* (3) *subject to* (2).

3.1.3. *Screening and risk aversion.* We have described three possible strategies that the lender may follow: pooling, exclusion, and screening. Given the focus of the article, the screening scenario is the most interesting of the three. Then the question arises: Does the lender ever use screening—that is, chooses $p \in (0, 1)$—in equilibrium?

Interestingly, if borrowers were risk neutral, lotteries (and hence screening) would never be utilized in equilibrium. To see this, notice that with a linear utility function, Equation (2) reduces to $R_S = pR_L + (1 - p)R_H$, and the lender's problem becomes

$$\max_{p\in[0,1]} pR_L + (1 - p)\gamma R_H.$$

Notice that the profit in the objective function is simply a linear combination of the profits under pooling and exclusion. That is, screening is always dominated by either pooling or exclusion (strictly so unless $R_L = \gamma R_H$). Thus, the lender does not benefit from using random contracts.

With risk-averse borrowers, however, there are parameter values for which screening gives the lender a strictly higher payoff than the pooling and exclusion alternatives. This happens, for example, when $R_L = \gamma R_H$. At that point, the lender is indifferent between pooling and exclusion, as well as any screening menu consisting of the lottery (p, R_L) and the deterministic offer $\bar{R}(p) = pR_L + (1-p)R_H$. Note that the low-type borrowers are not affected by the riskiness of the lottery, as both outcomes generate the same utility for them (equal to their value of bankruptcy). Note further that a risk-neutral high-income borrower would have been indifferent between the lottery (p, R_L) and the deterministic offer $\bar{R}(p)$. A risk-averse high-income borrower, however, strictly prefers the latter, and thus the lender is able to extract a higher payment $R_S(p) > \bar{R}(p)$ from her. As a result, the expected repayment is maximized by choosing some interior $p \in (0, 1)$.

Of course, there are parameter values for which either pooling or exclusion would be the lender's optimal strategies. In particular, exclusion (pooling) is attractive when γ is high (low) enough.

3.2. *One-Period Contracts: The General Case.* So far, we have characterized the optimal contracts in the case of debt overhang, when debt is so high that its level does not constrain the lender and is thus irrelevant for equilibrium repayments. We now turn to the general case with an arbitrary level of debt. Recall that the borrower always has an option to repay D to the lender. Thus, the level D imposes a restriction on how high a repayment the lender can demand in equilibrium. The following proposition describes the optimal contract in this case.

PROPOSITION 2. *Given a debt level D, the menu of contracts that maximizes the lender's profits consists of a deterministic repayment* $R_S^D = \min\{R_S^*, D\}$ *aimed at the high-income borrower and a lottery aimed at the low-income borrower. The lottery offers* $R_L^D = \min\{R_L, D\}$ *with probability* $p^D \in [0, 1]$ *and results in bankruptcy (i.e., offers any repayment above* R_H*) with probability* $(1 - p^D)$.

PROOF OF PROPOSITION 2. If $D \geq R_S^*$, then the debt level does not restrict the lender's problem, as both types of borrowers prefer their contracts in the debt-overhang case to repaying D. On the other hand, if $D \leq R_L$, then the outside option of bankruptcy becomes irrelevant, as all borrowers prefer simply repaying the full face value of debt. It is easy to verify that the lender can do no better than simply offer a deterministic repayment D to all borrowers.

The more interesting case of $D \in (R_L, R_S^*)$ closely resembles the debt-overhang case. First, it is never optimal to ask the low type for any repayment smaller than R_L. Combined with the previous point, this yields $R_L^D = \min\{R_L, D\}$. Second, it is never optimal to offer a lottery to the high type.[10] Hence, given a debt level D, the lender's problem can be written as

$$\max_{p \in [0,1], R_S^D} \gamma R_S^D + (1-\gamma) p R_L^D, \tag{4}$$

$$\text{s.t.} \quad u\left(I_H - R_S^D\right) = pu\left(I_H - R_L^D\right) + (1-p)u\left(I_H - R_H\right), \tag{5}$$

$$R_S^D \leq D. \tag{6}$$

[10] The arguments from the proof of Proposition 1 that establish these results apply directly to this case.

Note that for $D \geq R_L$, this problem is obtained by adding constraint (6) to the problem (3) subject to (2). Since the original problem is convex, the new constraint either has no impact (when $D \geq R_S^*$) or binds in the optimum. That is, if $D < R_S^*$, then $R_S^D = D$.[11] Combining these observations yields $R_S^D = \min\{R_S^*, D\}$. ■

Let p^D denote the lender's optimal choice of p in the generalized problem (4)–(6). Note that when constraint (6) binds, p^D is pinned down by Equation (5):

$$p^D = \frac{u(I_H - D) - u(I_H - R_H)}{u(I_H - R_L^D) - u(I_H - R_H)}. \tag{7}$$

When $D \leq R_L$, the above problem simply delivers $R_S^D = D$ and $p^D = 1$, which is a pooling contract where all borrowers fully repay their debt. When $D \geq R_S^*$, (6) does not bind, and thus the solution is the same as in the debt-overhang case, $R_S^D = R_S^*$ and $p^D = p^*$. The more interesting case is when $D \in (R_L, R_S^*)$.[12] In this case $R_L^D = R_L$, and, since (6) binds, $R_S^D = D$. Furthermore, as $R_L < D < R_S^* \leq R_H$, Equation (7) implies $p^D \in (0, 1)$. That is, the lender performs screening, where the high-income borrowers fully repay their debt, whereas the fraction p^D of delinquent borrowers receives an offer with a lower repayment of R_L. In particular, $p^D > 0$ means that the constrained lender never performs exclusion. Intuitively, suppose the lender chooses to perform exclusion under debt overhang (i.e., he extracts R_H from the high-income borrowers). He does not find screening attractive because the expected repayment from the low-income borrowers is not enough to offset the decrease in the repayment from the high-income borrowers. But when the debt level is between R_L and R_H, the lender can only extract D from them anyway (i.e., $R_S^D = D$). Therefore, he might as well offer R_L to the low type as long as the probability of doing so is low enough that the high type is still willing to repay D.

Note also that since p^* solves (7) when $D = R_S^*$, the probability p^D for $D \in (R_L, R_S^*)$ always exceeds p^*. In other words, a constrained lender sends a smaller fraction of borrowers to bankruptcy than an unconstrained lender. Moreover, for $D \in (R_L, R_S^*)$, Equation (7) immediately implies that p^D is strictly decreasing in D, so the higher the debt, the higher the probability of bankruptcy.

We summarize these findings in Corollary 2 below. Figure 1 further illustrates the results by depicting the types of contracts offered by the lender depending on the level of debt and on what he would have offered in the debt overhang case. The figure also plots the probability of bankruptcy $(1 - \gamma)(1 - p^D)$ as a function of the debt level D. In what follows, we will often refer to the probability of bankruptcy as the bankruptcy rate.

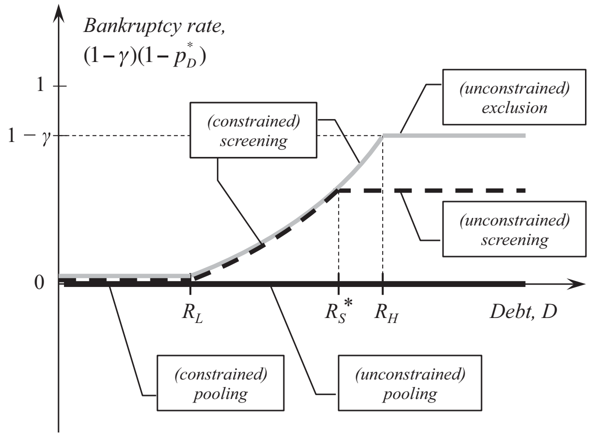
COROLLARY 2.

(i) *If $D \geq R_S^*$, then there is debt overhang, and the lender offers $(R_S^*, (R_L, p^*))$ that solves the unconstrained problem.*
(ii) *If $D \leq R_L$, then the lender demands repayment D, and all borrowers fully repay their debt.*
(iii) *If $D \in (R_L, R_S^*)$, then the lender performs screening. He offers $R_S^D = D$ to the high-income borrowers and (R_L, p^D) with $p^D > p^*$ to the low-income borrowers.*

To recap, when the face value of debt restricts how much the lender can extract from the borrower, the lender will never choose to go after the high-income borrowers only and will necessarily extract some repayment from the low-income borrowers. Furthermore, we obtain

[11] A mechanical proof can be obtained by contradiction: Suppose the solution to (4) subject to (5) and (6) is interior when $D < R_S^*$, that is, $R_S^D < D$. Consider an alternative menu that is a linear combination of $(R_S^D, (p^D, R_L^D))$ and $(R_S^*, (p^*, R_L^D))$ such that $R_S = D$. This linear combination yields higher profit to the lender, because $(R_S^*, (p^*, R_L^D))$ yields more than $(R_S^D, (p^D, R_L^D))$ and still satisfies all of the constraints. This contradicts the interior allocation being the solution of the profit maximization problem.

[12] Note that this interval is empty if the lender chooses pooling under debt overhang, that is, when $R_S^* = R_L$.



NOTES: The three lines correspond to different parameter values generating three possible cases obtained under debt overhang: exclusion (gray solid line), screening (black dashed line), and pooling (black solid line).

FIGURE 1

THE PROBABILITY OF BANKRUPTCY AS A FUNCTION OF THE DEBT LEVEL AND THE CORRESPONDING TYPES OF EQUILIBRIUM CONTRACTS

simple sufficient conditions under which screening is part of the optimal contract. Specifically, this happens whenever an unconstrained lender would not choose pooling (a sufficient condition for which is $\gamma > R_L/R_H$) and the debt level is in the intermediate range (R_L, R_H).[13] (Recall that R_L and R_H depend on the primitives of the model only.)

