Simple axiom systems for affine planes

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Summary. We give axiom systems for plane affine geometries, using the notion of parallelity, which are most simple in the sense that their axioms require, when written in prenex form, the least number of variables. By allowing the language to contain operation symbols we show that all interesting affine planes, with the exception of Desarguesian planes, are axiomatizable by axioms containing at most 4 variables.

1 Introduction

Given that for translation planes, as well as for Desarguesian and for Pappian planes, there are several possible configuration theorems characterizing these classes (cf. [3, p. 85], [4], [5], [7] and [1]), it is of some interest to determine whether the classical (and oldest) configuration theorems, namely the minor Desargues, the Desargues and the Pappus configurations are in any way simpler than their competitors.

In this paper we shall analyze to what extent this is true, based on a concept of simplicity that we have already used in the case of Euclidean geometry (cf. [11], [12], [13]): axioms that, roughly speaking, refer to n points are considered simpler than axioms that refer to more than n points.

Definition *A first-order theory T will be said to have* simplicity degree n *if n is the smallest number for which all of the axioms in some axiom system Σ for T contain, when written in prenex form, at most n variables. Such an axiom system Σ will be called* simple.

The affine geometries for which we shall determine the simplicity degree are being considered as parallelity geometries, i. e. they are expressed in a first-order language L with one sort of variables (to be interpreted as "points") and with a quaternary relation $\parallel$, where $ab \parallel cd$ is being read as "ab is parallel to cd".[1] As a convenient abbreviation, we shall define $L(abc) :\leftrightarrow ab \parallel ac$, which may be read as "the points a, b, c are collinear". The axiomatization of affine planes as parallelity planes was introduced by W. Szmielew in [17].

The simplicity degrees will turn out to be: 5 for the L-theories of parallelity planes, non-Fano parallelity planes, translation planes of characteristic $\neq 2$, and 6 for the L-theory of Pappian planes (both of characteristic $\neq 2$ and of unspecified characteristic). The simplicity degrees of the L-theory of translation planes of characteristic 2, as well as that of Moufang and Desarguesian affine geometry have not been determined. One would expect the last two to be 7 (the proof that they are at least 6 is contained in the proof of Theorem 3). The ordered version of any of these geometries turns out to have the same simplicity degree as the corresponding non-ordered one. In languages with operation symbols, all of these affine geometries, with the exception of the Desarguesian non-ordered planes, are shown to have simplicity degree 4.

[1] Theories will be referred to as "geometries" and models will be referred to as "planes": *parallelity planes* are models of *parallelity geometry*, or equivalently, the latter is the L-theory of the former. An axiom system for a certain geometry will also be called an axiom system for the corresponding class of models.

2 A simple axiom system for affine planes

Consider the following axioms (all having (in prenex form) at most 5 variables):

A 1 $ab \parallel cd \to ab \parallel dc$,

A 2 $ab \parallel cc$,

A 3 $ab \parallel ac \to ba \parallel bc$,

A 4 *(i)* $a \neq b \wedge ab \parallel dc \wedge ab \parallel ec \to dc \parallel ec$,
(ii) $a \neq b \wedge ab \parallel de \wedge ab \parallel ac \to de \parallel ac$,

A 5 $(\exists abc) \neg L(abc)$,

A 6 $(\forall abp)(\exists q)\,(ab \parallel pq \wedge p \neq q)$,

A 7 $(\forall abcd)(\exists p) \neg(ab \parallel cd) \to L(pab) \wedge L(pcb)$,

A 8 $ab \parallel cd \wedge ac \parallel bd \wedge ad \parallel bc \to L(abc)$.

Let $\Sigma = \{A1 - A7\}$ and $\Sigma' = \Sigma \cup \{A8\}$.

The axiom system Σ differs from the one given in [17, p. 19] in that it lacks two axioms: Ax 2.2.0, which states that $ab \parallel ba$, and Ax 2.2.2 which states that

$$a \neq b \wedge ab \parallel pq \wedge ab \parallel rs \to pq \parallel rs. \tag{1}$$

Ax 2.2.0. can be deduced from Σ, as A3 (with $c = a$) implies, since its antecedent is A2,

$$ba \parallel ba, \tag{2}$$

and using A1 we get Ax 2.2.0.

Using (2) and A4(ii) we can prove that, for $a \neq b$,

$$ab \parallel cd \to cd \parallel ab, \tag{3}$$

and, as an easy consequence of A2 and (3), we get, for $b \neq c$,

$$aa \parallel bc. \tag{4}$$

Notice that (3) and (4) are true for $a = b$ and $b = c$ as well (by A2).

We now turn to the proof of Ax 2.2.2 using Σ.

First, assuming $a \neq b$, $ab \parallel cd$ and $ab \parallel cf$, we want to deduce $cd \parallel cf$. We get, by A1, $ab \parallel dc$ and $ab \parallel fc$, hence, by A4(i), we deduce that $dc \parallel fc$, and by A1 that $dc \parallel cf$ as well. From (3) we further get $cf \parallel dc$, hence $cf \parallel cd$ (by A1), i. e. $cd \parallel cf$ (by (3)), q. e. d.

