Orthogonality as single primitive notion for metric planes

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Abstract

We provide a first order axiomatization for Bachmann's metric planes in terms of points and the ternary relation $\perp$ with $\perp(abc)$ to be read as 'a, b, c are the vertices of a right triangle with right angle at a'. The axioms can be chosen to be $\forall\exists$-statements.

1 Introduction

The concept of a *metric plane* grew out of the work of Hessenberg, Hjelmslev, and A. Schmidt, and was provided with a simple group-theoretic axiomatics by F. Bachmann. His axiomatics (cf. [2, §3,2, p. 33]) can be rephrased in a first-order language with points and lines as individual variables, and with a binary operation ϱ for reflections in lines, with $\varrho(l, P)$ denoting the point obtained by reflecting the point P in the line l, or with only one sort of variables, for lines, and a binary operation ρ, with $\rho(g, h)$ denoting the line obtained by reflecting line h in line g (cf. [3] for an axiom system in this language). Bachmann ([2, §2,3]) also described metric planes by an axiom system in a language with points and lines as individual variables, and point-line incidence, line-orthogonality, and mappings of models as non-logical notions (cf. also [1]). That axiom system cannot be rephrased in first-order logic, as it contains references to line reflections, which are defined as bijections of the collection of all points and lines, which preserve incidence and orthogonality, are involutory transformations, different from the identity, and fix all the points of a line. We shall nevertheless state that axiom system. Its axioms are (the words 'intersect', 'through', 'perpendicular', 'have in common' are the usual paraphrases):

MP 1 *There are at least two points.*

MP 2 *For every two different points there is exactly one line incident with those points.*

MP 3 *If a is orthogonal to b, then b is orthogonal to a.*

MP 4 *Orthogonal lines intersect.*

MP 5 *Through every point P there is to every line l a perpendicular, which is unique if P is incident with l.*

MP 6 *To every line there is at least a reflection in that line.*

MP 7 *The composition of reflections in three lines a, b, c which have a point or a perpendicular in common is a reflection in a line d.*

There are other axiom systems in the literature for non-elliptic metric planes (i. e. metric planes in which the composition of three reflections in lines is never the identity): (i) in terms of points and the quaternary relation of congruence $\equiv$ ([7], [4]), (ii) in terms of points and two ternary operations in [5], (iii) in terms of 'rigid motions', and a unary predicate symbol G, with $G(x)$ to be interpreted as 'x is a line-reflection', a constant symbol 1, to be interpreted as 'the identity', and a binary operation $\circ$, with $\circ(a, b)$, to be interpreted as 'the composition of a with b' ([6]); and (iv) in terms of the two sorts of variables, points and rigid motions, and a binary operation $\cdot$, the first argument of which is a rigid motion, the second argument a point, and whose value is a point, $\cdot(g, A)$ standing for 'the action of g on A' ([6]).

The aim of this paper is to show that Bachmann's metric planes can be axiomatized in terms of points and the notion of orthogonality as single primitive notion. By this we do not mean that the axiom system is simple or that it were preferable to its competitors, but simply that the theory of metric planes can be expressed in these very simple terms.

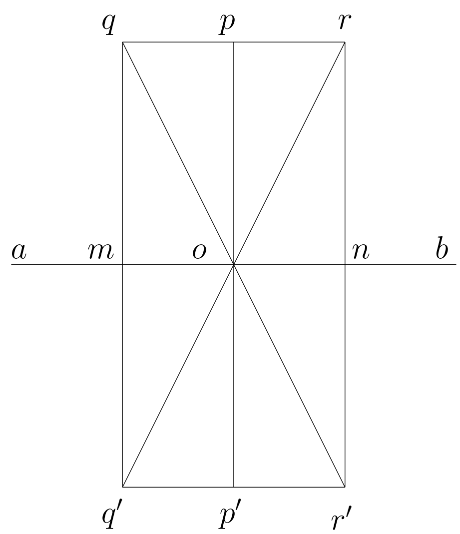

Figure 1: The reflection of p in the line ab obtained by means of φ.

2 The axiom system

The language in which we will express the axiom system for metric planes contains one sort of variables, standing for points, and a ternary relation $\perp$, with $\perp(abc)$ to be read as 'a, b, c are the vertices of a right triangle with right angle at a'.