Finally, note that the bankruptcy rate in the model is (weakly) increasing in the amount of debt. Specifically, Corollary 2 and Figure 1 illustrate that there are three "regions" of debt levels. When debt is sufficiently low ($D \leq R_L$), it is always repaid in full, and there is no bankruptcy in equilibrium. When debt is sufficiently high ($D \geq R_S^*$), its exact level is irrelevant and thus does not affect the bankruptcy rate within this region. For intermediate levels of debt, the bankruptcy rate is strictly increasing in debt. We summarize these findings in the following corollary.

COROLLARY 3. *The equilibrium bankruptcy rate,* $(1-\gamma)(1-p^D(D))$, *is increasing in the debt level D, strictly increasing for* $D \in (R_L, R_S^*)$.

3.3. *Sequential Interpretation of the Optimal Contract.* One of the central points of the article is that the simple screening mechanism described above generates the three stages of default in consumer credit—delinquency, renegotiation, and bankruptcy. In this subsection, we use a sequential setting to illustrate this point.

Suppose that instead of offering the two contracts simultaneously, the lender offers them sequentially. Assume also that the lender can commit ahead of time to (not) making offers. To be exact, he can commit to the probability of not making the second offer before the first offer is made. It is easy to see that under this assumption, the setup with sequential offers is equivalent to our original setup with simultaneous offers and that the lender's problem is still (4) subject to (5) and (6).

Consider the case when $D \in (R_L, R_S^*)$ and suppose the lender chooses screening. In the sequential setting, the optimal screening contract has the following interpretation: First, the lender asks the borrowers to repay the debt in full (recall from part (iii) of Corollary 2 that $R_S^D = D$ in this case), which only the high-income borrowers agree to. We interpret the low-income borrowers who refuse to repay the debt in full as *delinquent*. Next, the lender offers

[13] If the lender chooses screening under debt overhang, he will also use screening for all $D > R_L$. If the lender chooses exclusion under debt overhang, he will use screening for $D \in (R_L, R_H)$.

a lower repayment to—that is, *renegotiates* with—delinquent borrowers, but only with some probability. The borrowers with whom the lender renegotiates reach debt settlement, while the rest are subjected to late fees and penalties and declare *bankruptcy*.[14]

Notice that the assumption of commitment is crucial here. Without it, the lender would want to renegotiate with all borrowers who refused to make the initial high repayment. Of course, anticipating this, no one would make the high repayment to begin with.

When the face value of debt is small enough ($D \leq R_L$) all borrowers fully repay their debt, and delinquency and bankruptcy are altogether avoided. On the other hand, when the debt level is excessively large (so that there is debt overhang, $D > R_S^*$), there is initial debt forgiveness for all borrowers, as the lender never asks the borrowers to repay D, only R_S^*.[15]

4. EXTENSION TO TWO PERIODS

Our analysis so far has taken the level of debt as given. The goal of this section is to present a simple two-period framework that endogenizes borrowing in the first period and has our basic mechanism at work in the second period. We offer this analysis in two different contractual environments—one with standard debt contracts and one where lenders can ex ante commit to arbitrary menus of contracts for the future.

Consider an environment with one borrower and several identical lenders. There are two periods, $t = 1, 2$. Assume for simplicity that the borrower's endowment in period 1 is zero. Her endowment in period 2 is random—it equals I_H with probability γ and I_L with probability $(1 - \gamma)$. The borrower's endowment in period 2 is unknown to everyone in period 1 (i.e., at the time of contracting). Once the uncertainty is realized in period 2, the endowment is known to the borrower but not the lenders.

The borrower discounts time with the discount factor $\beta > 0$ and maximizes her expected discounted utility of consumption in the two periods. The lack of endowment in period 1 makes the borrower want to borrow against her future income. She can borrow from one of several competitive lenders, who maximize expected present discounted value of profits in the two periods and discount future profits at rate r.

4.1. *Standard Debt Contracts.* We first consider a situation where contracting in period 1 is restricted to specifying a transfer of resources to the borrower in period 1, denoted by c_1, and the face value of debt inperiod 2, D.[16] The lenders compete in period-1 contracts (c_1, D) that they offer to the borrower, and the borrower picks one contract or rejects all contracts (the latter option means living in autarky). In period 2, once uncertainty is realized, the borrower and the lender, whose contract the borrower accepted in period 1, interact in the environment described in Section 2 given the debt level D.

Competition between the lenders drives their expected profits to zero, and thus the equilibrium debt contract maximizes the borrower's expected discounted utility,

$$\max_{(c_1, D)} u(c_1) + \beta\left[\gamma u\left(I_H - R_S^D(D)\right) + (1-\gamma)\left(p^D(D) u\left(I_L - R_L^D(D)\right) + \left(1 - p^D(D)\right) v(I_L)\right)\right],$$

[14] If there are more than two types of borrowers, settings with simultaneous and sequential offers are no longer equivalent. Nevertheless, the generalization to more than two types is straightforward, as we demonstrate in the Appendix. In the sequential setting, the lender screens different types by making offers with progressively lower repayments that are advanced to delinquent borrowers with a progressively lower probability. Thus, a borrower with a lower income will have a longer expected delinquency duration and a higher probability of bankruptcy than a borrower with a higher income.

[15] In this case, we will call delinquent borrowers refusing the repayment of R_S^*. However, in the two-period model that endogenizes the choice of D, described in the next section, choosing any $D > R_S^*$ yields exactly the same allocation as $D = R_S^*$. Thus, without loss of generality, we can say that $D > R_S^*$ does not occur in equilibrium.

[16] In other words, c_1 is the amount borrowed and D/c_1 is the gross interest rate. Note that the interest rate depends on the loan size.

subject to the lenders' break-even condition

$$c_1 = \frac{\gamma R_S^D(D) + (1-\gamma)p^D(D)R_L^D(D)}{1+r},$$

where arguments reflect the dependence of the repayment scheme on the debt level D. Let (c_1^*, D^*) denote the solution to this problem.

The type of contract offered by the lender in period 2 will depend on the level D^* as described in Corollary 2. In particular, so long as $D^* \in (R_L, R_S^*)$, the optimal contract in period 2 is (constrained) screening involving delinquency, renegotiation, and bankruptcy. And if $D^* \geq R_S^*$, screening occurs in equilibrium for some parameter values as described at the end of Subsection 3.2. The borrower will find it optimal to borrow a high enough amount (so that the corresponding debt level is above R_L) if she is sufficiently impatient, $v(I_L)$ is not too low, and/or I_H is high enough. Thus, the screening mechanism we are emphasizing is used by the lender in period 2 given the debt level endogenously determined in period 1.

4.2. *Ex Ante Optimal Contracts.* The restriction imposed on the structure of contracts in Subsection 4.1 is not without loss. It assumes that lenders cannot commit in period 1 to the period-2 repayment scheme. That is, they maximize the expected repayment in period 2 subject to the level of debt. In this section, we consider ex ante optimal contracts, where, unlike in the previous setup, the lenders can commit in period 1 to the repayment terms in period 2. A contract offered by each lender now consists of a loan size c_1 and a menu of repayments (that the borrower will choose from) in period 2. The key difference with the previous setup is that now the lender can commit not to extract resources from the low-income state while still extracting resources from the high-income state. Notably, since the contract is still subject to adverse selection in period 2, the main features of the optimal mechanism remain the same. As we will demonstrate, the key insights of our previous analysis carry over to this alternative contractual environment.

We characterize the repayment menu in the ex ante optimal contract in two steps, as captured in the following two lemmata: Lemma 1 shows that the optimal contract still prescribes a deterministic repayment to the borrower with the high income.

LEMMA 1. *The optimal scheme assigns a deterministic repayment $R_S^o \leq R_H$ to the high-income borrower.*

PROOF. See the Appendix.

Next, suppose the lender wants to screen different types of borrowers and thus offers a lottery to the low type. Recall from Propositions 1 and 2 that, with standard debt contracts, utility of the low-income borrower was always equal to her reservation utility $u(I - R_L)$. With ex ante optimal contracts, the lender may want to extract less than R_L from her. Except for this difference, the structure of the lottery offered to the low-income borrower under the ex ante optimal contract is the same as under standard debt contracts. In particular, Lemma 2 shows that the lottery is still over only one repayment that the borrower is willing to make and the bankruptcy option. Unlike with standard debt contracts, this result requires an assumption that the borrower's preferences exhibit DARA.[17]

ASSUMPTION 1. *The borrower's preferences exhibit DARA.*

LEMMA 2. *Suppose Assumption 1 holds. Then the lottery aimed at the low-income borrower does not assign positive probability to more than one repayment that the borrower is actually willing to make.*

[17] We mean DARA in the weak sense that includes constant absolute risk aversion as a special case.

PROOF. See the Appendix.

The proof of Lemma 2 goes along the following lines: Since the high type is offered a deterministic repayment (by Lemma 1), the lottery can only be aimed at the low type. The assumption of DARA implies that the low type's willingness to pay to avoid a lottery exceeds that of the high type. Hence, if the random contract aimed at the low type were to involve a sublottery over repayments actually made in equilibrium, replacing that sublottery with its certainty equivalent would be incentive compatible and increase the ex ante utility of the borrower.

The above lemmata imply the following result.

PROPOSITION 3. *Suppose Assumption 1 holds. Then the period-2 repayment scheme of the ex ante optimal contract consists of a deterministic repayment* $R_S^o \leq R_H$ *aimed at the high-income borrower, and a lottery aimed at the low-income borrower. The lottery offers* $R_L^o \leq R_S^o$ *with probability* $p^o \in [0, 1]$ *and results in bankruptcy with probability* $(1 - p^o)$.