Suppose $a \neq b$, $e \neq c$, $ab \parallel cd$, $ab \parallel ef$, but $\neg(cd \parallel ef)$ (by A2 and (4) this implies $e \neq f$ and $c \neq d$). Then, by A7, there is a p such that $L(pcd) \wedge L(pef)$. By A3 we get $L(cpd)$ and $L(epf)$. By (3) we have $cd \parallel ab$ as well as $L(cdp)$, hence, with $cd \parallel ab \wedge L(cdp) \to ab \parallel cp$ (by A4(ii)), we get $ab \parallel cp$.

Using (3) we get $ef \parallel ab$ and $L(efp)$, hence, with $ef \parallel ab \wedge L(efp) \to ab \parallel ep$ (by A4(ii)), we get $ab \parallel ep$. From $ab \parallel cp$ and $ab \parallel ep$ we get, using A4(ii), $cp \parallel ep$.

Assuming $c \neq p$ and using $cp \parallel ep \wedge L(cpd) \to ep \parallel cd$ (by A4(ii)) we get $ep \parallel cd$. Assuming that $e \neq p$ as well and using $ep \parallel cd \wedge L(epf) \to cd \parallel ef$ (by A4(ii)), we finally get $cd \parallel ef$, contradicting our initial assumption.

We now need to show that the same contradiction follows if $c = p$ or $e = p$.

Suppose $c = p$. Then $L(cef)$, hence $L(ecf)$ (by A3). Since $ab \parallel ef$ and $L(ecf)$, and by (3) we also have $ef \parallel ab$ and $L(efc)$, we deduce from A4(ii) ($e \neq f \wedge ef \parallel ab \wedge L(efc) \to ab \parallel ec$) that $ab \parallel ec$. From $ab \parallel cd$ we get $ab \parallel dc$ (by A1). By A4(i) ($a \neq b \wedge ab \parallel dc \wedge ab \parallel ec \to dc \parallel ec$) we get $dc \parallel ec$, hence $ec \parallel dc$ (by (3)), and $ec \parallel cd$ as well (by A1). By A4(ii) ($e \neq c \wedge ec \parallel cd \wedge ec \parallel ef \to cd \parallel ef$) we obtain $cd \parallel ef$, q. e. d.

Suppose finally that $e = p$. We may also assume that $c \neq d$, since in the case $c = d$ the conclusion we wish to derive, namely $cd \parallel ef$, follows from (4). Then $L(ecd)$, hence $L(ced)$ (by A3), hence $L(cde)$ (by (3)). Since $cd \parallel ab$ as well (by (3)), $c \neq d \wedge cd \parallel ab \wedge L(cde) \to ab \parallel ce$ (by A4(ii)), we get $ab \parallel ce$. Since $ab \parallel ef$ as well, and hence also $ab \parallel fe$ (by A1), we deduce from A4(i) ($a \neq b \wedge ab \parallel ce \wedge ab \parallel fe \to ce \parallel fe$) that $ce \parallel fe$, and therefore $ce \parallel ef$ as well (by A1). Now A4(ii) ($c \neq e \wedge ce \parallel ef \wedge L(ced) \to ef \parallel cd$) gives $ef \parallel cd$, and from (3) we deduce $cd \parallel ef$, q. e. d.

We have thus proved

Theorem 1 *Σ is an axiom system for parallelity planes (i. e. for plane affine geometry) and Σ' is an axiom system for non-Fano parallelity planes.*

3 A simple axiom system for translation planes of characteristic $\neq 2$

Let now $\Delta' = \Sigma \cup \{\text{A9}\}$, where

A 9 $(\forall ab)(\exists c)(\forall de)\,[a \neq b \to L(abc) \wedge c \neq a \wedge (\neg L(abd) \wedge ab \parallel de \wedge ad \parallel be \to bd \parallel ce)$

As in [17, p. 28], let P be the operation of "completing the parallelogram": if $\neg L(abc)$, then $P(abc) = d \leftrightarrow ab \parallel cd \wedge ac \parallel bd$.

Theorem 2 *Δ' is an axiom system for translation planes of characteristic $\neq 2$.*

Proof. For any two points a and b with $a \neq b$, the point c given by A9 is unique — for, if d is any point for which $\neg L(abd)$ (the existence of such points follows from Σ), then c must be equal to $P(dbP(abd))$ — and will be denoted by $\sigma_b(a)$. We also set $\sigma_b(b) = b$, and want to show that, for every point b, the mapping σ_b is the reflection in b, i. e. that σ_b is an involutory homothety with centre b, which amounts to proving that

$$xy \parallel \sigma_b(x)\sigma_b(y) \text{ for all } x \text{ and } y, \tag{5}$$

since the fact that b is a fixpoint is in our definition of σ_b, the fact that it is the only fixpoint follows from the condition $c \neq a$ in A9, and the fact that σ_b is involutory also follows from A9.

If $L(xyb)$, then (5) follows from the fact that for all z we have $L(zb\sigma_b(z))$ by A9. Suppose now $\neg L(xyb)$. By A9 and (1) we have $bx \parallel \sigma_b(y)P(yxb)$ and $b\sigma_b(y) \parallel xP(yxb)$. We can now apply A9 again to get $bP(yxb) \parallel \sigma_b(x)\sigma_b(y)$. Since, by the definition of P, we also have $yx \parallel bP(yxb)$, we get $yx \parallel \sigma_b(x)\sigma_b(y)$, and hence (5).