To shorten the formal aspect of the axioms, we shall use the following abbreviations (see Fig. 1 for the definition of φ):

$$\begin{aligned}
L_e(abc) &:\Leftrightarrow \perp(abe) \wedge \perp(ace) \\
\varphi(abpmnoqq'rr'uvp') &:\Leftrightarrow a \neq b \wedge (o = a \vee \perp(oap)) \wedge (o = b \vee \perp(obp)) \\
&\quad \wedge L_o(pqr) \wedge q \neq r \wedge L_o(mqq') \wedge q \neq q' \\
&\quad \wedge L_o(nrr') \wedge r \neq r' \wedge L_o(p'q'r') \wedge L_m(opp') \wedge m \neq n \\
&\quad \wedge L_p(omn) \wedge L_u(orq') \wedge L_v(oqr') \\
R_{ab}(pp') &:\Leftrightarrow (\exists mnoqq'rr'uv)\,(((\perp(abp) \wedge \perp(bpa)) \vee (a \neq b \wedge (L_o(pab) \\
&\quad \vee p = a \vee p = b))) \wedge p' = p) \vee (\neg(\perp(abp) \wedge \perp(bpa)) \\
&\quad \wedge \varphi(abpmnoqq'rr'uvp'))
\end{aligned}$$

$L_e(abc)$ stands for 'a, b, c are three collinear points, with a different from b and c, and a, b, e are the vertices of a right triangle with right angle at a'; $R_{ab}(pp')$ stands, if a is different from b, for 'p' is the reflection of p in the line ab'.

The axioms are:

A 1 $\perp(abc) \rightarrow a \neq b \wedge b \neq c \wedge c \neq a,$

A 2 $\perp(abc) \rightarrow \perp(acb),$

A 3 $(\forall ab)(\exists c)\, a \neq b \rightarrow \perp(abc),$

A 4 $L_b(acd) \wedge \perp(ced) \rightarrow \perp(cea),$

A 5 $L_e(abc) \wedge \perp(abf) \rightarrow \perp(acf),$

A 6 $\perp(apb) \wedge \perp(bpa) \wedge L_e(abc) \rightarrow \perp(cpa),$

A 7 $(\forall abp)(\exists mnoqq'rr'p'uv)\, p = a \vee p = b \vee L_o(pab) \vee (\perp(apb) \wedge \perp(bpa))$
$\vee \varphi(abpmnoqq'rr'uvp'),$

A 8 $(\perp(apb) \wedge \perp(bpa)) \vee ((\bigwedge_{i=1}^{2} \varphi(abpm_in_io_iq_iq_i'r_ir_i'u_iv_ip_i)) \rightarrow p_1 = p_2),$

A 9 $\perp(xyz) \wedge R_{ab}(xx') \wedge R_{ab}(yy') \wedge R_{ab}(zz') \rightarrow \perp(x'y'z'),$

A 10 $(\forall abcefg)(\exists dd')(\forall pqrs)\, L_f(bac) \wedge \perp(aeb) \wedge \perp(cgb) \wedge R_{ae}(pq) \wedge R_{bf}(qr)$
$\wedge R_{cg}(rs) \rightarrow L_{d'}(dab) \wedge R_{dd'}(ps),$

A 11 $(\forall oabc)(\exists d)(\forall pqrs)\, R_{oa}(pq) \wedge R_{ob}(qr) \wedge R_{oc}(rs) \rightarrow R_{od}(ps),$

A 12 $(\exists ab)\, a \neq b.$

Somewhat informally (given that we refer to 'lines', which are not objects of our language), A1 states that if ab is orthogonal to ac, then a, b, c must be three different points; A2 states that if ab is orthogonal to ac, then ac is orthogonal to ab, A3 states that one can raise a perpendicular in a on a given line ab; A4 states that if a, c, d are three different collinear points, and ce is perpendicular on the line cd, then it is perpendicular on the line ca as well ('naturally', since the lines cd and ca are identical); A5 states that if a, b, c are three collinear points, and af is perpendicular to ab, then it is perpendicular to ac as well ('naturally', since the lines ab and ac are identical); A6 states that if both pa and pb are perpendicular to line ab, then the line pc is perpendicular to ca for any point c on the line ab; A7 states that if p is not on the line ab, and if pa and pb are not both perpendicular to ab, then there is a point p', which is the reflection of p in the line ab; A8 states that the point p' which A7 claims to exist, is unique; A9 states that reflections in lines preserve orthogonality; A10 states that the composition of reflections in the lines ae, bf, cg, which are perpendicular to the line on which a, b, c lie, is a reflection in a line, namely the reflection in the line dd'; A11 states that the composition of the reflections in the lines oa, ob, and oc, which have the point o in common, is a reflection in a line, namely the reflection in od.