PROOF. See the Appendix. ■

Having established the properties of the optimal repayment menu, we can now solve for the full lending contract by maximizing the ex ante utility of the borrower subject to the lenders' break-even condition, that is,

$$\max_{R_S^o, R_L^o, p^o} u\left(\frac{\gamma R_S^o + (1-\gamma) p^o R_L^o}{1+r}\right) + \beta\left[\gamma u\left(I_H - R_S^o\right) + (1-\gamma)\left(p^o u\left(I_L - R_L^o\right) + (1-p^o) v(I_L)\right)\right],$$

and subject to the borrower's period-2 incentive constraint,

$$u\left(I_H - R_S^o\right) = p^o u\left(I_H - R_L^o\right) + (1-p^o) u(I_H - R_H).$$

To summarize, the properties of the ex ante optimal contract are very similar to those of the optimal standard debt contract. The only difference is that the repayment made by the low type can in principle be lower than their willingness to pay. In fact, R_L^o is equal to R_L—and thus, the ex ante optimal contracts and the optimal standard debt contract are exactly the same—if the borrower is sufficiently impatient and/or $v(I_L)$ is not too low and I_L is high enough.

4.3. *No Screening without the Outside Option.* In addition to establishing robustness of the properties of our optimal repayment mechanism, analyzing ex ante optimal contracts allows us to make another important point.[18] The key feature of our optimal mechanism is that the lender uses lotteries to screen borrowers with different income levels. It is possible to do so because different types of borrowers value (the risk of) taking the outside option differently. But is the presence of the outside option really essential for screening? In particular, consider a situation where there is no bankruptcy option.[19] Specifically, suppose that the borrower can be forced to repay any amount up to her endowment level (which is still unobservable by the lender).[20]

Since borrowers with different levels of income have different attitude toward risk, their valuation of lotteries over repayments also differs. Thus, even in the absence of the outside option, it is *feasible* to use lotteries to screen different types of borrowers. But is it ever *optimal* to do so? The answer is "no."

[18] This point also applies to the environment with standard debt contracts but is rather obvious there.

[19] This setting captures some key aspects of the world with debtors' prisons, which existed before the institution of bankruptcy was introduced. In fact, Mann (2002) points out that debtors' prisons were sometimes used to elicit repayment from borrowers whose ability to repay was not observable by lenders (or a court).

[20] If the borrower agrees to a repayment that is greater than her endowment, she transfers the entire endowment to the lender and consumes zero.

Proposition 4. *Suppose that Assumption 1 is satisfied, and the borrower's outside option is $v(I) = u(0) = -\infty$. Then it is never optimal to use lotteries over repayments to screen borrowers with different incomes. In fact, the ex ante optimal contract prescribes the same repayment $R < I_L$ for both types.*[21]

Proof. See the Appendix.

Intuitively, with DARA preferences, the high-income borrower is more willing to bear risk than the low-income borrower. So, to separate the two types, one would want to impose (more) risk on the high-income type instead of the low-income type (which is the opposite of the optimal contracts described in this article). But since it is optimal (from the ex ante perspective) to repay more in the high- than in the low-income state, it is the incentive constraint of the high- and not the low-income borrower that binds in equilibrium. As a result, it is never optimal to impose risk on the high type (i.e., there is "no distortion at the top").

To summarize, even though it is feasible to screen different types of borrowers in the absence of the outside option, it is never optimal to do so. That is, the presence of the bankruptcy option is *essential* for screening and for the optimal contract to involve *delinquency, renegotiation*, and *bankruptcy*.[22]

5. APPLICATIONS: DEBT RESTRUCTURING AND FORECLOSURES

In this section, we adapt our model to analysis of mortgages and foreclosures. The purpose of this is twofold: to tie the model predictions to empirical observations and to highlight the importance of endogenous debt restructuring for studying the mortgage market and especially the recent housing crisis.

Reinterpreting our model in the context of mortgages (and secured debt in general) is straightforward—it simply comes down to subtracting the value of the collateral (house) from the mortgage. Specifically, let H denote the market value of the borrower's house and let M be the size of her mortgage. Then D in our model corresponds to the "underwater" portion of the mortgage $(M - H)$, and bankruptcy simply corresponds to foreclosure.

Our model matches some key empirical regularities. In particular, our results are consistent with the so-called "double-trigger" phenomenon documented, for example, by Foote et al. (2008) and Herkenhoff (2012b). These studies show that foreclosures are associated not just with negative home equity but also with a negative shock to the homeowner's income, for example, due to job loss. This is consistent with our result that foreclosures happen only if the underwater part of the mortgage is large enough ($D > R_L$) and if the borrower has the low income realization. Moreover, our model reproduces an empirical finding that foreclosure rates increase with the negative equity. (Recall from Corollary 2 that the probability of renegotiation is decreasing in D, and thus the probability of bankruptcy is increasing in D.)

The application of the model to mortgages helps highlight the importance of expilicitly modeling renegotiation. The endogeneity of the extensive margin (probability) of renegotiation, which is a crucial feature of our mechanism, is essential for capturing a number of important phenomena related to mortgage restructuring and foreclosures. We highlight two such phenomena in this section. First, we show how the endogenous renegotiation can generate amplification of aggregate house-price shocks in the presence of externalities. Second, we demonstrate that

[21] It is easy to see that under standard debt contracts (and preferences with or without DARA) in the absence of the outside option, it is always optimal for the lender to demand the full repayment D in period 2, so that the borrower with income I_j repays $\min\{D, I_j\}$. Then the lending contract that maximizes the borrower's ex ante utility is (c_1, D), where the debt level $D < I_L$ is always repaid, and, hence, $c_1 = D/(1 + r)$.

[22] Note that having the bankruptcy option can improve the ex ante welfare of the borrower. The presence of the outside option allows the borrower to transfer a larger amount of resources from the high state in period 2 to period 1, without immiserating herself in the low state in period 2. This point is reminiscent of the result that default can be welfare improving, which goes back to Zame (1993) and Dubey et al. (2005).

accounting for the endogenous renegotiation is critical in evaluating effects of a government intervention in debt restructuring.

5.1. *Amplification of House-Price Shocks and Foreclosures through Externalities.* A number of studies have argued that foreclosure imposes externalities on other homeowners— foreclosed houses in a neighborhood depress prices of other houses in that neighborhood (see, e.g., Harding et al., 2009; Campbell et al., 2011; Calomiris et al., 2013; and Hartley, 2014). In our model, this externality interacts with the endogenous mortgage renegotiation in an important way, leading to amplification of aggregate house-price shocks.

We introduce externality into our model by considering a large number of ex ante identical borrowers and by making the price of the (representative) house, denoted by H, a function of the economy-wide foreclosure rate f. That is, the more borrowers foreclose, the lower is the value of a house for each borrower. Specifically, let $H = Q(f)\varepsilon$, where Q is a decreasing function and ε is a house-price shock. We will think of the economy's "normal times" as having $\varepsilon = 1$ and a negative shock as having $\varepsilon < 1$.

For simplicity, we restrict our attention to the period-2 contracting described in Sections 2 and 3. Contracting in each lender–borrower relationship takes the market value of the borrower's house as given and, given D, determines the probability of renegotiation p^D as described in problem (4)–(6). The value of D is in turn affected by the average (over all borrowers) value of p^D through the economy-wide foreclosure rate $f = (1-\gamma)(1-p^D)$. That is, for a given value of ε, the value p^{D_ε} solves

$$p^{D_\varepsilon} = \begin{cases} 1, & \text{if } D_\varepsilon \le R_L, \\ \frac{u(I_H - D_\varepsilon) - u(I_H - R_H)}{u(I_H - R_L) - u(I_H - R_H)}, & \text{if } D_\varepsilon \in (R_L, R_S^*), \\ 0, & \text{if } D_\varepsilon \ge R_S^*, \end{cases} \tag{8}$$

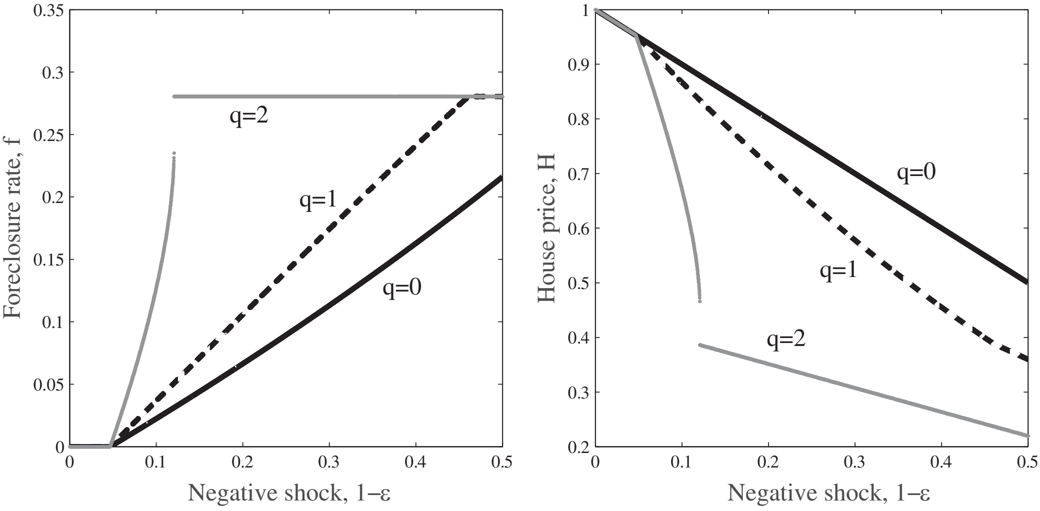
where

$$D_\varepsilon = M - Q((1-\gamma)(1-p^{D_\varepsilon}))\varepsilon. \tag{9}$$

Now consider the effects of a negative house-price shock in this environment; that is, let ε drop from one to some level below one. For simplicity, consider the case where $M \le R_L + Q(0)$; that is, there are no foreclosures in the absence of the price shock. Consider the following two scenarios: In the first scenario, suppose that the negative shock only affects one individual borrower. For example, a homeowner discovered mold in the basement of her house, which reduced the price of her (and only her) house. The negative shock leads to an increase in D, which in turn increases the probability of foreclosure for the affected borrower (i.e., her individual p falls). But since each borrower has a negligible effect on the average foreclosure rate, there are no further effects.

