AXIOMATIZATIONS OF HYPERBOLIC AND ABSOLUTE GEOMETRIES

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The two of us, the two of us, without return, in this world,
live, exist, wherever we'd go, we'd meet the same faraway point.
Yeghishe Charents, *To a chance passerby.*

Abstract A survey of finite first-order axiomatizations for hyperbolic and absolute geometries.

1. Hyperbolic Geometry

Elementary Hyperbolic Geometry as conceived by Hilbert

To axiomatize a geometry one needs a language in which to write the axioms, and a logic by means of which to deduce consequences from those axioms. Based on the work of Skolem, Hilbert and Ackermann, Gödel, and Tarski, a consensus had been reached by the end of the first half of the 20th century that, as Skolem had emphasized since 1923, "*if* we are interested in producing an axiomatic system, we can *only* use first-order logic" ([21, p. 472]).

The language of first-order logic consists of the logical symbols $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, a denumerable list of symbols called *individual variables*, as well as denumerable lists of n-ary *predicate (relation)* and *function (operation)* symbols for all natural numbers n, as well as *individual constants* (which may be thought of as 0-ary function symbols), together with two quantifiers, $\forall$ and $\exists$ which can bind only individual variables, but not sets of individual variables nor predicate or function symbols. Its axioms and rules of deduction are those of classical logic.

Axiomatizations in first-order logic preclude the categoricity of the axiomatized models. That is, one cannot provide an axiom system in first-order logic which admits as its only model a geometry over the field of real numbers, as Hilbert [31] had done (in a very strong logic) in his *Grundlagen der Geometrie*. By the Löwenheim-Skolem theorem, if such an axiom system admits an infinite model, then it will admit models of any given infinite cardinality.

Axiomatizations in first-order logic, which will be the only ones surveyed, produce what is called an *elementary* version of the geometry to be axiomatized, and in which fewer theorems are true than in the standard versions over the real (or complex) field. It makes, for example, no sense to ask what the perimeter of a given circle is in elementary Euclidean or hyperbolic geometry, since the question cannot be formulated at all within a first-order language.

This does, however, not mean that the axiom systems surveyed here were presented inside a logical formalism by the authors themselves. In fact, those working in the foundations of geometry, unless connected to Tarski's work, even when they had worked in both logic and the foundations of geometry (such as Hilbert, Bachmann, and Schütte), avoided any reference to the former in their work on the latter. Some of the varied reasons for this reluctance are: (1) given that the majority of 20th century mathematicians nurtured a strong dislike for and a deep ignorance of symbolic logic, it was prudent to stay on territory familiar to the audience addressed; (2) logical formalism is of no help in achieving the crucial foundational aim of proving a representation theorem for the axiom system presented, i. e. for showing that every model of that axiom system is isomorphic to a certain algebraic structure; (3) logical formalism is quite often detrimental to the readability of the axiom system.

The main aim of our survey is the presentation of the axiom systems themselves, and we are primarily concerned with formal aspects of possible axiomatizations of well-established theories for which the representation theorem, arguably one of the most difficult and imaginative part of the foundational enterprise, has been already worked out. It is this emphasis on the manner of narrating a known story which makes the use of the logical formalism indispensable.

We shall survey only finite axiomatizations, i. e. all our axiom systems will consist of finitely many axioms. The infinite ones are interesting for their metamathematical and not their synthetically geometric properties, and were comprehensively surveyed by Schwabhäuser in the second part of [71]. All of the theories discussed in this paper are undecidable, as proved by Ziegler [88], and are consistent, given that they have consistent, complete and decidable extensions. The consistency proof can be carried out inside a weak fragment of arithmetic (as shown by H. Friedman (1999)).

Elementary hyperbolic geometry was born in 1903 when Hilbert [32] provided, using the end-calculus to introduce coordinates, a first-order axiomatization for it by adding to the axioms for plane absolute geometry (the plane axioms contained in groups I (Incidence), II (Betweenness), III (Congruence)) a *hyperbolic parallel axiom* stating that

HPA. *From any point P not lying on a line l there are two rays r_1 and r_2 through P, not belonging to the same line, which do not intersect l, and such that every ray through P contained in the angle formed by r_1 and r_2 does intersect l.*

Hilbert left many details out. The gaps were filled by Gerretsen (1942) and Szász [82], [83] (cf. also Hartshorne [26, Ch. 7, §41-43]), after initial attempts by Liebmann (1904), [49] and Schur (1904). Gerretsen, Szász, and Hartshorne succeeded in showing how a hyperbolic trigonometry could be developed in the absence of continuity, and in providing full details of the coordinatization. Different coordinatizations were proposed by de Kerékjártó (1940/41), Szmielew [85] and Doraczyńska [18] (cf. also [71, II.2]).