3 Proof of the main result

We now proceed to prove that the axioms A1-A12 axiomatize Bachmann's metric planes.

Lemma 1 *If $a \neq b \wedge (L_p(oab) \vee (o = a \wedge \perp (opb)) \vee (o = b \wedge \perp (opa)))$, then, for no x can we have $L_x(pab)$.*

Proof. Assume $a \neq b$, $L_p(oab)$, and $L_x(pab)$. Let e be such that $\perp (abe)$ (such an e exists by A3). By A4,

$$L_x(pab) \wedge \perp (aeb) \rightarrow \perp (aep)$$
$$L_p(oab) \wedge \perp (aeb) \rightarrow \perp (aeo)$$

Given that the hypotheses of the above implications hold, we must have $\perp (aep)$ and $\perp (aeo)$, and thus, by A2, $\perp (aoe)$ as well, which means that $L_e(aop)$ holds. By A4, we have

$$L_e(aop) \wedge \perp (oap) \rightarrow \perp (oaa),$$

contradicting A1.

Assume $o = a$, $\perp (opb)$, as well as $L_x(pab)$. By A4, $L_x(pab) \wedge \perp (apb) \rightarrow \perp (app)$, and since the hypothesis holds, so must the conclusion, i. e. $\perp (app)$, contradicting A1.

Assume $o = b$, $\perp (opa)$, as well as $L_x(pab)$. Again A4 leads to $\perp (bpp)$, contradicting A1. $\square$

Notice also that, by A1, $L_p(oab) \rightarrow p \neq a \wedge p \neq b$. Thus:

(1) $$\varphi(abpmnoqq'rr'uvp') \rightarrow \neg(L_x(pab) \vee p = a \vee p = b).$$

We now turn to the proof of

(2) $$L_b(acd) \wedge \perp (cea) \wedge c \neq d \rightarrow \perp (ced)$$

Proof. Since $c \neq d$, by A3 and A2, we have $(\exists f) \perp (cfd)$. By A4, $L_b(acd)$ and $\perp (cfd)$ imply $\perp (cfa)$. Since we have both $\perp (cfa)$ and $\perp (cea)$, we have $L_a(cfe)$. Since we also have $\perp (cfd)$, we get, using A5, $\perp (ced)$. $\square$

Let us define a new predicate λ, with $\lambda(abc)$ to be read as 'a, b, c are (not necessarily different) collinear points', defined by

$$\lambda(abc) :\Leftrightarrow (\exists e)\, L_e(abc) \vee a = b \vee a = c. \tag{3}$$

To show that the axiom system A1-A12 axiomatizes metric planes, we need to define the notions of *line*, *point-line incidence*, and *line-orthogonality*, and to show that these notions, as well as the line-reflections that are induced by these notions satisfy the axioms MP1-MP7.

For any two different points a and b, we define a new object, the *line* $\overline{ab}$, and we say that a point x is *incident* with $\overline{ab}$ if and only if $\lambda(abx)$. We say that two lines $\overline{ab}$ and $\overline{cd}$ are *equal* if and only if they are incident with the same points, and we say that they are *orthogonal* if and only if there is a point o incident with both lines, and there are points p on $\overline{ab}$ and q on $\overline{cd}$, such that $\perp (opq)$.

Notice that, from the very definition of L we have

$$L_e(abc) \rightarrow L_e(acb). \tag{4}$$

We now turn to proving that

$$\lambda(abc) \rightarrow \lambda(cba) \wedge \lambda(bac). \tag{5}$$

Proof. By (3) and $\lambda(abc)$, $L_e(abc)$ for some e, or $a = b$ or $a = c$. Notice that $\lambda(abb)$ holds for all a and b by (3) and A3, thus, in case $a = b$ or $a = c$, the conclusion of (5) holds. Suppose $a \neq b$ and $a \neq c$, and $L_e(abc)$. By A3 $(\exists f) \perp (cbf)$. Given that, by (4) and A2, we have $L_e(acb)$ and $\perp (cfb)$, and os, by A4 and A1, $\perp (caf)$. Together with $\perp (cbf)$, this gives $L_f(cba)$, thus $\lambda(cba)$. Since $a \neq b$, by A3, $(\exists g) \perp (bag)$. By (2) and A2, given $L_e(abc)$ and $\perp (bag)$, we get $\perp (bcg)$, so $L_g(bac)$, i. e. $\lambda(bac)$. $\square$

