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Early Examples of Resource-Consciousness

Abstract. As with the development of several logical notions, it is shown that the concept of resource-consciousness, i. e. the concern over the number of times that a given sentence is used in the proof of another sentence, has its origin in the foundations of geometry, pre-dating its appearence in logical circles as BCK-logic or affine logic.

Keywords: BCK logic, affine logic, projective geometry, Pappus and Desargues axioms, Pythagorean theorem.

1. Introduction

The question regarding the number of uses of a certain sentence in a proof of another sentence first surfaced in logic in the mid 1930s with the work of Fitch and Tarski, which may be considered to belong to what came to be known after Meredith and Prior (1963) as implicational BCK logic. In case the sentence is one of predicate logic, a logical framework sensitive to this question was developed by Grishin (1974), and a similar logic has become a subject of intense investigation after the publication of J.-Y. Girard's [7], where interest is motivated by actual computing, where hypotheses are considered resources that are used up in the process of deduction (as one would use up money in the process of purchasing goods), which has resulted in the conclusion. Each access to a given resource has its own cost, or requires a certain energy-consumption, so the classic approach where unlimited access to a given hypothesis (resource) is allowed does not reflect the actual computing practice. Girard proposes two variants, linear logic, which prohibits both the rule of contraction

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

and the rule of weakening,

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

Presented by **Heinrich Wansing**; *Received* April 2003

Studia Logica **68**: 1–6, 2001.

and affine logic, which prohibits only the former, and thus is related to BCK logic (see [6] for a history of the subject).

Looking at the literature in the foundations of geometry we found several instances of explicit mention of resource-consciousness. The first one dates back to 1907, where the first and only theorem concerning a resource-optimal proof is given. Another group of mentions is connected with the group of theorems of the *Pappus (Pascal) implies Desargues* type, in which a configuration theorem is shown to derive from another one, and goes back to 1905. The authors of papers on dependence of configuration theorems routinely explicitely mention the number of times one of them has to be used to prove the other one, and in a 1931 survey paper Dehn asks the question of resource-optimality in such proofs.

2. An Optimal Proof of the Pythagorean Theorem

In his doctoral dissertation of 1907, published the following year, H. Brandes [3] provides a first and forgotten example of resource-consciousness, in the sense that he asks for the minimal number of times that a certain axiom has to be used for a proof of a certain form of the Pythagorean theorem, while allowing any number of uses of the other axioms.

He starts his investigation motivated by the question of the simplicity of a proof of a certain theorem φ from a given axiom system Σ, which consists of a finite number of axioms $\sigma_1, \dots, \sigma_n$. A proof p of φ from Σ is simplest with respect to σ_i if it uses σ_i m times and this is the minimum number of uses of σ_i (one is allowed to use all the other axioms an unlimited number of times) in any proof of φ from Σ. For a given sentence φ and axiom system Σ there are thus, in general, n "simplest" proofs, $p_1, \dots, p_n$ with respect to each of $\sigma_1, \dots, \sigma_n$. If it happens that there is a proof that is simplest with respect to each of the σ_i, then it is called the absolutely simplest proof.

As an important example of proof simplicity he studies the simplicity of proofs by decomposition of the Pythagorean theorem. In other words, given a right triangle ABC, with right angle in B, consider the set S of all sentences stating that there exist squares s_1, s_2, s_3, with sides congruent to BA, BC, and AC, and there are points $P_1, \dots P_n$ on the sides or in the interior of these squares, such that, under a certain triangulation of each square formed by these points and the vertices of s_1, s_2, s_3, the triangles formed in the triangulations of s_1 and s_2 are congruent, under a certain pairing, to the triangles formed by the triangulation of s_3. Congruence of triangles is defined to mean that all sides and all correspondinmg angles are congruent.

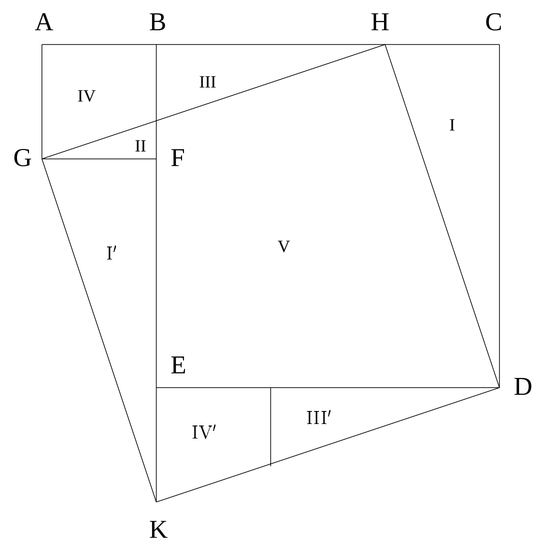

Figure 1. The sum of the areas of $BCDE$ and $ABFG$ is equal to the area of $GHDK$.