Tarski's language and axiom system. Given that Hilbert's language is a two-sorted language, with individual variables standing for *points* and *lines*, containing point-line incidence, betweenness, segment congruence, and angle congruence as primitive notions, there have been various attempts at simplifying it. The first steps were made by Veblen (1904, 1914) and Mollerup (1904). The former provided in 1904 an axiom system with *points* as the only individuals and with betweenness as the only primitive notion, arguing that segment and angle congruence may be defined in Cayley's manner in the projective extension, and thus, in the absence of a precise notion of elementary (first-order) definability, deemed them superfluous. In 1914 he provided an axiom system with *points* as individual variables and betweenness and equidistance as the only primitive notions. Mollerup (1904) showed that one does not need the concept of angle-congruence, as it can be defined by means of the concept of segment congruence. This was followed by Tarski's [86] most remarkable simplification of the language and of the axioms, a process started in 1926-1927, when he delivered his first lectures on the subject at the University of Warsaw, by both turning, in the manner of Veblen, to a one-sorted language, with *points* as the only individual variables — which enables the axiomatization of geometries of arbitrary dimension, without having to add a new type of variable for every dimension, as well as that of dimension-free geometry (in which there is only a lower-dimension axiom, but no axiom bounding the dimension from above) — and two relation symbols, the same used by Veblen in 1914, namely betweenness and equidistance.

We shall denote Tarski's first-order language by $\mathcal{L}(B, \equiv)$: there is one sort of individual variables, to be referred to as *points*, and two relation symbols, a ternary one, B, with $B(abc)$ to be read as 'point b lies between a and c', and a quaternary one, $\equiv$, with $ab \equiv cd$ to be read as 'a is as distant from b as c is from d', or equivalently 'segment ab is congruent to segment cd'. For improved readability, we shall use the following abbreviation for the concept of collinearity (we shall use the sign $:\Leftrightarrow$ whenever we introduce abbreviations, i. e. defined notions):

$$L(abc) :\Leftrightarrow B(abc) \vee B(bca) \vee B(cab). \tag{1}$$

In its most polished form (to be found in [71] (cf. also [87] for the history of the axiom system)), the axioms corresponding to the plane axioms of Hilbert's groups I, II, III, read as follows (we shall omit to write the universal quantifiers for universal axioms):

A 1.1. $ab \equiv ba$,

A 1.2. $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,

A 1.3. $ab \equiv cc \rightarrow a = b$,

A 1.4. $(\forall abcq)(\exists x)\, B(qax) \wedge ax \equiv bc$,

A 1.5. $a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd'$
$\wedge bd \equiv b'd' \rightarrow cd \equiv c'd'$,

A 1.6. $B(aba) \rightarrow a = b$,

A 1.7. $(\forall abcpq)(\exists x)\, B(apc) \wedge B(bqc) \rightarrow B(pxb) \wedge B(qxa)$,

A 1.8. $(\exists abc)\, \neg L(abc)$,

A 1.9. $p \neq q \wedge ap \equiv aq \wedge bp \equiv bq \wedge cp \equiv cq \rightarrow L(abc)$.

A1.4 is a segment transport axiom, stating that we can transport any segment on any given line from any given point; A1.5 is the five-segment axiom, whose statement is close to the statement of the side-angle-side congruence theorem for triangles; A1.7 is the Pasch axiom (in its *inner form*); A1.8 is a lower dimension axiom stating that the dimension is ≥ 2; A1.9 is an upper-dimension axiom, stating that the dimension is ≤ 2. We denote by $\mathcal{A}_2$ the $\mathcal{L}(B, \equiv)$-theory axiomatized by A1.1-A1.9, and by $\mathcal{A}$ the one axiomatized by A1.1-A1.8, i. e. $\mathcal{A}_2 := Cn(A1.1 - A1.9)$, where $Cn(\Sigma)$ stands for the set of logical consequences of Σ.