We have thus shown that for every point of a model of Δ' there is an involutory homothety with that point as centre. According to a theorem of R. Baer (cf. [14, p. 213] or [8]), such an affine plane is a translation plane of characteristic $\neq 2$.

□

Let A10, A11 stand for the minor Desargues axiom (des in [17, p. 41]) and the Pappus axiom (Papp in [17, p. 45]) and let $\Delta = \Sigma \cup \{\text{A10}\}$, $\Pi = \Sigma \cup \{\text{A11}\}$, $\Pi' = \Sigma' \cup \{\text{A11}\}$. Here are, for convenience, the statements of these axioms:

A 10 $\neg L(abp) \wedge \neg L(abr) \wedge ab \parallel pq \wedge ab \parallel rs \wedge ap \parallel bq \wedge ar \parallel bs \to pr \parallel qs$,

A 11 $p_1 \neq p_2 \wedge p_2 \neq p_3 \wedge p_3 \neq p_1 \wedge q_1 \neq q_2 \wedge q_2 \neq q_3 \wedge q_3 \neq q_1 \wedge L(p_1p_2p_3) \wedge L(q_1q_2q_3) \wedge \neg L(p_1q_1q_2) \wedge \neg L(p_2q_1q_2) \wedge \neg L(p_3q_1q_2) \wedge \neg L(q_1p_1p_2) \wedge \neg L(q_2p_1p_2) \wedge \neg L(q_3p_1p_2) \wedge \neg(p_1p_2 \parallel q_1q_2) \wedge p_1q_2 \parallel p_2q_1 \wedge p_2q_3 \parallel p_3q_2 \to p_1q_3 \parallel p_3q_1$.

We then have the following

Theorem 3 *Σ, Σ', Δ', Π and Π' are simple axiom systems (and the corresponding theories have simplicity degrees 5, 5, 5, 6 and 6 respectively) for parallelity planes, non-Fano parallelity planes, translation planes of characteristic $\neq 2$, Pappian planes and Pappian planes of characteristic $\neq 2$.*

Proof. We have to prove that:

(i) for $Cn(\Sigma)$, $Cn(\Sigma')$ and $Cn(\Delta')$, there is no axiom system, all of whose axioms contain, when written in prenex form, at most 4 variables;

(ii) for $Cn(\Pi)$ and $Cn(\Pi')$ there is no axiom system, all of whose axioms contain, when written in prenex form, at most 5 variables.

In order to prove each of these statements, we shall provide two models. One of them will be a model of the particular theory $\mathcal{T}$ for which we claim to have provided a simple axiom system, the other will not be a model of $\mathcal{T}$, such that both models satisfy the same 4-variable statements (for (i)), or the same 5-variable statements (for (ii)).

Denoting by L_n the language that contains the same symbols as L, except that there are not countably many, but only n individual variables, let $\mathcal{T}_n := Cn(\{\varphi \mid \varphi \in \mathcal{T} \cap \mathsf{L}_n, \varphi \text{ is written in prenex form}\})$.

In order to prove that, for a given theory $\mathcal{T}$ and a particular natural number n, $\mathcal{T} \neq \mathcal{T}_n$, we shall use the model-theoretic method of EHRENFEUCHT-FRAÏSSÉ games, as described in [6]. The method allows us to prove that two models $\mathfrak{A}$ and $\mathfrak{B}$, with $\mathfrak{A} \in Mod(\mathcal{T})$ and $\mathfrak{B} \notin Mod(\mathcal{T})$, satisfy the same L_n-sentences of a particular quantifier-type, and so, by using it for all possible quantifier-types, that the two models satisfy the same prenex L_n-sentences (it, in effect, implies more than that the two models satisfy the same *prenex* sentences with n quantifiers, but this is all that we need for our present purpose). This would prove that $\mathcal{T}_n \neq \mathcal{T}$.

To prove (i), let $\mathfrak{A} = \langle \mathbb{Q} \times \mathbb{Q}, \|_{\mathbb{Q}} \rangle$ be the parallelity plane over the field of rational numbers, and let $\mathfrak{B} = \langle \mathbb{Q} \times \mathbb{Q}, \|_{\mathfrak{B}} \rangle$, where $\mathbf{ab} \|_{\mathfrak{B}} \mathbf{cd}$ iff $\mathbf{ab} \|_{\mathbb{Q}} \mathbf{cd}$ or $\{\mathbf{a}, \mathbf{b}\} = \{(0,0),(1,0)\}$ and $\{\mathbf{c}, \mathbf{d}\} = \{(0,1),(1,2)\}$ or $\{\mathbf{c}, \mathbf{d}\} = \{(0,0),(1,0)\}$ and $\{\mathbf{a}, \mathbf{b}\} = \{(0,1),(1,2)\}$.