Now consider the second scenario, where the negative price shock is economy-wide, so that the value of each borrower's house falls. The initial effect is the same as in the first scenario, except every borrower is now affected. But since the probability of foreclosure increases for all borrowers, the house prices fall even further due to the externality. This has a further effect on the probability of foreclosure for each borrower, which further lowers prices, and so on and so forth. The new equilibrium foreclosure rate that solves (8) is a fixed point of this process. Thus, this simple extension of our model illustrates how in the presence of externality, the endogenous renegotiation *amplifies* the effect of a negative house-price shock on the foreclosure rate.

Figure 2 graphically illustrates the amplification using a numerical example. In all computations presented in this section we use logarithmic utility function $u(c) = \ln c$ and the value of bankruptcy given by $v(I) = u(I(1-\theta))$, where $\theta \in (0, 1)$. The parameters for this example are listed in the note to the figure, and they not calibrated in any way other than to permit comparisons across different cases. We set the deterministic component of the market value of the house to be $Q(f) = 1 - qf$, where $q \ge 0$ is a parameter that captures the strength of the



NOTES: Parameter values: $\theta = 2/3$, $I_H = 2.5$, $I_L = 1$, $\gamma = 0.4$. The initial (before the shock) level of debt is set to the optimal period-1 choice of D given $\beta = 0.9$ and $r = 0.05$ and is equal to $D = 0.62$, which is smaller than $R_L = 2/3$. The value of the mortgage is $M = D + 1$.

FIGURE 2

THE FORECLOSURE RATE AND THE VALUE OF THE HOUSE AS FUNCTIONS OF THE NEGATIVE SHOCK, $(1 - \varepsilon)$, FOR DIFFERENT VALUES OF THE STRENGTH OF THE EXTERNALITY, q

externality. The figure plots the foreclosure probability (rate) and the value of the house H as functions of the negative shock calculated as $(1 - \varepsilon)$ for different parameter values of q, namely, $q = 0$ (no externality), $q = 1$ (weaker externality), $q = 2$ (stronger externality). The $q = 0$ case corresponds to the idiosyncratic price shock (the first scenario discussed earlier), while with $q > 0$ we have the responses to the economy-wide shock with externalities.

The graphs show that our mechanism generates large amplification of the effects of negative house-price shocks both on the foreclosure rate and the house prices. Since the effects of the house prices on the foreclosure rate operate through increases in D, these effects are bounded by debt overhang. This is the top flat portion on the left panel of Figure 2. Note that externalities in the model may lead to multiple equilibria.[23] In the figure, we always plot the one with the lowest foreclosure rate. The switch to the debt overhang being the only equilibrium (outcome) is responsible for the discontinuity for $q = 2$. Just to the left of the discontinuity, there are two equilibria, one of which corresponds to the debt overhang. The other equilibrium, which is the one we depict in the figure, has a lower foreclosure rate than the debt overhang equilibrium. To the right of the discontinuity, the debt overhang is the only equilibrium outcome.

5.2. *Government Intervention in Debt Restructuring.* In this section, we demonstrate that understanding the workings of private debt renegotiation is crucial for analyzing effects of a government intervention in debt restructuring. Consider, for instance, a government intervention in the form of a mortgage modification program that aims at lowering the foreclosure rate. The motivation for the government intervention may come, for example, from the presence of externalities described in the previous subsection. One example of such a program is HAMP, introduced in the United States in 2009. We will analyze effects of a program of this sort through the lens of our model and show that the program may have unintended consequences if its design is naive and ignores the effects on private debt restructuring.

[23] A low (high) foreclosure rate can be self-sustaining as it generates low (high) level of D and thus a high (low) probability of renegotiation. That is, the system of equations (8)–(9) may have multiple solutions.

As a benchmark, it will be useful to introduce a government intervention in the absence of any externalities with the purpose of separating the effects of the intervention from the amplification mechanism described in the previous subsection. We will discuss most of our results for this simpler case and at the end will discuss how they are affected if externalities are included.

Within our framework, we will assume that the government steps in if bankruptcy is initiated, that is, if private renegotiation has been unsuccessful (i.e., did not take place). To keep the analysis simple, we model the intervention as the government making an offer to a delinquent borrower with probability $p_G \leq 1$ to make a repayment R_G. If the borrower accepts the offer and makes the repayment, the repayment is transferred to the lender.

We first analyze the impact of the government intervention in the one-period setting and then go on to investigate the effects of the government intervention on the endogenous borrowing decision and ex ante welfare in the two-period model in the case when the intervention is anticipated.

5.2.1. *Government intervention in the one-period model.* We begin by analyzing the simplest case where the government intervention is deterministic, that is, $p_G = 1$, and the borrower is in debt overhang.[24] We illustrate some of the results in this simple case, and then show that some additional insights can be obtained in the case of a random intervention and the endogenous debt level.

First, note that if the repayment R_G offered by the government exceeds R_H, then the intervention is completely irrelevant, because no borrower will ever want to make such a repayment. Thus, we can view the case of $R_G \geq R_H$ as the no-intervention benchmark.

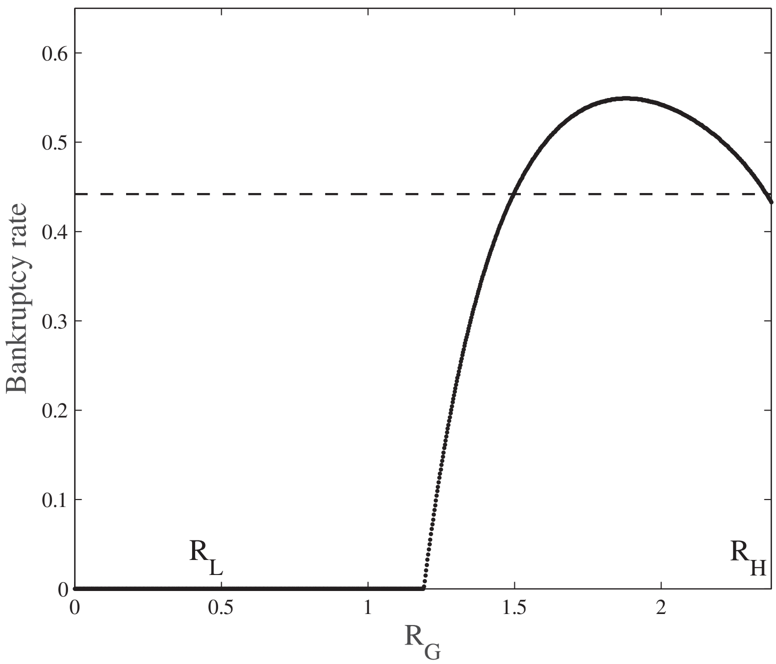
Consider next what happens if $R_G \leq R_L$, that is, if the government offers a repayment that is lower than the lender's offer to delinquent borrowers in absence of an intervention. Clearly, such an intervention constrains the lender because no borrower would accept a higher repayment knowing that she would be offered the more favorable R_G upon rejecting the lender's offer. Thus, the effect of the intervention in this case is similar to the effect of lowering the debt level to $D = R_G \leq R_L$: A pooling outcome is achieved (i.e., all borrowers repay R_G), and the bankruptcy rate inevitably drops to zero. Thus, in this case, the government policy is (trivially) effective, as it prevents all bankruptcies in equilibrium.

Finally, consider the less trivial case of $R_G \in (R_L, R_H)$, where the repayment offered by the government exceeds the willingness to pay of the low-income borrowers, but is acceptable to the high-income borrowers. In this case, the government intervention only restricts the lender's ability to extract repayment from the high-income borrowers.

Recall from Subsection 3.2 that when $D \in (R_L, R_S^*)$, the restriction that the borrower can always just pay the face value of debt forces the lender to renegotiate more often than he would have under debt overhang and thus reduces the bankruptcy rate. Since the lender's ability to extract repayment from the high type is limited anyway, he can extract repayment from a higher fraction of the low type without distorting the incentives of the high type. By analogy, one might infer that the government intervention with $R_G \in (R_L, R_H)$ would have a similar effect and reduce the bankruptcy rate. However, it turns out that the restriction imposed on the lender by the government is in fact quite different from the one imposed by the value of debt. As we show in the Appendix, the probability of renegotiation is generally nonmonotone in R_G. Most interestingly, it decreases, and thus the bankruptcy rate rises, in response to the government intervention if $R_G(< R_H)$ is close enough to I_H. That is, the government intervention aimed at preventing foreclosures may actually lead to an increase in foreclosures in equilibrium.

Figure 3 demonstrates the nonmonotonicity of the bankruptcy rate as a function of R_G when $p_G = 1$ using in a numerical example. The solid and dashed lines correspond to the bankruptcy rates with and without the government intervention, respectively. Since the government intervention is irrelevant when $R_G = R_H$, the two lines coincide at that point.

[24] With appropriate modifications, the analysis extends to the general case with a given debt level.



NOTE: Parameter values: $\theta = 0.95$, $I_H = 2.5$, $I_L = 0.5$, $\gamma = 0.35$. The debt-overhang case is depicted.