The following axiom does not follow from Hilbert's axioms of groups I, II, III, but it nevertheless states a property common to Euclidean and hyperbolic geometry, usually called the Circle Axiom, which states that a circle intersects any line passing through a point which lies inside the circle.

CA. $(\forall abcpqr)(\exists x)\, B(cqp) \wedge B(cpr) \wedge ca \equiv cq \wedge cb \equiv cr \rightarrow cx \equiv cp \wedge B(axb)$.

The exact statement **CA** makes is: "If a is inside and b is outside a circle (with centre c and radius cp), then the segment ab intersects that circle."

All those involved in the coordinatization of elementary plane hyperbolic geometry proved a version of the following

REPRESENTATION THEOREM 1. *$\mathfrak{M}$ is a model of $\mathcal{H}_2 := \mathrm{Cn}(\mathcal{A}_2 \cup \{$**HPA**$\})$ if and only if $\mathfrak{M}$ is isomorphic to the Klein plane over a Euclidean ordered field (or, historically more accurate, the Beltrami-Cayley-Klein plane). These are the planes described in Pejas's classification of models of Hilbert's absolute geometry as planes of type III with $J = (0)$ and K a Euclidean ordered field.*

The end-calculus, which is the method developed by Hilbert [32] to prove the above theorem, uses the notion of limiting parallel ray, defined on the basis of **HPA**, to introduce the notion of an *end*, which is an equivalence class of limiting parallel

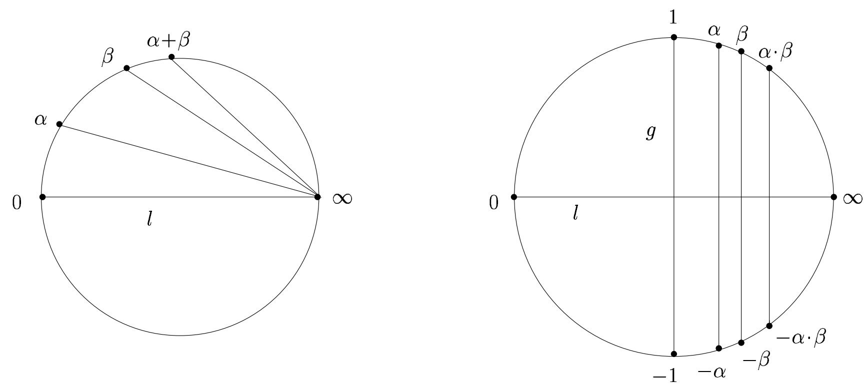

Figure 1. The end-calculus

rays. When we say that a line l has two ends α and β, we are saying that the two opposite rays in which the line l can be split, belong to the equivalence classes α and β. On the set of all ends, from which one end, denoted ∞, has been removed, one defines, with the help of the three-reflection theorem (which allows one to conclude that the composition of certain three reflections in lines is a reflection in a line), an addition and a multiplication operation, as well as an ordering, which turn the set of ends without ∞ into a Euclidean ordered field.

One starts by fixing a line l, and labeling its ends 0 and ∞ (see Fig. 1). Given ends α, β not equal to ∞, we let σ_a, σ_b denote the reflections in the lines having the ends ∞, α, and ∞, β respectively. We define $\alpha + \beta$ to be the end ξ, which is the end different from ∞ on the line x, for which $\sigma_x = \sigma_a\sigma_l\sigma_b$ (the fact that the composition of the three reflections $\sigma_a\sigma_l\sigma_b$ is a reflection in a line can be proved from the axioms in [4, §11,1], significantly weaker than those assumed here). To define multiplication, let h be a line perpendicular to l, and let 1 and -1 denote its two ends. Given two ends α and β different from both 0 and ∞, we let a, b denote the perpendiculars to l with ends α, respectively β. Then $\alpha \cdot \beta$ is defined to be that end of the line c for which $\sigma_c = \sigma_a\sigma_h\sigma_b$ which lies (i) on the same side of l in which 1 lies, provided that α and β lie on the same side of l; (ii) on the same side of l in which -1 lies, provided that α and β lie on different sides of l (by "an end ϵ lies on the side $\mathbf{s}$ of a line l" we mean to state that "there is a ray belonging to ϵ which lies completely in $\mathbf{s}$"). The existence of the line c for which $\sigma_c = \sigma_a\sigma_h\sigma_b$ follows from the three-reflection theorem for three lines with a common perpendicular, i. e. A2.18. An end is positive if it lies on the same side of l as 1, and negative if it lies on the same side of l as -1, and zero if it is 0.