Next, we prove that

$$a \neq b \wedge \lambda(abc) \wedge \lambda(abd) \rightarrow \lambda(acd). \tag{6}$$

Proof. Suppose $a \neq b$, $\lambda(abc)$ and $\lambda(abd)$. By (3), $a = c$ or there is an e such that $L_e(abc)$, and $a = d$ or there is an f such that $L_f(abd)$. If $a = c$ or $a = d$, there is nothing to prove, since $\lambda(acd)$ follows from (3). Suppose $L_e(abc)$ and $L_f(abd)$. By A5, we deduce from $L_e(abc)$ and $\perp(abf)$ that $\perp(acf)$ holds. From this and $\perp(adf)$ we get $L_f(acd)$, thus $\lambda(acd)$. $\square$

We now check the validity, with our defined notions, of the axioms MP1-MP7.

MP1 holds by A12. The existence part of MP2, i. e. the existence of a line incident with two different points a and b follows from the fact that we have $\lambda(aba)$ and $\lambda(abb)$ by (3), thus a and b are incident with the line $\overline{ab}$. To see that the uniqueness part of MP2 holds, we need to show that

$$u \neq v \wedge a \neq b \wedge L(uva) \wedge L(uvb) \rightarrow (L(uvx) \leftrightarrow L(abx)). \tag{7}$$

Proof. If $a = u$ or $a = v$ or $b = u$ or $b = v$, then (7) follows from applying once or twice (6). Suppose now $a \neq u, b \neq u, a \neq v, b \neq v$, $u \neq v$, $a \neq b$, $\lambda(uva)$, $\lambda(uvb)$, $\lambda(uvx)$. By (6) we have $\lambda(uab)$ and $\lambda(uax)$, thus $\lambda(bua)$ and $\lambda(aux)$ (by (5)). By (5) we also get $\lambda(buv)$ and $\lambda(auv)$. Since $b \neq u$, $\lambda(buv)$ and $\lambda(bua)$ imply $\lambda(bva)$ (by (6)), and, since $a \neq u$, $\lambda(aux)$ and $\lambda(auv)$ imply $\lambda(avx)$. By (5) we also have $\lambda(avb)$. Since $a \neq v$, $\lambda(avb)$ and $\lambda(avx)$ imply $\lambda(abx)$ (by (6)). Suppose now $a \neq u, b \neq u, a \neq v, b \neq v$, $u \neq v$, $a \neq b$, $\lambda(uva)$, $\lambda(uvb)$, $\lambda(abx)$. If $x = a$, then $\lambda(uvx)$ and we are done. Suppose $x \neq a$. From $u \neq v$, $\lambda(uva)$, $\lambda(uvb)$ we get $\lambda(uab)$ (by (6)) and $\lambda(vab)$ (by (5) and (6)). By (5) we have $\lambda(abu)$, $\lambda(abv)$ and $\lambda(abx)$, which together with $a \neq b$ give us $\lambda(aux)$ and $\lambda(avx)$ (by (6)). By (5) we have $\lambda(xau)$ and $\lambda(xav)$, thus, since $x \neq a$, we have $\lambda(xuv)$ as well by (6)), so $\lambda(uvx)$ (by (5)). $\square$

By (3) and (5) we have

$$\text{If } a \neq b \text{ and } c \neq d, \text{ then } \overline{ab} = \overline{cd} \text{ if and only if } \lambda(abc) \text{ and } \lambda(abd), \tag{8}$$

By A5, (8), and A2 we get

$$\begin{gathered} \text{If } a \neq b, a \neq c, a \neq b', a \neq c', \overline{ab} = \overline{ab'} \text{ and } \overline{ac} = \overline{ac'}, \\ \text{then } \perp(abc) \text{ if and only if } \perp(ab'c'), \end{gathered} \tag{9}$$

and by the definition of line perpendicularity and (9) we get

$$\begin{gathered} \text{If } a \neq b, a' \neq b', c \neq d, c' \neq d', \overline{ab} = \overline{a'b'} \text{ and } \overline{cd} = \overline{c'd'}, \\ \text{then } \overline{ab} \text{ is orthogonal to } \overline{cd} \text{ if and only if } \overline{a'b'} \text{ is orthogonal to } \overline{c'd'}. \end{gathered} \tag{10}$$

Thus orthogonality is well-defined as a binary relation between lines. That it satisfies MP4, i. e. that orthogonal lines intersect, is part of the definition of line-orthogonality. By A2 and the definition of line-orthogonality, we deduce that MP3 holds as well.