FIGURE 3

THE BANKRUPTCY (FORECLOSURE) RATE AS A FUNCTION OF R_G, WHERE $p_G = 1$

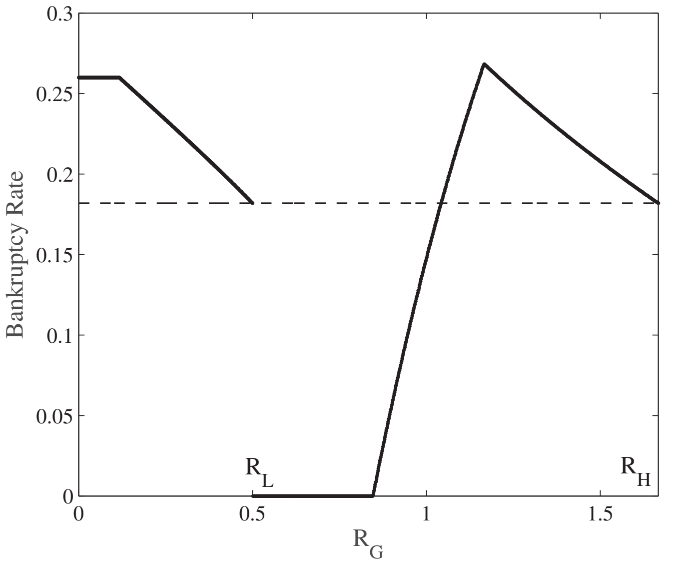
Two scenarios illustrated in Figure 3 are of particular interest. First, consider the case where R_G is greater than R_L but is sufficiently close to it. In this case, the government intervention is completely successful in preventing foreclosures, despite appearing irrelevant—the repayment offered by the government is greater than that offered by the lender, as the lender offers R_L to all borrowers. That is, in equilibrium the government is not actively involved in restructuring mortgages. Yet, absent the intervention, the bankruptcy rate would have been strictly positive.

The second scenario occurs when R_G is below R_H but is sufficiently close to it. In this case, the government intervention "backfires"—it leads to an increase instead of a decrease in foreclosures. As we have already pointed out, the government's offer is never accepted in equilibrium in this case. Later in this section, we present a case where the government intervention backfires even though borrowers, who receive the government's offer, accept it.

Next, we consider the case of a random government intervention, $p_G < 1$, which can be interpreted as the borrower not being certain whether she is eligible for the government program. We will illustrate two additional insights that we gather in this case that were absent in the case of the deterministic intervention.

The first insight is that private lenders may entirely offset the government intervention, and thus the policy is totally ineffective (i.e., it does not change the bankruptcy rate) although the government is busy preventing foreclosures. In the second scenario, the policy again backfires (leads to more foreclosures), but unlike in the case of the deterministic intervention, the government's offer is accepted by some borrowers.

These scenarios are illustrated in Figure 4. First, consider the case where $R_G = R_L$ and $p_G \leq p^D$ (which is the case presented in the figure). Without the intervention, the lender sets p equal to p^D. In the presence of the intervention, the lender simply adjusts the probability of renegotiation to offset the intervention, that is, $\hat{p} + (1 - \hat{p})p_G = p^D$. The resulting bankruptcy



NOTES: Parameter values: $\theta = 2/3$, $I_H = 2.5$, $I_L = 0.75$, $\gamma = 0.35$. The debt level is set to the optimal period-1 choice of D in the laissez faire case given $\beta = 0.9$ and $r = 0.05$ and is equal to $D = 0.93$, which is smaller than $R_S^* = 1.44$. The probability of the government's offer is $p_G = 0.6$, which is smaller than $p^D = 0.72$.

FIGURE 4

THE BANKRUPTCY (FORECLOSURE) RATE AS A FUNCTION OF R_G, WITH $p_G < 1$

rate, $(1 - \gamma)(1 - p^D)$, is the same as the laissez faire one, and thus the intervention is ineffective. In this case, the government is busy preventing foreclosures, but its net effect is exactly nil.[25]

Next, consider the case where $R_G < R_L$. Recall that when $p_G = 1$, such an intervention always leads to pooling, that is, reduces the bankruptcy rate to zero. This is not necessarily the case when $p_G < 1$. In fact, as Figure 4 shows, a random offer from the government with a repayment that is lower than that offered to delinquent borrowers by the lender ($R_G < R_L$) can lead to an increase in the number of foreclosures.[26] In this case, the government program once again backfires. The government is actively participating in reducing foreclosures, as its offers (of a lower repayment) are accepted in equilibrium by the delinquent borrowers. Yet, the foreclosure rate is higher than it would have been in the absence of the intervention.

Finally, when $R_G \in (R_L, R_H)$, the equilibrium bankruptcy rate can also be lower or higher than without intervention. Figure 4 illustrates both cases. Intervention with R_G slightly higher than R_L induces pooling (and prevents all bankruptcies), and intervention with higher R_G backfires, just like in the corresponding cases with deterministic intervention.

The results presented in this section indicate that explicit modeling of the private sector debt restructuring is key for analyzing the effects of a government intervention. In particular, the failure to understand how private lenders renegotiate with delinquent borrowers can lead to the policy having the opposite effect from the one intended.

The welfare effects in the one-period model are rather trivial. Since the government's offer restricts the lender's ability to extract repayment, the intervention makes the lender (weakly)

[25] Note that when $R_G > R_L$, the government's offer would not be accepted by the (delinquent) low-type borrowers. Thus, with $R_G > R_L$ the government is no longer effectively renegotiating on behalf of the lender but instead distorts the lender's ability to extract repayment from the high type without actually generating any revenue from the low type. For values of R_G just above R_L, this distortion leads the lender to switch to the pooling contract (just like in the deterministic-intervention case). This switch is reflected in the discontinuity at $R_G = R_L$ in Figure 4.

[26] The argument behind this result is similar to the argument behind the nonmonotonicity result in the case of the deterministic intervention except now the government's offer affects the lender's ability to extract repayment not only from the high type but also from the low type.

worse off. The high-income borrower is (weakly) better off with the intervention, while the low-income borrower is better off if $R_G < R_L$.[27]

Of course, in the presence of an externality (like the one described in the previous section), the intervention can be Pareto improving, but only if it lowers the equilibrium foreclosure rate. On the other hand, any backfiring would be exacerbated in the presence of externalities.

The effects of the intervention in the one-period model should be interpreted as effects of an *unanticipated* intervention. We think of it not just as a useful benchmark, but also as a very plausible empirical scenario. Thus, Figure 4 reports the effect of the intervention for the debt level that would have been chosen optimally in period 1 of the two-period model (if neither borrowers nor lenders anticipated the intervention). The next section takes the natural next step of studying the effects of the intervention that is fully anticipated.

5.2.2. *Government intervention in the two-period model.* Now consider the two-period model with standard debt contracts as presented in Subsection 4.1. If the government intervention is unanticipated ex ante, it does not affect the level of D chosen in the first period. But suppose the intervention is anticipated in period 1. We will show that the endogenous borrowing choice and ex ante welfare are then affected in a nontrivial way.

With the anticipated intervention, the ex post improvement in the borrower's utility comes at a cost. The lender anticipates that for a given debt level, his ability to extract repayment will be reduced due to the government intervention. As a result, the price of any level of debt decreases. That is, for a given D, the amount c_1 that lenders are willing to advance in period 1 (in exchange for the promise of D) is (weakly) lower with the government intervention than without it.

Does that mean that the borrower is necessarily worse off ex ante when the government intervention is anticipated? Recall that with standard debt contracts, the lenders cannot commit not to extract the maximum possible repayment in period 2. It is thus conceivable that the government intervention could mitigate the commitment problem by restricting the lender's ability to extract that repayment. However, the borrower does not need to rely on the government to restrict the repayment extraction—she can do so herself by simply borrowing less.

Indeed, one can show that no intervention with $R_G \geq R_L$ can be welfare improving ex ante. To see this, consider the allocation obtained as the equilibrium under such an intervention. Denote the corresponding debt level as D^G. The borrower could have replicated this allocation in the laissez faire setting by promising to repay D^G. The period-1 consumption level that the borrower would have obtained for that promise is (weakly) higher in the laissez faire setting. Thus, the proposed laissez faire allocation would have made the borrower at least as well off as the equilibrium allocation under the intervention.