Brandes proves that, among all proofs of any sentence in S from Hilbert's [9] axiom system for elementary Euclidean geometry (no Archimedean axiom and no completeness axiom), the simplest with respect to the plane congruence axiom (PCA), by which he understands the conjunction of axioms III4, III5, and III6 from p. 9 of [9], uses PCA 7 times (see Figure 1, in which $ABFG$, $BCDE$, and $GHDK$ are the squares s_1, s_2, and s_3, whereas BF, BC are the sides of the right triangle FBC, the square s_3 being drawn on GH, which is congruent to FC (the decomposition goes back to Al-Nairizi, who attributes it to Thabit ibn Qurra (9th century)[1]).

The PCA is made up of three statements: One is the side-angle-side congruence theorem, the other two state that angle congruence is an equivalence relation. If triangle congruence were not defined as above, but only by requiring the sides of the two triangles to be congruent (as was first suggested in [12], and then used systematically by Tarski and his school (cf. [16])) then the number of times the axiom is used in the case presented by Brandes as minimal would be only 4, since 3 of his triangles coincide in the triangulations of s_1, s_2 and s_3, so that the only use made of PCA is in proving that the angles of those triangles are congruent to themselves.

It would be interesting to find out what the minimum number of uses of Mollerup's five segment axiom (cf.[16]), which is the substitute of the

[1] I thank Douglas Rogers and Greg Frederickson for this historical reference.

side angle side congruence axiom, III6 in [9] (III5 in the 12th edition of the *Grundlagen*), would be for a proof of any sentence in S from Tarski's axiom system.

3. Three Times Pappus Implies Desargues

In 1905 G. Hessenberg proved that the threefold application of Pappus axiom, together with the trivial axioms for plane projective geometry, implies Desargues axiom. It follows that this holds for the affine case as well. All known proofs require a threefold use of Pappus' axiom (cf. [10]), and this threefold application is mentioned in all accounts, including the original review of Hessenberg's paper in JFM, and in all subsequent works dealing with configurations, such as [17, p. 132], [13, p. 1], [11, Satz 5 (p. 130), Satz 6 (p. 132), Satz 9 (p. 134)] (where the relationship between several configuration theorems of affine geometry is studied, and one finds the same *durch dreimalige Anwendung von* scrupulously mentioned in every instance), and [1, S4,4 (Satz 6)] (where we find mention of a sixfold use of one statement in the proof of another one).

In these works it is not investigated whether the stated number of applications of a particular axiom is minimal, and in none of the theorems proved is it known whether this actually is the case. However, this question, as well as another one regarding equivalent configuration theorems, have been formulated in full generality by Dehn [5, p. 72]:

> Wie oft muss man mindestens den Schnittpunktsatz S anwenden, um aus ihm den Schnittpunktsatz T zu folgern? T folge durch n-malige Anwendung aus S, S folge durch m-malige Anwendung aus T. Besteht zwischen n und m irgend welche Abhängigkeit?

These questions (such as 'Can fewer than three times Pappus imply Desargues?') can be asked in a precise manner inside a sequent calculus of the type proposed by [14], the details of which can be found in [15], so there is no need for a move into affine or linear logic to make sense of such questions. The fact that they were stated early on indicates the concern with such fine points of proof theory from those working in the foundations of geometry.

A similar statement appeared in Blaschke's lecture *Waben und Gruppen* held in 1951, and annexed to [2] as its last chapter. He writes:

> Es genügt, dreimal die Thomsen-Bedingung anzuwenden, um eine Reidemeister-Figur zu erhalten. ([2, p. 190]).

And in the review of [2] for *Mathematical Reviews*, Coxeter [4] wrote:

> The author draws attention to a mysterious analogy relating projective geometry to net-theory. The Pappus configuration, which exists when the field of the geometry is commutative, corresponds to the Thomsen figure, which closes when the group of the net is Abelian. Just as three applications of Pappus's theorem are needed to prove Desargues's theorem, so three applications of the closing of Thomsen's figure are needed to establish the closing of another important figure, named after Reidemeister.

Although Blaschke draws analogies between projective geometry and nets, he does not mention the analgy between the triple use of Pappus and Thomsen. It is Coxeter's own observation.

4. Conclusion

As is the case with so many logical notions, whether that of a model, of independence of an axiom from other axioms, it was shown that the concern over the number of uses of hypotheses, considered as resources, originates not in computer science, nor in pure logic, but in the foundations of geometry, and it seems that the gradual separation of logic from the foundations of geometry has lead to long delays in investigating issues that research in the foundations of geometry had raised.

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