The EHRENFEUCHT-FRAÏSSÉ game to be used in order to prove that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same prenex L_4-sentences of a certain prefix can be described as follows:

In this game, there are two players, I and II, that alternate in making choices from the universes of the two models, $u(\mathfrak{A})$ and $u(\mathfrak{B})$, (it depends on the prefix which set a player is supposed to choose from at the n^{th} move; a universal quantifier in the n^{th} position forces I to choose from $u(\mathfrak{B})$, an existential one forces I to choose from $u(\mathfrak{A})$). The choice of I at the n^{th} move will be denoted by $\mathbf{x}_n$, the choice of II at the n^{th} move by $\mathbf{y}_n$. Let $\{\mathbf{a}_n\} = \{\mathbf{x}_n, \mathbf{y}_n\} \cap u(\mathfrak{A})$ and $\{\mathbf{b}_n\} = \{\mathbf{x}_n, \mathbf{y}_n\} \cap u(\mathfrak{B})$. Player II wins the game, which in our case consists of 4 moves, if at the end of the game the function f, defined by $f(\mathbf{a}_n) = \mathbf{b}_n$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. The fact that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same prenex sentences of that prefix containing 4 quantifiers follows from the existence of a winning strategy for II in the corresponding game.

In the EHRENFEUCHT-FRAÏSSÉ game for $\mathfrak{A}$ and $\mathfrak{B}$ the strategy for II would be, regardless of prefix, the following:

For the first three moves II chooses the point with identical coordinates to the one chosen by I. In the 4^{th} move II chooses the point whose coordinates are identical to the one chosen by I *unless*:

(α) I has chosen a point from $u(\mathfrak{B})$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\} = \{(0,0),(1,0),(0,1),(1,2)\}$, in which case II chooses a point in $u(\mathfrak{A})$ such that $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ form a parallelogram and the corresponding relations hold;

(β) I has chosen a point from $u(\mathfrak{A})$ and $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \{(0,0),(1,0),(0,1),(1,2)\}$, in which case II chooses a point in $u(\mathfrak{B})$ such that $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_4$ form a trapezium that is not a parallelogram, and the corresponding relations hold.

Since $\mathfrak{A}$ is a model of $Cn(\Pi')$, hence a fortiori of $Cn(\Sigma)$, $Cn(\Sigma')$ and $Cn(\Delta')$, none of these theories can be axiomatized by prenex sentences in L_4, which proves (i).

To prove (ii), let $\mathfrak{A}$ be the affine plane over $GF(9)$ and $\mathfrak{B}$ be the exceptional nearfield plane of order 9 (cf. [9, II.7-8] for a definition). $\mathfrak{B}$ is a non-Fano translation plane and $\mathfrak{A}$ is a non-Fano Pappian plane. The winning strategy for II in a game with 5 moves would be:

For clarity's sake, let's denote by $\mathbf{p}_i$ (for $i = 1, 2, \ldots, k-1$) the previous choices (of either player) that were made from the model from which II has to choose at move k, and by $\mathbf{o}_i$ (for $i = 1, 2, \ldots, k$) the choices made in the other model, where the indices denote the move at which the choice was made.

In the first two moves the choices are arbitrary, subject to the only condition that $\mathbf{p}_2 = \mathbf{p}_1$ if and only if $\mathbf{o}_2 = \mathbf{o}_1$ (since both $\mathfrak{A}$ and $\mathfrak{B}$ have doubly transitive collineation groups, there is no distinguished pair of points). In the third move, if $\neg\mathbf{L}(\mathbf{o}_1\mathbf{o}_2\mathbf{o}_3)$ holds, then II chooses any $\mathbf{p}_3$ for which $\neg\mathbf{L}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)$ holds. If $\mathbf{L}(\mathbf{o}_1\mathbf{o}_2\mathbf{o}_3)$ holds, but $\mathbf{o}_3$ is not a midpoint of $\mathbf{o}_1\mathbf{o}_2$ (note that this means that none of the $\mathbf{o}$'s is a midpoint of the pair formed by the remaining two $\mathbf{o}$'s), then II chooses $\mathbf{p}_3$ such that $\mathbf{L}(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)$ and the condition of it not being a midpoint of $\mathbf{p}_1\mathbf{p}_2$ holds. If $\mathbf{o}_3$ is a midpoint of $\mathbf{o}_1\mathbf{o}_2$, then choose for $\mathbf{p}_3$ the midpoint of $\mathbf{p}_1\mathbf{p}_2$. In the fourth move the choice will be such that parallelity (under which we subsume collinearity as well) or the absence thereof and the midpoint-relation or the absence thereof in the set $\{\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3, \mathbf{o}_4\}$ will be respected by the choice of $\mathbf{p}_4$. The choice in the fifth move will be such as to produce a partial isomorphism. That this is possible follows from our previous strategy. This proves (ii). □

It is natural to ask whether A9 could be replaced by a set of configuration theorems (*Schließungssätze*), as it replaces the minor Desargues axiom. By analyzing all the possible 5-point configurations in non-Fano affine planes, one notices that there is none that is true in translation planes of characteristic $\neq 2$, but not in all non-Fano affine planes as well, i. e. $\Delta' \setminus \Sigma'$ contains no universal 5-variable prenex sentence. However this can not be seen by playing an EHRENFEUCHT-FRAÏSSÉ game, because player I will have a winning strategy in a game with 5 moves for any two structures $\mathfrak{A}$ and $\mathfrak{B}$, the first a translation plane of characteristic $\neq 2$, the second not a translation plane. The reason behind this is that there is a universal 5-variable sentence, which is not in prenex form, that is equivalent to A9, namely (with $Par(abcd) :\leftrightarrow \neg L(abc) \wedge ab \parallel cd \wedge ac \parallel bd$):
$(\forall abc)L(abc) \wedge a \neq b \wedge b \neq c \wedge c \neq a \rightarrow ((\forall xy)Par(abxy) \rightarrow Par(bcxy)) \vee ((\forall xy)Par(abxy) \rightarrow \neg Par(bcxy))$.
Written in prenex form, it would require 7 variables.