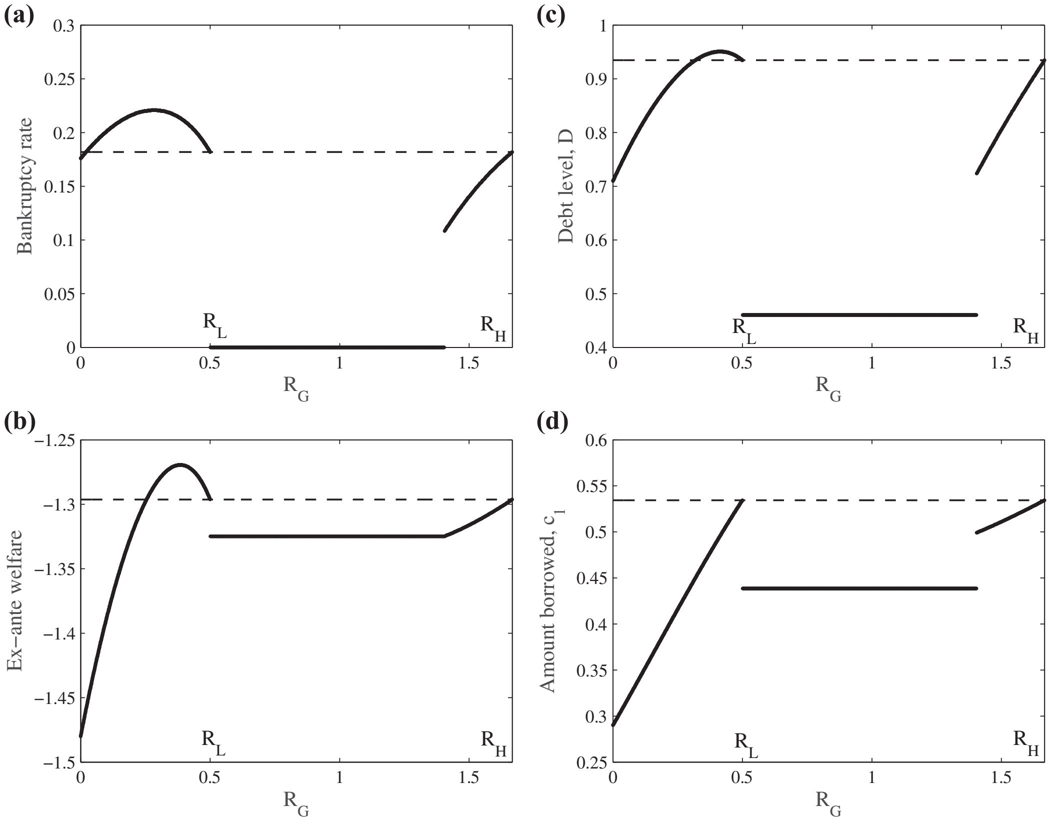
The intuition for this result is rather straightforward. As long as $R_G \geq R_L$, the government intervention does not (cannot) affect the ex post welfare of borrowers with the low income realization. Thus, any government intervention merely changes the ability of the borrower to transfer resources from the high income state in period 2 to period 1 (it does not improve the borrower's ability to transfer resources across states in period 2). But in the setting with standard debt contracts, the face value of debt (in the absence of the government intervention) successfully serves the same purpose.

This argument implies that an anticipated intervention may be ex ante welfare improving *only if* $R_G < R_L$ and $p_G < 1$.[28] In this case, the intervention allows the borrower to repay less than R_L in the low-income state (while repaying more in the high-income state), thereby improving her ability to transfer income across states in period 2.[29] That is, by doing something that the

[27] It is worth pointing out that, absent any externalities, in the one-period model a government intervention is never Pareto improving, because the equilibrium allocation is constrained Pareto efficient.

[28] Recall that a deterministic intervention with $R_G < R_L$ leads to pooling, which can also be replicated in the laissez faire case by simply setting $D = R_G$.

[29] The low-income borrower is offered a repayment below R_L by the lender because she is now tempted by the government's offer of $R_G < R_L$ (with probability $p_G < 1$), and hence her incentive constraint binds.



NOTES: The bankruptcy rate (panel a), ex ante welfare (panel b), debt (panel c), and period-1 consumption (panel d) as functions of R_G. Parameter values: $\theta = 2/3$, $I_H = 2.5$, $I_L = 0.75$, $\gamma = 0.35$, $\beta = 0.9$, $r = 0.05$, $p_G = 0.6$.

FIGURE 5

EFFECTS OF THE ANTICIPATED GOVERNMENT INTERVENTION

lender would never find optimal to do ex post (renegotiating the repayment of delinquent, low-income borrowers to a level below their willingness to pay, R_L), the government moves private contracting "closer" to the ex ante optimal contracts and may improve the borrower's welfare ex ante.[30]

Figure 5 illustrates the effects of the anticipated government intervention with $p_G < 1$ on the foreclosure rate (panel a), ex ante welfare (panel b), the level of debt D (panel c), and the borrowed amount c_1 (panel d). Parameter values (shown in the note to the figure) are the same as those in Figure 4. Two scenarios shown on the figure are of particular interest. First, when R_G is to the left of R_L and close enough to it, the intervention backfires (increases foreclosures) ex post, yet increases the borrower's welfare ex ante. It is associated with the borrower taking out a bigger loan (greater c_1) and promising to repay more (greater D). Second, for R_G to the right of R_L, the intervention can be successful at reducing foreclosures (sometimes eliminating them altogether), yet necessarily reduces welfare ex ante.

The first discontinuity on Figure 5 (the one at $R_G = R_L$) is a familiar one—the intervention is entirely undone at R_G equal to R_L and prevents all foreclosures just to the right of R_L. In this example, the intervention with $R_G > R_L$ close enough to R_L leads to pooling in period 2 (for any debt level), and thus the borrower can only borrow safely, that is, $D \leq R_L$. Inability to borrow (or, in other words, credibly promise to repay) more than R_L results in reduction in

[30] See the Appendix for the explicit statement of the problem of finding the optimal standard debt contract in the case of the government intervention with $R_G < R_L$ and $p_G < 1$.

the borrower's ability to move resources from period 2 to period 1, thus reducing her ex ante welfare. For larger values of R_G, the intervention is less restrictive ex post, and the borrower switches back to risky borrowing. This switch is reflected in the second discontinuity in this example.

The effects of the anticipated government intervention are quite similar in the presence of the externality, like the one described in Subsection 5.1. As we mentioned earlier, any ex post backfiring (i.e., an increase in the foreclosure rate) would be amplified. As a result, the welfare improvements from the intervention with $R_G < R_L$ seen in Figure 5 would be diminished by the welfare losses from the greater foreclosure rate. On the other hand, the welfare losses from intervention with $R_G > R_L$ would also be diminished, again due to the change in the foreclosure rate.

It is important to point out that the equilibrium response to the intervention in our model does not come through changing the borrower's incentive to default (become delinquent), but rather through changing the lender's incentives to renegotiate. Thus, the primary effect is through the intensive margin (of the probability) of renegotiation, instead of the extensive margin of delinquency. In contrast, existing papers, such as Benjamin and Mateos-Planas (2012) and Yue (2010), focus on how renegotiation affects the borrower's incentives to default.

6. CONCLUSIONS

We propose a simple model of consumer credit where a lender demands repayments from an indebted borrower, and the borrower's alternative to making a repayment is to declare bankruptcy. The main friction in the model is that the borrower's income is her private information.

We characterize the optimal contract in this environment. We show that the lender may choose to screen borrowers with different income levels using lotteries over repayments. The optimal screening contract has a natural economic interpretation as it generates three stages of default—delinquency, bankruptcy, and renegotiation. Specifically, the lender first offers a high repayment that only borrowers with the high-income accept. The low-income borrowers refuse to make this payment, and are thus considered delinquent. The lender then renegotiates by offering a lower repayment, but only with a fraction of the delinquent borrowers, while the rest end up in bankruptcy.

The application of the model to mortgages and foreclosures yields a number of interesting insights. First, our mechanism generates amplification of house-price shocks in the presence of externalities. Second, a government intervention in debt restructuring can have nontrivial consequences due to the endogenous response of the lenders. We show that a program aiming to reduce foreclosures that overlooks the response of the private debt restructuring may lead to an increase instead of a reduction in the bankruptcy rate. Yet, if such an intervention is anticipated at the loan origination stage, it may actually be welfare improving ex ante.

APPENDIX

A.1. *Omitted Proofs.*

PROOF OF PROPOSITION 1. The first observation that will be helpful in this proof is that the lender will extract at least as much (in expectation) from a high-income borrower as from a low-income borrower. If that were not the case, then the revenue could have been improved by offering only the low type's contract, as the high type accepts any repayment that the low type accepts, i.e., $R_H > R_L$.

The second basic observation is that the low-type borrower is made indifferent between her prescribed repayment and bankruptcy. Obviously, she would not accept anything that would yield lower utility than bankruptcy. Keeping her strictly above her outside option, on the other hand, is not optimal for the lender. If the low-type borrower is offered any repayment less than R_L, the revenue could be improved by raising that repayment to R_L. If this change makes the low-type borrower prefer the high type's contract, then the high type's contract was not revenue maximizing—it must have assigned positive probability to payments below R_L. The expected repayment could have been further increased by substituting the repayment lottery for the high type with its certainty equivalent (from the high type's perspective). The latter alteration leaves the high type's incentive constraint unchanged, makes the high type's contract unattractive to the low type, as the certainty equivalent is necessarily higher than what was collected from the low type (see the first observation above), and increases the expected revenue collected from the high type (due to the borrower's risk aversion). So, the only repayment that the low type may be making in the optimal scheme is R_L (though she may be induced to file for bankruptcy with a positive probability).

The next key point is that the optimal menu does not result in the high-income borrower facing any probability of bankruptcy. If it did, it could have been improved upon by simply assigning the probability of bankruptcy to repayment R_H. That improves the revenue collected and leaves all incentive constraints unchanged.

In fact, the high-income borrower does not face any uncertainty in the optimal menu. A menu with a lottery for the high type could be improved upon by replacing that lottery with its certainty equivalent (from the point of view of the high-type borrower). The new menu yields higher revenue (since the certainty equivalent is greater than expected revenue from the lottery due to the borrower's risk aversion), the incentive constraint of the high type is unaffected, and the incentive constraint of the low type is not violated either (if it were, the lender would been better off offering the new deterministic contract to both types).

Finally, if the lottery offered to the low type induces bankruptcy with positive probability, it does so by demanding repayment that *neither* type would be willing to make. While demanding $R' \in (R_L, R_H)$ would be sufficient to drive the low type to bankruptcy, asking for a larger repayment is not payoff equivalent—it makes such a lottery less attractive to the high-type borrower (i.e., relaxes the high type's incentive constraint), allowing the lender to successfully demand larger repayment from the high type.