We can now extend the set of points of the hyperbolic plane by first adding all ends to it (see Fig. 2). The set of all perpendiculars to a line g of the hyperbolic plane will be called a *pole* of g, and will be denoted by $P(g)$. We shall treat poles as points, and add these new points to our plane, calling them *exterior* points. The

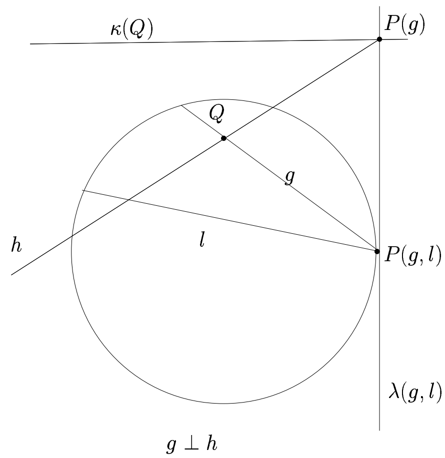

Figure 2. The projective extension

extended plane thus consists of the points of the hyperbolic plane, to be referred to as *interior* points, of ends, to be referred to as *absolute* points, and of exterior points. We also extend the set of lines with two kinds of lines: *absolute* lines and *exterior* lines, in one-to-one correspondence with absolute respectively interior points. The absolute point uniquely determined by two different (interior) lines g and h passing through it will be denoted by $P(g,h)$, and its associated absolute line by $\lambda(g,h)$; the exterior line associated with the the interior point P will be denoted by $\kappa(P)$. The incidence structure of the extended plane is given by the following rules: (i) interior points are not on absolute or exterior lines; (ii) absolute points are not on exterior lines; (iii) an absolute point $P(g,h)$ is on an interior line l if and only if l, g, h go through the same end; (iv) the absolute point $P(a,b)$ is on the absolute line $\lambda(g,h)$ if and only if $P(a,b) = P(g,h)$; (v) an exterior point $P(l)$ is on an interior line g if and only if g is perpendicular to l; (vi) an exterior point $P(l)$ is on an exterior line $\kappa(Q)$ if and only if $Q \in l$; (vii) an exterior point $P(l)$ is on an absolute line $\lambda(g,h)$ if and only if l passes through $P(g,h)$. The extended plane turns out to be a projective plane coordinatized by the ordered field of ends, and the correspondence between points and lines we have just defined turns out to be a hyperbolic projective polarity, with the set of ends as its *absolute conic*. The hyperbolic plane is thus the interior of the absolute conic, in other words, the Klein plane over the field of ends.

Synthetic proofs that **CA**, as well as the Two Circle Axiom (**TCA**), stating that two circles, one of which has points both inside and outside the other circle, intersect, follow from $\mathcal{H}_2$, were provided by Schur (1904), Szász (1958) and Strommer [80], [81]. That $\mathcal{A}_2 \vdash \mathbf{TCA} \rightarrow \mathbf{CA}$ was proved in [42, p. 168f], and the fact that $\mathbf{CA} \rightarrow \mathbf{TCA}$ can also be proved based on $\mathcal{A}_2$ was shown by Strommer (1973).

That **HPA** can be replaced by the weaker requirement that

HPA$_0$. *There is a point P and a line l, with P not incident with l, and there are two rays r_1 and r_2 through P, not belonging to the same line, which do not intersect l, and such that every ray through P contained in the angle formed by r_1 and r_2 does intersect l.*

has been shown independently by Strommer (1962), Piesyk (1961), and Baumann and Schwabhäuser (1970). These authors were probably aware that at the time of their writing the problem of finding equivalents of **HPA** had been reduced to that of checking whether a certain statement holds in a certain algebraically described coordinate geometry, since Pejas [66] had succeeded in describing algebraically all models of $\mathcal{A}_2$, so their aim was to provide meaningful synthetic proofs for the equivalence. The same holds for the synthetic proofs of **CA** and **TCA**, and the proof of their equivalence in $\mathcal{A}_2$.