We do not know whether Δ is a simple axiom system for translation planes, since it is not known whether there are planes satisfying Fano's axiom ($\neg$A8) that are not translation planes (GLEASON [2] showed that all finite Fano planes must be Pappian, in particular, must be translation planes, therefore any non-translation plane satisfying $\neg$A8 must be infinite.) If there are such planes, then Δ is simple; if there are no such planes, then Δ is not simple, and the simplicity degree of translation planes would be 5, since $\neg$A8 $\vee$ A9 — which can be stated in prenex form by using only 5 variables — would, together with Σ axiomatize translation planes (of unspecified characteristic).

4 Ordered affine planes

By enlarging the language L by a ternary relation symbol B standing for the betweenness relation and adding to any of the axiom systems Σ', Δ', and Π' the order axioms

O 1 *(i)* $L(abc) \rightarrow B(abc) \vee B(bca) \vee B(cab)$,
(ii) $B(abc) \rightarrow L(abc)$,

O 2 $B(abc) \rightarrow B(cba)$,

O 3 $B(abc) \wedge B(acd) \rightarrow B(bcd)$,

O 4 $\neg L(abb') \wedge L(ab'c') \wedge bb' \parallel cc' \wedge B(abc) \rightarrow B(ab'c')$,

we obtain simple axiom systems for ordered non-Fano parallelity planes, ordered translation planes and ordered Pappian planes, respectively. The simplicity degree of the ordered theories will remain the same as before. To see this, one needs to modify the strategy of player II in the relevant games in Theorem 3, by adding the condition that II's choice should also match the order relation that exists among the $\mathbf{o}$'s for all moves beginning with the third one.

5 The simplest possible axiom systems

A natural question to ask in this context would be whether there is any language for affine geometry in which the simplicity degrees of the various theories analyzed in the previous sections are lower than the ones obtained when these geometries are expressed using the single quaternary relation of parallelity. The notions of such a language for affine geometry should be required to be invariant under collineations. If the language is supposed to contain only relation symbols, then we know of no language for which the simplicity degree of any of the affine geometries previously referred to would be less than the simplicity degree of the corresponding L-theory. However, if we allow operation symbols in the language, then the simplicity degree of all but Desarguesian affine geometry (both of characteristic $\neq 2$ and of unspecified characteristic) will be shown to be 4, which is the absolute minimum for any axiomatization of affine geometry with only one sort of variables, to be interpreted as "points", and without constants.

The language L_{con} in which these axiom systems will be expressed consists of a ternary relation symbol L, a ternary operation symbol P, both having the same intended interpretation as the respective defined notions in the axiom systems given above, and a quaternary operation symbol I, with the intended interpretation: '$I(abcd)$ is the intersection point of ab and cd, provided that ab and cd are not parallel, and arbitrary, otherwise'. Consider the following axioms

C 1 $L(aba)$,

C 2 $L(abc) \to L(cba) \wedge L(bac)$,

C 3 $a \neq b \wedge L(abc) \wedge L(abd) \to L(acd)$,

C 4 $P(abc) = P(acb)$,

C 5 $\neg L(abc) \wedge \neg L(cdP(bac)) \to L(I(abcd)ab) \wedge L(I(abcd)cd)$,

C 6 $L(cP(abc)d) \wedge L(abd) \to L(abc)$.

Let $\Sigma_{con} = \{\mathrm{C1} - \mathrm{C6}, A5\}$. For two given points $\mathbf{a}$ and $\mathbf{b}$ of a model $\mathfrak{M}$ of Σ_{con}, we define the line $[\mathbf{ab}]$ to be the set $\{\mathbf{x} \in \mathfrak{M} \,|\, \mathbf{L}(\mathbf{abx})\}$ and we say that $\mathbf{x}$ is incident with (or lies on) $[\mathbf{ab}]$ if $\mathbf{x} \in [\mathbf{ab}]$. It is easy to see that the usual axioms of incidence for rudimentary affine geometry (cf. [3, p. 16]) hold in any model of Σ_{con}, in which the notions of line and incidence have been defined as above. This means that Σ_{con} is an axiom system for affine geometry, i. e., with $\|$ defined by

$$ab \parallel cd \leftrightarrow L(cdP(bac)), \tag{6}$$

we get the following

Theorem 4 $\Sigma_{con} \cup \{(6)\} \vdash \Sigma$.

In order to obtain an axiom system for translation planes, all we need is a variant of the minor Desargues axiom, which can be expressed as

C 7 $P(abd) = P(cP(abc)d)$.

A more elegant axiom system for translation planes can be obtained by replacing C6 by the following three axioms (for convenience, we shall also use the abbreviation σ, defined as above by $\sigma(ba) = P(abb)$):

C 8 $P(abc) = c \to a = b$,

C 9 $L(abc) \to L(a'P(aba')P(aca'))$,

C 10 $L(ab\sigma(ab))$.