We have thus established that the optimal repayment scheme consists of a deterministic repayment $R_S \geq R_L$ for high type and a lottery between R_L and an implausibly large repayment (inducing bankruptcy) for the low type. (Clearly, R_S cannot exceed R_H, for otherwise the (high-type) borrower would not be willing to make that repayment.) ■

Proof of Lemma 1. Suppose not, and the high-income borrower is offered a lottery. If the lottery is over bankruptcy and repayment, eliminate the probability of bankruptcy. If there are multiple repayments (weakly below R_H), then replace them with their certainty equivalent from the high-income borrower's point of view (denote that repayment by $\hat{R}_S$). The former makes the high-income borrower better off. The latter keeps the high-income borrower indifferent. Both generate strictly greater expected revenue for the lender in the high-income state. That translates into higher consumption of the borrower in period 1. Thus, we have found a contradiction to the lottery for the high-income borrower being a part of the optimal contract, as long as we can establish that the new allocation does not violate the low-income borrower's incentive constraint.

If the low type (strictly) prefers $\hat{R}_S$ to her prescribed allocation, we come to another contradiction. Either demanding $\hat{R}_S$ from all borrowers ex post is a welfare improvement (both ex post and ex ante) or the lenders collect more than $\hat{R}_S$ from the low-income borrowers (in expectation). But that means the lender was collecting more in expectation from the low- than from the high-income borrower. Then there is a welfare improvement on the candidate allocation that simply assigns the same deterministic repayment to all borrowers—make that repayment $R' = \gamma\hat{R}_S + (1-\gamma)\hat{R}_L$, where $\hat{R}_L$ is defined as a certainty equivalent of the repayments made by the low-income borrower (from her standpoint). This new pooling contract generates at least as much ex post revenue for the lender (and thus no lower consumption in the first period) and improves consumption smoothing in the second period across the two income states. The key to making this simple deviation possible is that $\hat{R}_S < R' < \hat{R}_L \leq R_L < R_H$, which implies that all borrowers would rather make the payment R' than go through bankruptcy. ■

PROOF OF LEMMA 2. Suppose not, and the lottery offered to the low-income borrower (as part of the optimal menu) assigns positive probability to multiple repayments that the borrower is willing to make. That is, the lottery includes $N > 1$ different repayment $\{R_1, \ldots, R_N\}$ that do not exceed R_L. Denote the probabilities assigned to these repayments by $\{p_1, \ldots, p_N\}$, respectively. Consider an alternative contract that replaces these elements of the lottery with a single repayment R' with probability p', where $p' = \sum_{i=1}^{N} p_i$ and R' is such that

$$p'u(I_L - R') = \sum_{i=1}^{N} p_i u(I_L - R_i).$$

(That is, R' is the certainty equivalent of the sub lottery over $\{R_1, \ldots, R_N\}$ evaluated from the low type's point of view.) Since the borrower is risk averse, this alternative lottery yields a greater expected repayment to the lender ($p'R' > \sum_{i=1}^{N} p_i R_i$) and thus greater first-period consumption and ex ante utility for the borrower. Moreover, under DARA, the new contract is incentive compatible, as the high-income borrower does not find the low type's new contract any more attractive than the low type's old contract. This is because by the definition of DARA, the high-income borrower is not willing to pay as much to get rid of the uncertainty of the sub-lottery as the low-income borrower.[31] Thus, the original lottery could not have been part of the optimal menu.[32] ■

PROOF OF PROPOSITION 3. Lemma 1 characterized the (deterministic) contract aimed at the high type, and Lemma 2 established basic properties of the contract aimed at the low type in the optimal repayment menu. The only remaining claim, that $R_L^o \leq R_S^o$, simply follows from the observation that otherwise no borrower would pick the lottery over the deterministic R_S^o. ■

PROOF OF PROPOSITION 4. The argument from the proof of Lemma 1 applies directly to establish that the contract aimed at the high type is a deterministic repayment. The argument from the proof of Lemma 2 guarantees that there cannot be multiple repayments below I_L in the contract aimed at the low type. The only remaining point is that no repayment weakly greater than I_L is demanded (from the low type). That simply follows from the assumption that $v(I) = u(0) = -\infty$, which insures that the ex ante optimal contract does not assign positive probability to extracting the entire endowment from the borrower. ■

A.2. *Government Intervention.*

A.2.1. *Deterministic intervention in the one-period model.* For simplicity of exposition, we will restrict our attention to the debt-overhang case, i.e., $D \geq R_S^*$, where the laissez faire outcome is screening. With appropriate modifications, the analysis extends to the general case with a given debt level. When $R_G \in (R_L, R_H)$ and $p_G = 1$, the lender's problem becomes

$$\max_{p \in [0,1]} \gamma \hat{R}_S(p) + (1 - \gamma) p R_L, \tag{A.1}$$

[31] See, for example, Kreps (1990, chapter 3), and Kreps (1988, chapter 6).

[32] Alternatively, we could have constructed the new contract with the certainty equivalent R' being defined from the high-income borrower's point of view. With the DARA assumption, this modification leaves the high type just indifferent between the low type's old and new contracts, but improves the low type's utility. Hence the modified contract leads to both an ex ante improvement through a greater average repayment and ex post welfare improvement in the low-income state.

where $\hat{R}_S(p)$ is given by

$$u(I_H - \hat{R}_S) = pu(I_H - R_L) + (1-p)u(I_H - R_G). \tag{A.2}$$

Note that the problem is identical to the familiar (3) subject to (2), where R_H has been replaced by R_G. That is, the government intervention basically amounts to lowering the high-income borrowers' willingness to repay, R_H. We denote the solution to problem (A.1)–(A.2) by $\hat{p}$.

Notice that in equilibrium no borrower actually makes the repayment offered by the government. The low-income borrowers reject the government's offer because R_G exceeds their willingness to pay, and the high-income borrowers never receive the offer in the first place, because the lender makes them an offer that they prefer to delinquency. Thus, all renegotiation is performed by the lender, and the equilibrium bankruptcy rate is $(1-\gamma)(1-\hat{p})$.

To understand the effects of the intervention, we will study comparative statics of $\hat{p}$ with respect to R_G, keeping in mind that $R_G \geq R_H$ corresponds to the laissez faire case. We will then compare the bankruptcy rate obtained under $R_G \in (R_L, R_H)$ with that under $R_G = R_H$.

To this end, consider the first-order condition of the lender's problem (A.1)–(A.2). It can be written as

$$(1-\gamma)R_L = \gamma \underbrace{\frac{u(I_H - R_L) - u(I_H - R_G)}{u'(I_H - \hat{R}_S(p; R_G))}}_{=\frac{d\hat{R}_S}{dp}}, \tag{A.3}$$

where $\hat{R}_S(p; R_G)$ is defined by (A.2). The left-hand side of the above equation is the marginal benefit of increasing p—it corresponds to an increase in the lender's profits due to a higher total repayment from the low-income borrowers (and is unaffected by R_G). The right-hand side is the marginal cost of an increase in p—it reflects the fact that $\hat{R}_S$ must be reduced as p increases to keep the incentive constraint (A.2) satisfied.

The rate at which $\hat{R}_S$ can be "exchanged" for p, $d\hat{R}_S/dp$, depends on R_G through two channels. First, as R_G falls, the high-income borrowers' utility from the lottery increases, and thus a smaller increase in utility $u(I_H - \hat{R}_S)$ is needed to keep (A.2) satisfied as p increases. This effect is reflected in the numerator of the right-hand side of (A.3) being increasing in R_G. The second effect, working in the opposite direction, comes from the fact that as R_G falls, so does $\hat{R}_S$, which lowers the marginal utility $u'(I_H - \hat{R}_S)$. This in turn increases the rate at which an increase in $u(I_H - \hat{R}_S)$ translates into a decrease in $\hat{R}_S$. This second effect is reflected in the denominator of the right-hand side of (A.3) being increasing in R_G.

Whether the marginal benefit of an increase in p, $\gamma d\hat{R}_S/dp$, increases or decreases with R_G depends on which of the two effects dominates. Suppose, for example, that R_H is very close to I_H, and R_G decreases from R_H marginally. Since bankruptcy is arbitrarily costly for the high-income borrowers, even a small probability of bankruptcy is enough to make delinquency unattractive for them and to induce them to make the prescribed payment.[33] This implies that $u'(I_H - \hat{R}_S(p; R_G))$ is very responsive to the change in R_G, so that the negative effect dominates, and thus the probability of renegotiation decreases as R_G decreases. But as R_G falls close to R_L, the numerator on the right-hand side of (A.3) becomes small, and the benefit of increasing p becomes greater than the cost. Thus, for R_G close enough to R_L, the positive effect dominates, and the intervention causes the lender to choose pooling as the optimal contract, that is, $\hat{p} = 1$.

We have thus established that $\hat{p}$ is generally nonmonotone in R_G. Most interestingly, $\hat{p}$ decreases, and thus the bankruptcy rate rises, in response to the government intervention if $R_G (< R_H)$ is close enough to I_H. That is, the government intervention aimed at preventing foreclosures may actually lead to an increase in foreclosures in equilibrium.

[33] This follows from the assumption that the utility function satisfies the Inada condition.

A.2.2. *Ex ante effects of a random intervention with $R_G < R_L$.* The period-2 problem of the lender under a random government intervention with $R_G < R_L$ can be written as

$$\pi_G(D) = \max_{(p,\hat{R}_S,\hat{R}_L)} \gamma \hat{R}_S + (1-\gamma)\left[p\hat{R}_L + (1-p)p_G R_G\right]$$

subject to

$$u(I_L - \hat{R}_L) \geq p_G u(I_L - R_G) + (1-p_G)u(I_L - R_L), \tag{A.4}$$

$$u(I_H - \hat{R}_S) \geq p u(I_H - \hat{R}_L) + (1-p)\left[p_G u(I_H - R_G) + (1-p_G)u(I_H - R_H)\right], \tag{A.5}$$

$$\hat{R}_S \leq D,\ \ \hat{R}_L \leq D,$$

$$p \in [0,1].$$

Denote by $\hat{R}_L^*$ the level of $\hat{R}_L$ that makes constraint (A.4) hold with equality. Note that the constraint will hold with equality at the optimum for all $D > R_L^*$. (For $D \leq R_L^*$, the optimum is, rather trivially, $\hat{R}_L = \hat{R}_S = D$ and $p = 1$.) Thus, for $D > \hat{R}_L^*$, the optimal level of $\hat{R}_L$ is pinned down by Equation (A.4) and does not respond to (small enough) changes in D.