To see that MP5 holds, notice that, by A7, for $a \neq b$, and p not on $\overline{ab}$, we have either $\perp(apb)$ and $\perp(bpa)$ or $\varphi(abpmnoqq'rr'uvp')$. If $\perp(apb)$, then the lines $\overline{pa}$ and $\overline{ab}$ are orthogonal, and $\overline{pa}$ passes through p. Suppose we have $\varphi(abpmnoqq'rr'uvp')$. Then $\overline{po}$ is a line that passes through p and is orthogonal to $\overline{ab}$, given that o is on both lines, and that $\perp(oap)$ or $\perp(obp)$ must hold by the definition of φ and A2. Should p be on $\overline{ab}$, the the existence of a line through p orthogonal to $\overline{ab}$ follows from A3. The uniqueness of the orthogonal to $\overline{ab}$ through p in this case is a consequence of our definition of L, (3), and A2.

We now define the reflection ϱ_{ab} in the line $\overline{ab}$, with $a \neq b$, by assigning to each point p the point p', for which $R_{ab}(pp')$ holds. This point is p in case one of $\perp(abp) \wedge \perp(bpa)$ or $\lambda(pab)$ holds, and thus is unique. Notice that, by A6 and (8), the choice of p' as p does not depend on the particular points a and b we have chose to represent the line $\overline{ab}$. If neither $\perp(abp) \wedge \perp(bpa)$ nor $\lambda(pab)$ hold, then, by A7, there must be $m, n, o, q, q', r, r', p', u, v$ such that $\varphi(abpmnoqq'rr'uvp')$. Thus, according to the definition of R, p' must, in this case be the point for which $\varphi(abpmnoqq'rr'uvp')$ (notice that, by (1), we cannot have both $\lambda(pab)$ and $\varphi(abpmnoqq'rr'uvp')$, so that ϱ is well-defined). The point p' is unique in this case as well, by A8. Notice again that, given p, the point p' is determined by the *line* $\overline{ab}$, and not by the particular choice of a and b used to represent it. This can be seen by noticing that, in the definition of φ the only occurrence of a and b is in $a \neq b \wedge ((L_p(oab) \vee (o = a \wedge \perp(opb)) \vee (o = b \wedge \perp(opa)))$, and that, by A5, (6), (5), (3), we have

$$a \neq b \wedge (L_p(oab) \vee (o = a \wedge \perp(opb)) \vee (o = b \wedge \perp(opa))) \wedge \lambda(abc) \wedge \lambda(abd)$$
$$\wedge c \neq d \rightarrow L_p(ocd) \vee (o = c \wedge \perp(opd)) \vee (o = d \wedge \perp(opc)).$$

Thus, using A6 as well, we have

$$\neg(\perp(apb) \wedge \perp(bpa)) \wedge \varphi(abpmnoqq'rr'uvp') \wedge \lambda(abc) \wedge \lambda(abd) \wedge c \neq d$$
$$\rightarrow \neg(\perp(cpd) \wedge \perp(dpc)) \wedge \varphi(cdpmnoqq'rr'uvp'),$$

showing that the point p' depends, in case p is not such that there are two orthogonal from it to $\overline{ab}$, only on p and the line $\overline{ab}$.

The map ϱ_{ab} is orthogonality-preserving by A9, and thus, given our definitions of L and λ, collinearity-preserving as well. It fixes all the points on the line $\overline{ab}$, and it is involutory, given that $\varphi(abpmnoqq'rr'uvp') \rightarrow \varphi(abp'mnoq'qr'ruvp)$.

MP6 and MP7 follow from A10 and A11.