In the presence of these axioms, C1 becomes redundant and C2 can be weakened to
C 2′ $L(abc) \to L(bac)$.

Let $\Delta_{con} = \{\mathrm{C2}', \mathrm{C3}-\mathrm{C5}, \mathrm{C7}-\mathrm{C10}, \mathrm{A5}\}$. In order to prove that Δ_{con} is an axiom system for translation planes, we shall prove that, with parallelity defined by (6), all the axioms of Δ can be derived from Δ_{con}, i. e. that

Theorem 5 $\Delta_{con} \cup \{(6)\} \vdash \Delta$.

Proof. Suppose $P(abx) = P(aby)$. We want to conclude that $x = y$. By C7 and C4 we have $P(abx) = P(yxP(aby))$, hence $P(yxP(aby)) = P(aby)$. Therefore, by C8,

$$P(abx) = P(aby) \to x = y. \tag{7}$$

By C7 we have $P(abb) = P(aP(aba)b)$, so using C4 and (7) we get

$$P(aba) = P(aab) = b. \tag{8}$$

As a special case of (8) we get

$$\sigma(aa) = a. \tag{9}$$

We also notice that, from the definition of σ and C8 we deduce

$$\sigma(ab) = a \to a = b. \tag{10}$$

With $c = b$ and $d = a$, C7 becomes $P(bP(abb)a) = P(aba)$, which, by (8) and C4, implies

$$P(ba\sigma(ba)) = b. \tag{11}$$

Let $a' = \sigma(ba)$. By C7 we have $P(a'ba') = P(bP(a'bb)a')$, i. e. $P(bP(a'bb)a') = b$ (by (8)). By (11) this means that $P(bP(a'bb)a') = P(baa')$. Using C4 and (7) we conclude that $P(a'bb) = a$, i. e. that

$$\sigma(b\sigma(ba)) = a. \tag{12}$$

We now turn to the proof of the deleted part of axiom C2, i. e. to $L(abc) \to L(cba)$. We shall first prove it for $a \neq b$ and $a \neq c$. Suppose $L(abc)$ and $a \neq b$. By C10 we have $L(ab\sigma(ab))$, whereas C3 gives $a \neq b \wedge L(ab\sigma(ab)) \wedge L(abc) \to L(a\sigma(ab)c)$. Since the antecedent of the above implication holds, its consequent, $L(a\sigma(ab)c)$, holds as well. By C10 we have $L(a\sigma(ab)\sigma(a\sigma(ab)))$, i. e. , since $\sigma(a\sigma(ab)) = b$, we have $L(a\sigma(ab)b)$. Since $a \neq \sigma(ab)$ (by (10)), the antecendent of $a \neq \sigma(ab) \wedge L(a\sigma(ab)c) \wedge L(a\sigma(ab)b) \to L(acb)$ (which holds by C3) holds, and so its consequent, $L(acb)$, holds as well. This proves

$$a \neq b \wedge a \neq c \wedge L(abc) \to L(acb). \tag{13}$$

Applying C2′ with antecedent $L(acb)$, we get $L(cab)$. Since $a \neq c$, we can apply (13) with $L(cab)$ as antecendent to get $L(cba)$. Notice that, if $a = c$, then $L(abc) \to L(cba)$ is a tautology. We have thus shown that

$$a \neq b \wedge L(abc) \to L(cba). \tag{14}$$

We now turn to the proof of C1. Let $a \neq b$. Then $\sigma(ab) \neq a$ (by (10)). We have $L(ba\sigma(ba)$ (by C10)), therefore $L(b\sigma(ba)a)$ (by (13). By C3, $a \neq b \wedge L(b\sigma(ba)a) \wedge L(b\sigma(ba)a) \to L(baa)$, therefore $L(baa)$; hence $L(aba)$ (by C2), which is C1 for $a \neq b$. For $a = b$, C1 is a consequence of C10 and of (9).

Apply now C2′ with $L(aca)$ (which holds, by C1) as antecedent to get $L(caa)$. This shows that (14) holds without the condition $a \neq b$ as well, and proves that C2 holds in Δ_{con}.

Since $P(baP(abc)) = P(bP(acb)a) = P(aca) = c$ (the first equation follows from C4, the second from C7 and the third from (8)) and
$P(bac) = P(P(abc)P(baP(abc))c)$ (by C7), we get

$$P(bac) = \sigma(cP(abc)). \tag{15}$$

We are now ready to prove Ax 2.2.0 of [17, p. 19], i. e. $ab \parallel ba$. According to (6) $ab \parallel ba$ is equivalent to $L(baP(bab))$, i. e. to $L(baa)$ (by (8)), which is a consequence of C1 and C2.

To see that A2 follows from $\Delta_{con} \cup \{(6)\}$, notice that $ab \parallel cc$ means, by (6), that $L(ccP(bac))$, which follows from C1 and C2.