Among the weaker versions of **HPA**, there are a number of axioms which have been of interest. The first is the axiom characterizing the metric, and not the behaviour of parallels, as non-Euclidean. Among the many equivalent statements, "There are no rectangle" ($\neg\mathbf{R}$) is the most suggestive. A strengthening of $\neg\mathbf{R}$, stating that the metric is hyperbolic, can be expressed as "The midline of a triangle is less that half of that side of the triangle whose midpoint is not one of the endpoints of the midline" ($\mathbf{HM}$). A weakening of **HPA**, which is stronger than $\mathbf{HM}$, and was introduced by Bachmann [5], is the negation of his *Lotschnittaxiom* ($\mathbf{A}$), stating that "There exists a quadrilateral with three right angles which does not close" (or, put differently, "There is a right angle and two perpendiculars on the sides of it which do not intersect"). Axiom $\neg\mathbf{A}$ is equivalent to an existential statement (cf. [57]). We have the following chain of implications, with no reverse implication holding $\mathcal{A}_2 \vdash \mathbf{HPA} \to \neg\mathbf{A} \to \mathbf{HM} \to \neg\mathbf{R}$ and none of the reverse implications holds in $\mathcal{A}_2 \vdash \mathbf{HPA} \to \neg\mathbf{A} \wedge \mathbf{CA} \to \mathbf{HM} \wedge \mathbf{CA} \to \neg\mathbf{R} \wedge \mathbf{CA}$ either (cf. [23]).

It is natural to ask what the missing link is, that one would need to add to $\mathcal{A}_2$, $\neg\mathbf{R}$, and $\mathbf{CA}$ to obtain an axiom system for hyperbolic geometry. As proved by Greenberg [24], it is for Aristotle's axiom $\mathbf{Ar}$, stating that "The lengths of the perpendiculars from from one side to the other of a given angle increase indefinitely, i. e. can be made longer than any given segment", that we have

$$\mathcal{A}_2 \vdash \mathbf{HPA} \leftrightarrow \neg\mathbf{R} \wedge \mathbf{CA} \wedge \mathbf{Ar}. \tag{2}$$

It follows from (2), and has been pointed out in [46], that $\mathcal{H}_2$ admits a $\forall\exists$-axiom system, i. e. one in which for all axioms, when written in prenex form, all universal quantifiers (if any) precede all existential quantifiers (if any). It was shown by Kusak (1979) that one could replace **CA** in (2) by Liebmann's [49] axiom **L**, best expressed by using perpendicularity, defined by $\perp(abc) :\Leftrightarrow (\exists c')B(cac') \wedge ac \equiv ac' \wedge bc \equiv bc'$, as

L. $(\forall abcd)(\exists x)\ \perp(bac) \wedge \perp(cbd) \wedge \perp(dca) \to B(axd) \wedge ab \equiv cx,$

i. e. "In a quadrangle $abcd$ with three right angles b, c, d, the circle with centre c and radius ab intersects the side ad". Thus (2) becomes $\mathcal{A}_2 \vdash \mathbf{HPA} \leftrightarrow \neg\mathbf{R} \wedge \mathbf{L} \wedge \mathbf{Ar}$.

The Menger-Skala axiom system

Menger (1938) has shown that in hyperbolic geometry the concepts of betweenness and equidistance can be defined in terms of the single notion of point-line incidence, and thus that plane hyperbolic geometry can be axiomatized in terms of this notion alone by rephrasing a traditional axiom system in terms of incidence alone. This was one of the most important discoveries, for it shed light on the true nature of hyperbolic geometry, which moved nearer to projective geometry than to its one-time sister Euclidean geometry, in partial opposition to which it had been born. Menger even claimed that this fact alone proved — pace Poincaré — that hyperbolic geometry was actually *simpler* than Euclidean geometry.

Since an axiom system obtained by replacing all occurrences of betweenness and equidistance with their definitions in terms of incidence would look highly unnatural, its axioms long and un-intuitive, expressing properties of incidence in a roundabout manner, Menger and his students have looked for a more natural axiom system, that should not be derived from a traditional one, but should isolate some fundamental properties of incidence in plane hyperbolic geometry from which all others derive. This task was carried out by Menger, whose last word on the subject was [52], and by his students Abbott, DeBaggis and Jenks, but even their most polished axiom system contained a statement on projectivities that was not reducible to a first-order statement. Skala [73] showed that that axiom can be replaced by the axioms of Pappus and Desargues for the hyperbolic plane, thus accomplishing the task of producing the first elegant first-order axiom system for hyperbolic geometry based on incidence alone.