To show that the two axiom systems axiomatize the same class of models, we need to define in the language of the axioms MP1-MP7, the notion $\perp$, and to show that, with that definition, the axioms A1-A12 can be derived from MP1-MP7. The definition of $\perp$ (abc) is, as expected, '$a \neq b$, $a \neq c$, and the lines $\overline{ab}$ and $\overline{ac}$ are orthogonal'. By the main theorem of [2, §6,§8]:

Representation Theorem. *Every model of a metric plane (i. e. of* MP1-MP7*) can be represented as an embedded subplane (i. e. containing with every point all the lines of the projective-metric plane that are incident with it) that contains the point* $(0,0,1)$ *of a projective-metric plane* $\mathfrak{P}(K,\mathfrak{f})$ *over a field* K *of characteristic* $\neq 2$*, from which it inherits the collinearity and orthogonality relations.*

By *projective-metric* plane $\mathfrak{P}(K,\mathfrak{f})$ over a field K of characteristic $\neq 2$, with $\mathfrak{f}$ a symmetric bilinear form, which may be chosen to be defined by $\mathfrak{f}(\mathbf{x},\mathbf{y}) = \lambda x_1 y_1 + \mu x_2 y_2 + \nu x_3 y_3$, with $\lambda\mu \neq 0$, for $\mathbf{x},\mathbf{y} \in K3$ (where $\mathbf{u}$ always denotes the triple (u_1,u_2,u_3), line or point, according to context), we understand a set of points and lines, the former to be denoted by (x,y,z) the latter by $[u,v,w]$ (determined up to multiplication by a non-zero scalar, not all coordinates being allowed to be 0), endowed with a notion of incidence, point (x,y,z) being incident with line $[u,v,w]$ if and only if $xu+yv+zw=0$, an orthogonality of lines defined by $\mathfrak{f}$, under which lines $\mathbf{g}$ and $\mathbf{g}'$ are orthogonal if and only if $\mathfrak{f}(\mathbf{g},\mathbf{g}')=0$.

Thus, all we need to check is that the axioms A1-A12 hold in these embedded subplanes of projective-metric planes.[1]

The only axioms that need to be checked, the others being known to hold in metric planes, are A7 and A8. To simplify computations, we will assume, to prove that both of these axioms hold, that the line $\overline{ab}$ is the line $[0,1,0]$, and that $p=(0,\alpha,1)$, for some $\alpha \in K \setminus \{0\}$, the metric plane being denoted by $\mathfrak{M}$. This choice of p is possible whenever we know that there do not exist two different lines through p that are orthogonal to $[0,1,0]$ (if two such perpendiculars exist, then p would have to be

[1]It would have been preferable to have a synthetic proof that the axioms A7 and A8 can be derived from Bachmann's axioms for metric planes, but we could not find such a proof for A8 (see the Appendix for synthetic proofs).

$(\alpha, \beta, 0))$, and that p does not lie on $\overline{ab}$. That there exist $m, n, o, q, q', r, r', p', u, v$ such that $\varphi(abpmnoqq'rr'uvp')$ can be seen by taking $m = (x, 0, 1)$, $n = (-x, 0, 1)$, $o = (0, 0, 1)$, $q = (x, \alpha, 1)$, $q' = (x, -\alpha, 1)$, $r = (-x, \alpha, 1)$, $r' = (-x, -\alpha, 1)$, $p' = (0, -a, 1)$, u any point, different from $(0, 0, 1)$, on the line $[\alpha, x, 0]$, and v any point, different from $(0, 0, 1)$, on the line $[\alpha, -x, 0]$, where $x \in K \setminus \{0\}$ is such that $q = (x, \alpha, 1)$ is a point of $\mathfrak{M}$ (such an x must exist, given that there must be a second point on the line $[0, 1, -\alpha]$, which is a line of $\mathfrak{M}$, given that it passes through a point of $\mathfrak{M}$, namely p, and the requirement that $\mathfrak{M}$ be an embedded subplane. To see that, with $p = (0, \alpha, 1)$ and $\overline{ab} = [0, 1, 0]$, the point p' given by $\varphi(abpmnoqq'rr'uvp')$ is unique in case there do not exist two perpendiculars through p to $\overline{ab}$ (we know by (1) that p cannot be on $\overline{ab}$), we notice that the conditions $L_o(pqr)$, $q \neq r$, $L_o(mqq')$, $q \neq q'$, $L_o(nrr')$, $r \neq r'$, $L_o(p'q'r')$, $L_m(opp')$, $m \neq n$,$L_p(omn)$, from the definition of $\varphi(abpmnoqq'rr'uvp')$, imply that $m = (x, 0, 1)$, $n = (y, 0, 1)$, $o = (0, 0, 1)$, $q = (x, \alpha, 1)$, $q' = (x, \beta, 1)$, $r = (y, \alpha, 1)$, $r' = (y, \beta, 1)$, $p' = (0, \beta, 1)$, with $x \neq y$, $x \neq 0$, $y \neq 0$, $\alpha \neq 0$, $\beta \neq 0$, and $\alpha \neq \beta$. The last two condition, that, for some u and v, we have $L_u(orq')$ and $L_v(oqr')$, imply that the points o, r, q' are collinear, and that the points o, q, r' are collinear. Let $[i, j, k]$ be the line on which o, r, q' lie, and $[i', j', k']$ the line on which o, q, r' lie. Since both lines pass through $(0, 0, 1)$, we must have $k = k' = 0$. The remaining four incidences give $iy + j\alpha = 0$, $ix + j\beta = 0$, $i'x + j'\alpha = 0$, $i'y + j'\beta = 0$. This is a homogeneous linear system in the unknowns x, y, α, β, and it has a solution $(x, y, \alpha, \beta) \neq (0, 0, 0, 0)$ if and only if the determinant of the matrix of this system is zero. This means that