Let now $a \neq b$, $ab \parallel cd$ and $ab \parallel ef$. We want to show that $cd \parallel ef$, thus proving Ax 2.2.2 in [17, p. 19]. If $c = d$, then $cc \parallel ef$ is a consequence of $\Delta_{con} \cup \{(6)\}$, being equivalent to $L(efP(cce))$, hence to $L(efe)$ (by (8)), which is C1. Suppose now $c \neq d$. By (6) our hypothesis amounts to $a \neq b$, $c \neq d$, $L(cdP(bac))$ and $L(efP(bae))$. From C9 applied to $L(cdP(bac))$ we get $L(eP(cde)P(cP(bac)e))$, i. e., since $P(cde) = \sigma(eP(dce))$ (by (15)) and $P(cP(bac)e) = P(bae)$ (by C7), we have $L(e\sigma(eP(dce))P(bae))$. Since we also have $L(eP(dce)\sigma(eP(dce))$ (by C10) and $\sigma(eP(dce)) \neq e$ (by $c \neq d$ and (10)), using C2 and C3 we conclude that $L(eP(dce)P(bae))$. Since we also have $L(efP(bae))$ from our hypothesis and $P(bae) \neq e$ (by C8), using C2 and C3 we get $L(efP(dce))$, i. e. $cd \parallel ef$ (by (6)). This proves Ax 2.2.2.

We now turn to the proof of A3. Let $ab \parallel ac$ and suppose a, b and c are different (else $ba \parallel bc$ will be true by A2, or by $cc \parallel ef$ which was proved above, or by Ax 2.2.0 and Ax 2.2.2). By (6) we have that $L(acP(baa))$, that is $L(ac\sigma(ab))$. Since we also have that $L(ab\sigma(ab))$ and $\sigma(ab) \neq a$ (by (10)), we get $L(abc)$ (by C2 and C3). From $L(abc)$ and $L(ba\sigma(ba))$ (by C10) we get, by using C2 and C3, $L(bc\sigma(ba)$, i. e. $ba \parallel bc$ (by (6)). This proves A3.

To see that A6 follows from $\Delta_{con} \cup \{(6)\}$, notice that, with $q = P(abp)$, $ab \parallel pq$ becomes (by (6)) $L(pP(abp)P(bap))$, which in turn becomes (since by (15) we have $P(bap) = \sigma(pP(abp))$) $L(pP(abp)\sigma(pP(abp)))$, which follows from C10. If $a \neq b$, we also conclude that $p \neq q$ (by C8). If $a = b$, any q with $q \neq p$ satisfies the conditions of A6.

Now A7 follows from C5 and (6), with $p = I(abcd)$, so all the axioms in [17, p. 19] are consequences of $\Delta_{con} \cup \{(6)\}$, which has been therefore shown to be an axiom system for affine planes, which means that $\parallel$ (as defined by (6)) and L satisfy all the axioms in Σ (notice also that $L(abc) \leftrightarrow ab \parallel ac$ is also a consequence of $\Delta_{con} \cup \{(6)\}$).

We now turn to the proof that A10 is also a consequence of $\Delta_{con} \cup \{(6)\}$. We first notice that

$$\neg L(abc) \wedge ab \parallel cd \wedge ac \parallel bd \to d = P(abc) \tag{16}$$

and

$$ac \parallel bP(abc) \tag{17}$$

are consequences of $\Delta_{con} \cup \{(6)\}$. That this is so can be seen by noticing that (17) requires that $L(bP(abc)P(cab))$, i. e. (by (15)) that $L(bP(abc)\sigma(bP(acb)))$, which is a consequence of C10 and C4. On the other hand (16) states that $P(abc)$ is the unique d satisfying the antecedent of (16). This is true since, according to (17) and C4, the antecedent of (16) is satisfied with $d = P(abc)$, and this is the unique point satisfying it, since d is the intersection point of the parallel from d to ac with the parallel from c to ab, two lines that must be distinct, as $\neg L(abc)$.

By (16), the antecedent of A10 can be rewritten as $\neg L(abp) \wedge \neg L(abr) \wedge q = P(abp) \wedge s = P(abr)$. Now, by C7, we have $P(abr) = P(pP(abp)r)$, i. e $s = P(pqr)$. By (17) we have $pr \parallel qP(pqr)$, i. e $pr \parallel qs$, which is the consequent of A10.

$\square$

For any strong left quasi-field F (cf. [17, p. 8] for a definition) let $\mathfrak{A}_{con}(F)$ be the structure $\langle F^2, \mathbf{L}_F, \mathbf{P}_F \rangle$, where $\mathbf{L}_F(\mathbf{abc})$ iff $a_1 = b_1 = c_1 \vee (\exists u \in F)\, u(a_1 - b_2) = (a_2 - b_2) \wedge u(a_1 - c_1) = a_2 - c_2$ and $\mathbf{P}_F(\mathbf{abc}) = \mathbf{b} + \mathbf{c} - \mathbf{a}$ (here $\mathbf{x} = (x_1, x_2)$). From Theorem 5 we deduce the following

Corollary *$\mathfrak{M} \in Mod(\Delta_{con})$ iff $\mathfrak{M} \simeq \mathfrak{A}_{con}(F)$, for some strong left quasi-field F.*

In order to eliminate the possibility that the translation plane is of characteristic 2, we need to add to Δ_{con} the axiom

C 11 $\sigma(ab) = b \to a = b$.