Denote by $\hat{R}_S^*$ the optimal repayment demanded from the high type under debt overhang (e.g., when $D \geq R_S^*$). Note that for $D \in (\hat{R}_L^*, \hat{R}_S^*)$, the solution to the lender's problem is characterized by $\hat{R}_S = D$, $\hat{R}_L = \hat{R}_L^*$ and the optimal probability of private debt renegotiation can be obtained from solving constraint (A.5) (holding with equality) for p. (For $D \geq \hat{R}_S^*$, the lender will choose to demand repayments $\hat{R}_S^*$ from the high type and $\hat{R}_L^*$ from the low type.)

Solving for the equilibrium allocation in the environment where lenders compete in period 1 (and are subject to the government intervention in period 2) simply amounts to solving the borrower's utility maximization problem subject to the lenders' break-even condition,

$$U = \max_D u\left(\frac{\pi_G(D)}{1+r}\right) + \beta\left[\gamma u\left(I_H - \hat{R}_S^o(D)\right) + (1-\gamma)u\left(I_L - \hat{R}_L^o(D)\right)\right],$$

where $\hat{R}_S^o$ and $\hat{R}_L^o$ are the optimal solutions for $\hat{R}_S$ and $\hat{R}_L$ in the lender's period-2 problem.

The problem can be broken down into two subproblems—one with risk-free loans and one with risky loans:

$$U_{safe} = \max_{D \leq \hat{R}_L^*} u\left(\frac{D}{1+r}\right) + \beta\left[\gamma u(I_H - D) + (1-\gamma)u(I_L - D)\right],$$

$$U_{risky} = \max_{D \in (\hat{R}_L^*, \hat{R}_S^*]} u\left(\frac{\pi_G(D)}{1+r}\right) + \beta\left[\gamma u(I_H - D) + (1-\gamma)u\left(I_L - \hat{R}_L^*\right)\right],$$

$$U = \max\{U_{safe}, U_{risky}\}.$$

The solution to this problem is plotted on Figure 5.

A.3. *Extension to Three Types.*

Suppose there are three possible income realizations, $I_L < I_M < I_H$, and γ_j is the probability of j's income realization, $j \in \{L, M, H\}$. Let R_j be the willingness to pay of the borrower with income I_j as defined by (1). For simplicity, we will restrict our attention to one-period contracts and the case of debt overhang.

Using arguments similar to those in the proof of Proposition 1, one can show that the contract aimed at the high type is a deterministic repayment $\tilde{R}_H \in [R_L, R_H]$, and the contract aimed at the low type is a lottery over repaying R_L with probability p_L and taking the bankruptcy option with probability $(1 - p_L)$. The middle type is also offered a lottery, as described in the following claim.

CLAIM 1. *Suppose Assumption* 1 *holds. Then the contract aimed at the middle type is a lottery over repaying $\tilde{R}_M \in [R_L, R_M]$ with probability p_M and taking the bankruptcy option (i.e., repayment in excess of R_H) with probability* $(1 - p_M)$.

PROOF. Just as in proof of the Proposition 1, the first observation is that the expected repayment generated by the middle type is no smaller than that coming from the low type and no greater than that extracted from the high type (otherwise the expected revenue could have been increased by offering eliminating one of the contracts—the one yielding lower expected revenue while being aimed at a higher type).

Next, we establish that the contract offered to the middle type in the optimal menu does not assign positive probability to multiple repayments that the middle type would be willing to make. Suppose that were not the case, and the contract aimed at the middle type included $N > 1$ different repayment $\{R_1, \ldots, R_N\}$ that did not exceed R_M. Denote the probabilities assigned to these repayments by $\{p_1, \ldots, p_N\}$, respectively. Consider an alternative contract that replaces these elements of the lottery with a single repayment R' with probability p', where $p' = \sum_{i=1}^{N} p_i$ and R' is such that

$$p' u(I_M - R') = \sum_{i=1}^{N} p_i u(I_M - R_i).$$

(That is, R' is the certainty equivalent of the sublottery over $\{R_1, \ldots, R_N\}$ evaluated from the middle type's point of view.) Since the borrower is risk averse, the expected repayment from this alternative lottery is greater, as $p'R' > \sum_{i=1}^{N} p_i R_i$. Since the alternative contract leaves the utility of the middle type unchanged, their incentive constraint is still satisfied. We just have to verify that the incentive constraints of the other borrowers are not violated under this alternative menu. The first observation in this proof ensures that $R' > R_L$, and, thus, the low-type borrowers do not prefer the new contract to theirs. Finally, due to DARA preferences, the high-type borrowers (who are not willing to pay as much to avoid risk) actually found the original lottery offered to the middle type more attractive than the newly constructed one. And since their incentive constraint was not violated before, it is still satisfied.

Finally, if the lottery offered to the middle type induces bankruptcy with positive probability, it does so by demanding repayment that *neither* type would be willing to make. While demanding $R \in (R_M, R_H)$ would be sufficient to drive the middle type to bankruptcy, asking for a larger repayment is not payoff equivalent—it makes such a lottery less attractive to the high type (i.e., relaxes the high type's incentive constraint) allowing the lender to successfully demand larger repayment from the high type. ■

Thus, the lender's problem can be written as follows:

$$\max_{\tilde{R}_H, \tilde{R}_M, p_M, p_L} \gamma_H \tilde{R}_H + \gamma_M p_M \tilde{R}_M + \gamma_L p_L R_L$$

$$\begin{aligned}
\text{s.t. } & u(I_H - \tilde{R}_H) \geq p_M u(I_H - \tilde{R}_M) + (1 - p_M) u(I_H - R_H), \\
& p_M u(I_M - \tilde{R}_M) + (1 - p_M) u(I_M - R_M) \geq p_L u(I_M - R_L) + (1 - p_L) u(I_M - R_M), \\
& u(I_H - \tilde{R}_H) \geq p_L u(I_H - R_L) + (1 - p_L) u(I_H - R_H), \\
& p_M, p_L \in [0, 1].
\end{aligned}$$

The second constraint can be rewritten as

$$p_M\left[u(I_M-\tilde{R}_M)-u(I_M-R_M)\right]\geq p_L\left[u(I_M-R_L)-u(I_M-R_M)\right].$$

Rearranging it further and keeping in mind that $\tilde{R}_M\in[R_L,R_M]$, we obtain

$$\frac{p_L}{p_M}\leq\frac{u(I_M-\tilde{R}_M)-u(I_M-R_M)}{u(I_M-R_L)-u(I_M-R_M)}\leq 1.$$

That is, the middle type is renegotiated with more often than the low type. Put differently, the fraction of middle-type borrowers who declare bankruptcy, $(1-p_M)$, is lower than that of the low-type borrowers, $(1-p_L)$.

It is important to note that with more than two types, the optimal mechanism (where the lender offers all repayment options simultaneously) is no longer equivalent to the sequential setting (where he offers the repayments sequentially and can commit not to renegotiate with some probability). The reason is that in the sequential setting, it is impossible for the lender to exclude the middle-type borrowers who were not renegotiated with (which is a fraction $\gamma_M(1-p_M)$ of all borrowers) from participating in the lottery designed for the low-type borrowers.

Using similar arguments as before, one can show that in the sequential setting the lender first asks for a high deterministic repayment (which only the high type makes), then with some probability he asks for a lower repayment (which only the middle type makes), and finally with some probability he asks for the lowest repayment R_L (which is accepted by the low-type borrowers and the middle-type borrowers who were not renegotiated with in the second stage). Thus the lender's problem in the sequential setting is

$$\begin{aligned}
&\max_{\tilde{R}_H,\tilde{R}_M,p_M,p_L}\ \gamma_H\tilde{R}_H+\gamma_M p_M\tilde{R}_M+(\gamma_M(1-p_M)+\gamma_L)p_L R_L\\
\text{s.t. }\ & u(I_H-\tilde{R}_H)\geq p_M u(I_H-\tilde{R}_M)+(1-p_M)[p_L u(I_H-R_L)+(1-p_L)u(I_H-R_H)],\\
& u(I_M-\tilde{R}_M)\geq p_L u(I_M-R_L)+(1-p_L)u(I_M-R_M),\\
& p_M,p_L\in[0,1].
\end{aligned}$$

Naturally, since the lender is more constrained in the sequential setting than he is in the simultaneous one, his profits are (weakly) lower.

Note that, in the sequential setting, the middle-type borrowers have two chances to be renegotiated with. First, fraction p_M of them receive (and accept) the offer of $\tilde{R}_M$, and then fraction p_L of the remaining ones receive an offer of R_L. Thus, the probability of bankruptcy for the middle-type borrower is $(1-p_M)(1-p_L)$, which is lower than that for the low-type borrowers, $(1-p_L)$.

So, in both the sequential and simultaneous setting, the middle-type borrowers are renegotiated with more often (end up in bankruptcy less frequently) and repay more, even conditional on renegotiation, than the low-type borrowers.

One can interpret the lower repayment as the intensive margin of renegotiation and the higher probability (of being offered a lower repayment) as the extensive margin of renegotiation. That is, the lender uses the intensive margin more with the middle type and the extensive margin more with the low type.

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