$$\left(\frac{j'}{i'}\right) 2 - \left(\frac{j}{i}\right) 2 = 0,$$

i. e. that $\frac{j'}{i'} = \pm\frac{j}{i}$. Since $\frac{j'}{i'} = \frac{j}{i}$ leads to $\alpha = \beta$, we must have $\frac{j'}{i'} = -\frac{j}{i}$, and this leads to $\beta = -\alpha$ and $y = -x$, implying the uniqueness of p', which must be $(0, -\alpha, 1)$ regardless of the intermediate points $m, n, o, q, q', r, r', u, v$.

4 $\forall\exists$-axiomatizability

There is a problem of a logical complexity nature regarding our axiom system A1-A12. Two of the axioms, A10 and A11 have quantifier complexity $\forall\exists\forall$, all the other axioms being $\forall\exists$-axioms (i. e. all universal quantifier (if any) precede all existential quantifiers (if any)). However, they can be replaced with axioms which are $\forall\exists$-axioms, to obtain an axiom system, A1-A9, A12, A13-A16, all of whose axioms are

$\forall\exists$-statements. A10 can be replaced by the two axioms

A 13 $(\forall abcefg)(\exists dd')\, L_f(bac) \wedge \perp(aeb) \wedge \perp(cgb) \wedge R_{bf}(ee') \wedge R_{cg}(e'e'')$
$\to L_{d'}(dab) \wedge R_{dd'}(ee'')$,

A 14 $L_f(bac) \wedge \perp(aeb) \wedge \perp(cgb) \wedge R_{bf}(ee') \wedge R_{cg}(e'e'') \wedge L_{d'}(dab) \wedge R_{dd'}(ee'')$
$\wedge R_{ae}(pq) \wedge R_{bf}(qr) \wedge R_{cg}(rs) \to L_{d'}(dab) \wedge R_{dd'}(ps)$,

and A11 by the two axioms

A 15 $(\forall oabc)(\exists d)\, o \neq a \wedge R_{ob}(aa') \wedge R_{oc}(a'a'') \to R_{od}(aa'')$,

A 16 $o \neq a \wedge R_{ob}(aa') \wedge R_{oc}(a'a'') \wedge R_{od}(aa'') \wedge R_{oa}(pq) \wedge R_{ob}(qr) \wedge R_{oc}(rs) \to R_{od}(ps)$.

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5 Appendix: Synthetic Proofs

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We present here the synthetic proofs that the author thought preferable to the algebraic proof, but could not find for A8, of the fact that the axioms A7 and A8 hold in Bachmann's metric planes. This proof removes the need to refer to the representation theorem for Bachmann's metric planes, relying instead only on the fact that metric planes can be embedded in Pappian Fanoian projective planes (i. e. projective planes that can be coordinatized by fields of characteristic $\neq 2$).

Lemma 2 *Metric planes satisfy A7.*

Proof. Let a, b, p be three given points, satisfying the hypothesis of A7, and let o be the foot of the perpendicular from p to $\overline{ab}$, and q a point different from p on the line through p, which is orthogonal to $\overline{op}$ (see Figure 1). Let $p' := \varrho_{ab}(p)$ and $q' := \varrho_{ab}(q)$, where ϱ_{ab} denotes, as previously defined, the reflection in the line $\overline{ab}$. Let $r := \varrho_{op}(q)$ and $r' := \varrho_{ab} = r$. The lines $\overline{p'r'}$ and $\overline{pp'}$ are orthogonal. The points q', r, and o lie on the line $\varrho_{ab}(\overline{qr'})$, being images under ϱ_{ab} of the points q, r', and o. If we denote by m and n the feet of the perpendiculars from q and respectively from r on $\overline{ab}$, we get $\varphi(abpmnoqq'rr'uvp')$. □

Lemma 3 *Metric planes satisfy A8.*

Proof. We will present two proofs for this lemma.