This axiom system is formulated in a two-sorted first-order language, with individual variables for *points* (upper-case) and *lines* (lower-case), and a single binary relation $|$ as primitive notion, with $P|l$ to be read 'point P is incident with line l'. To shorten the statement of some of the axioms we define: (1) the notion of betweenness β, with $\beta(A, B, C)$ ('B lies between A and C') to denote 'the points A, B, and C are three distinct collinear points and every line through B intersects at least one line of each pair of intersecting lines which pass through A and C; (2) the notions of ray and segment in the usual way, i. e. a point X is on (incident with) a ray $\overrightarrow{AB}$ (with $A \neq B$) if and only if $X = A$ or $X = B$ or $\beta(A, X, B)$ or $\beta(A, B, X)$, and a point X is incident with the segment AB if and only if $X = A$ or $X = B$ or $\beta(A, X, B)$; (3) the notion of ray parallelism, for two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ not part of the same line by the condition that every line that meets one of the two rays meets the other ray or the segment AC; two lines or a line and a ray are said to be parallel if they contain parallel rays; (4) the notion of a rimpoint as a pair (a, b) of parallel lines, which is said to be incident with a line l if $l = a$, $l = b$ or l is parallel to both a and b, and there exists a line that intersects a, b, and l; a rimpoint (a, b) is identical with a rimpoint (c, d) if both c and d are incident with (a, b). Rimpoints will be denoted in the sequel by capital Greek characters. A point of the closed hyperbolic plane (i. e. a point or a rimpoint) will be denoted in the sequel by capital Latin characters with a bar on top, and will be referred to as a *Point*. If Π_1 and Π_2 are rimpoints, then $\Pi_1\Pi_2$ denotes the line l incident with both Π_1 and Π_2, and $\Pi_1 P$ denotes the line l incident with Π_1 and P.

The axioms, which we present in informal language, their formalization being straightforward, are:

A 1.10 *Any two distinct points are on exactly one line.*

A 1.11 *Each line is on at least one point.*

A 1.12 *There exist three collinear and three non-collinear points.*

A 1.13 *Of three collinear points, at least one has the property that every line through it intersects at least one of each pair of intersecting lines through the other two.*

A 1.14 *If P is not on l, then there exist two distinct lines on P not meeting l and such that each line meeting l meets at least one of those two lines.*

A 1.15 *Any two non-collinear rays have a common parallel line.*

A 1.16 *(Pascal's theorem on hexagons inscribed in conics) If Π_i ($i = 1, \dots, 6$) are rimpoints and M, N, P are the intersection points of the lines $\Pi_1\Pi_2$ and $\Pi_4\Pi_5$, $\Pi_2\Pi_3$ and $\Pi_5\Pi_6$, $\Pi_3\Pi_4$ and $\Pi_6\Pi_1$, then M, N, and P are collinear.*

A 1.17 *(Pappus) Let a and b be different lines containing Points $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ and $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$ respectively, with $\overline{A}_i \neq \overline{B}_j$ for all $i, j \in \{1, 2, 3\}$ and $\overline{A}_i \neq \overline{A}_j$, $\overline{B}_i \neq \overline{B}_j$ for $i \neq j$. If $\overline{M}$ lies on the lines $\overline{A}_1\overline{B}_2$ and $\overline{A}_2\overline{B}_1$, $\overline{N}$ lies on the lines $\overline{A}_1\overline{B}_3$ and $\overline{A}_3\overline{B}_1$, and $\overline{P}$ lies on the lines $\overline{A}_2\overline{B}_3$ and $\overline{A}_3\overline{B}_2$, then $\overline{M}$, $\overline{N}$, and $\overline{P}$ are collinear.*