In order to get an axiom system for Moufang planes of characteristic $\neq 2$, we need to add to Δ_{con}, in addition to C11, the axiom *Parallelogram-Desargues* (cf. [15, Satz 9]), which in L would read

A 12 $\neg L(abp) \wedge \neg L(abr) \wedge o \neq a \wedge o \neq b \wedge o \neq p \wedge o \neq q \wedge o \neq r \wedge o \neq s \wedge L(oab) \wedge L(opq) \wedge L(ors) \wedge ap \parallel or \wedge ar \parallel op \wedge ap \parallel bq \wedge ar \parallel bs \rightarrow pr \parallel qs$,

but which could be expressed with fewer variables in L_{con}. In expressing it we shall also use (besides the abbreviation $\parallel$, defined in (6)) the abbreviation (cf. [17, p. 29])

$$T_o(abc) = I(ocbP(abc)). \tag{18}$$

It thus becomes

C 12 $\neg L(aoq) \wedge o \neq a \wedge \wedge o \neq p \wedge o \neq q \wedge L(opq) \rightarrow pP(poa) \parallel qP(qoT_o(pqa))$.

Similarly A11 becomes

C 13 $p_1 \neq p_2 \wedge q_2 \neq q_3 \wedge \neg(p_1p_2 \parallel q_2q_3) \wedge \neg L(p_1q_2q_3) \wedge \neg L(p_2q_2q_3) \wedge \neg L(q_2p_1p_2) \wedge \neg L(q_3p_1p_2) \rightarrow p_1q_3 \parallel I(p_1p_2q_2P(q_3q_2p_2))I(q_2q_3p_2P(p_1p_2q_2))$.

Another simple axiom system for Moufang planes of characteristic $\neq 2$ is obtained, as shown in [10], by adding to Δ_{con} the axiom C11 and the following special case of the Desargues axiom (which is denoted in [4, p. 127f.] by (d''))

A 13 $\neg L(abp) \wedge \neg L(abr) \wedge o \neq a \wedge o \neq b \wedge o \neq p \wedge o \neq q \wedge o \neq r \wedge o \neq s \wedge L(oab) \wedge L(opq) \wedge L(ors) \wedge ap \parallel bq \wedge ar \parallel bs \wedge L(arq) \rightarrow pr \parallel qs$,

which differs from the Desargues axiom by having $L(arq)$ added to its antecedent. Expressed it L_{con}, with q, b, and s standing for $I(arop)$, $I(oaqP(pqa))$, and $I(orbP(arb))$ respectively, it reads

C 14 $\neg L(oap) \wedge \neg L(oar) \wedge o \neq a \wedge \wedge o \neq p \wedge o \neq r \rightarrow pr \parallel qs$.

D. Scott [16] has shown that the simplicity degree of plane Euclidean geometry, axiomatized with only one sort of variables, whose intended interpretation is "points", is at least 4.[2] This is so because any axiom which contains at most 3 variables, which holds in the plane would have to hold in all higher dimensions. The same result is valid for affine geometry, since all the affine notions are Euclidean as well (i. e. invariant under isometries).

We have thus shown that

Theorem 6 *Δ_{con} and $\Delta_{con} \cup \{C13\}$ are simple axiom systems in L_{con} for translation planes and Pappian planes (of unspecified characteristic). Adding C11 to them one obtains simple axiom systems for the same structures with characteristic $\neq 2$. $\Delta_{con} \cup \{C11, C14\}$ (or $\Delta_{con} \cup \{C11, C12\}$) is a simple axiom system for Moufang planes of characteristic $\neq 2$. Σ_{con} is a simple axiom system for affine planes. The simplicity degree of the corresponding L_{con}-theories is 4. This is best possible in the sense of* D. Scott*'s Theorem.*

In order to axiomatize ordered affine planes, we need to add a new ternary predicate B to L_{con}, but we may drop L from the list of symbols of L_{con}. Let L_o be the first-order language containing the ternary relation symbol B, the ternary operation symbol P and the quaternary operation symbol I. Every occurrence of $L(xyz)$ in any axiom in this section, as well as in the definition (6) is to be replaced by $B(xyz) \vee B(yzx) \vee B(zxy)$. The resulting axioms and axiom systems will be distinguished from the one containing L by adding a $'$ (Δ_{con} thus becomes Δ'_{con}). Noticing that O4 can be expressed in L_o with only 4 variables as

O 5 $\neg L(abb') \wedge B(abc) \rightarrow B(ab'T_a(bcb'))$.

[2] Scott also requires that the language contains no function symbols, but Theorem 2.2, on which his result depends, remains valid in languages that contain operation symbols.

and that ordered Moufang planes are Desarguesian (cf. [14, p. 240]), we get the following

Theorem 7 *$\Delta'_{con} \cup \{C11, O2, O3, O5\}$, $\Delta'_{con} \cup \{C11, C14', O2, O3, O5\}$ (or $\Delta'_{con} \cup \{C11, C12', O2, O3, O5\}$), $\Delta'_{con} \cup \{C11, C13, O2, O3, O5\}$ are simple axiom systems for ordered translation planes, ordered Desarguesian planes and ordered Pappian planes respectively. The corresponding L_o-theories thus have simplicity degree 4, which is best possible in the sense of* D. Scott*'s Theorem.*

Open Problem. What is the simplicity degree of the L-theory and of the L_{con}-theory of Desarguesian planes (of unspecified characteristic or of characteristic $\neq 2$)?

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