1 (by Rolf Struve). Let $a, b, p, m, n, o, q, q', r, r', u, v, p'$ be points of a metric plane $\mathfrak{M}$ with $\varphi(abpmnoqq'rr'uvp')$, and such that there is only one perpendicular from p to $\overline{ab}$, i. e. points as shown in Figure 1. We will show, that p' is $\varrho_{ab}(p)$, a uniquely determined point. According to Bachmann's [2] Main Theorem, $\mathfrak{M}$ can be embedded in a Pappian Fanoian projective plane $\mathfrak{P}$. Let z_∞ and a_∞ be the points in $\mathfrak{P}$ that lie on all lines perpendicular to $\overline{ab}$ and $\overline{pp'}$ respectively. Let ϱ be the uniquely determined homology with axis $\overline{ab}$ and centre z_∞ that maps p into p'. The point q will be mapped by ϱ into a point on the perpendicular from q to $\overline{ab}$, which is incident with $\overline{p'a_\infty}$ (given that a_∞ is the intersection point of $\overline{pq}$ and $\overline{ab}$) thus in q'. Analogously, one shows that $\varrho(r) = r'$. We also have $\varrho(q') = q$, given that q' is incident with the perpendicular from q' to $\overline{ab}$ and with $\overline{or}$, the point $\varrho(q')$ must be the point of intersection of the perpendicular from q to $\overline{ab}$ with $\overline{or'}$. Thus $\varrho(\varrho(q)) = \varrho(q') = q$. The projective collineation $\varrho \circ \varrho$ thus fixes q, z_∞, and the line $\overline{ab}$ pointwise, and must thus be the identity, so ϱ is involutory. In a Pappian Fanoian projective plane there exists only one involutory homology with given centre and axis. Thus p' is uniquely determined: it is the image of p under the reflection ϱ_{ab}.

2 (by Horst Struve). Under the same assumptions regarding the points a, b, p, m, n, o, q, q', r, r', u, v, p', let $p'^* := \varrho_{ab}(p)$, $r^* := \varrho_{op}(q)$, $q'^* := \varrho_{ab}(q)$, $r'^* := \varrho_{ab}(r^*)$. With n^* standing for the foot of the perpendicular from r^* to $\overline{ab}$, we have $\varphi(abpmn^*oqq'^*r^*r'^*u^*v^*p'^*)$. We will show, that $p' = p'^*$ i. e. that $p' = \varrho_{ab}(p)$.

To this end, we consider the following Desargues configuration: through the point q, the centre of the configuration, pass the three lines $\overline{qp}, \overline{qo}$, and $\overline{qq'}$. On these lines lie the vertices of the two triangles $rr'q'$ and $r^*r'^*q'^*$. Suppose $r \neq r^*$. Then the vertices of the two triangles are pairwise different, i. e. $r' \neq r'^*$ and $q' \neq q'^*$ as well. Thus, according to the Desargues axiom (which holds in $\mathfrak{P}$, in which the Pappus axiom holds) the two triangles must be perspective from a line as well, i. e. the intersection points of $\overline{rr'}$ and $\overline{r^*r'^*}$, of $\overline{q'r'}$ and $\overline{q'^*r'^*}$ and of $\overline{rq'}$ and $\overline{r^*q'^*}$ must be collinear. Both $\overline{rr'}$ and $\overline{r^*r'^*}$ are incident with the pole z_∞ of line $\overline{ab}$, both $\overline{q'r'}$ and $\overline{q'^*r'^*}$ are incident with the pole a_∞ of line $\overline{pp'}$, both $\overline{rq'}$ and $\overline{r^*q'^*}$ are incident with o. These three points cannot be collinear, for else, we would have $\overline{oz_\infty} = \overline{oa_\infty}$, and thus $\overline{pp'} = \overline{ab}$, contradicting the fact that $\overline{pp'}$ is perpendicular to $\overline{ab}$. This means that two corresponding vertices of the two triangles $rr'q'$ and $r^*r'^*q'^*$ must coincide, and thus all three corresponding vertices coincide, given the definition of the points with an asterisk. Thus also $p' = p'^* = \varrho_{ab}(p)$.

□

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