A 1.18 *(Desargues) Let a, b, c be three different lines, $\overline{O}$ a Point incident with each of them, each containing pairs of distinct Points $(\overline{A}_1, \overline{A}_2)$, $(\overline{B}_1, \overline{B}_2)$, and $(\overline{C}_1, \overline{C}_2)$ respectively. If $\overline{M}$ lies on the lines $\overline{A}_1\overline{B}_1$ and $\overline{A}_2\overline{B}_2$, $\overline{N}$ lies on the lines $\overline{A}_1\overline{C}_1$ and $\overline{A}_2\overline{C}_2$, and $\overline{P}$ lies on the lines $\overline{B}_1\overline{C}_1$ and $\overline{B}_2\overline{C}_2$, then $\overline{M}$, $\overline{N}$, and $\overline{P}$ are collinear.*

Although this axiomatization is simpler than any possible one for Euclidean geometry , by being based on point-line incidence alone, there is no $\forall\exists\forall$-axiom system for hyperbolic geometry formulated only in terms of incidence or collinearity (L), as noticed by Pambuccian (2004). All the axioms of the Menger-Skala axiom system can be formulated as $\forall\exists\forall\exists$-axioms, this being the simplest possible one for hyperbolic geometry expressed by means of L alone, as far as quantifier-complexity is concerned.

Since in hyperbolic geometry of any dimension three points a, b, c are collinear if and only if there is a point d such that, for all x we have $\sigma(a\sigma(b\sigma(cx))) = \sigma(dx)$, where $\sigma(xy)$ stands for the point obtained by reflecting y in x (i. e. three points are collinear if and only if the composition of the reflections in those points is a reflection in a point, cf. [39], [41]), we can rephrase any axiom system of hyperbolic geometry expressed in terms of collinearity into one expressed in terms of the binary operation of point reflection σ (or equivalently in terms of the midpoint operation). However, one can provide an axiom system for hyperbolic geometry of any finite dimension in terms of σ of lower quantifier complexity than that of any axiom system based on collinearity. A $\forall\exists$-axiom system for it has been provided in [62].

In the above two theories for plane hyperbolic geometry presented so far, the one expressed in $\mathcal{L}(B, \equiv)$ and the one expressed in terms of collinearity (L) alone, the primitive notions are different, although the intended interpretation of the individual variables is the same (*points* in both cases). In this case, the equivalence or synonymity of two theories can be defined by stipulating that there exists a definition of each predicate and operation symbol of one theory in terms of the primitive notions of the other, and that the theory expressed in the language that contains all the primitive notions of the two theories, containing all the sentences true in both theories, as well as all the definitions referred to above is consistent (in other words, if the two theories have a common definitional extension).

In case the individual variables of two theories do not necessarily have the same intended interpretation (and such a case would be hyperbolic geometry expressed in $\mathcal{L}(B, \equiv)$ and expressed in a two-sorted language by means of the single binary relation $\in$ of point-line incidence), we say that two theories $\mathcal{T}_1$ and $\mathcal{T}_2$, in languages $\mathcal{L}_1$ and $\mathcal{L}_2$, which we shall, for simplicity's sake (the general case being a straightforward extension), consider to be one-sorted (they each have only one sort of individual variables), axiomatize the 'same geometry', if the following conditions hold:

There are natural numbers k_i for $i = 1, 2$ such that:

(i) one can identify the individuals X of L_1 with any $k = k_1$-tuple $(x_1, \ldots, x_k)$ of individuals from L_2 which satisfies a certain formula with k free variables $\varphi(x_1, \ldots, x_k)$ of L_2;

(ii) there is a definition for the equivalence of two k-tuples, in terms of an L_2 formula ψ with $2k$ free variables, such that $(x_1, \ldots, x_k) \equiv (y_1, \ldots, y_k)$ if and only if $\psi(x_1, \ldots, x_k, y_1, \ldots y_k)$ holds;

(iii) for every n-ary relation symbol π of L_1, there is an L_2-formula δ_π with kn free variables, such that $\pi(X_1, \ldots, X_n)$ holds if and only if
$\delta_\pi(x_{1,1}, \ldots x_{1,k}, \ldots, x_{n,1}, \ldots, x_{n,k})$ holds, where $(x_{i,1}, \ldots, x_{i,k})$ is a k-tuple associated via (i) to X_i; analogously for operation symbols;

(iv) For every formula θ, if $\mathcal{T}_1 \vdash \theta$ then $\mathcal{T}_2 \vdash \overline{\theta}$, where $\overline{\theta}$ is the L_2-formula obtained by replacing all of its individual variables with k-tuples satisfying φ, all equality symbols with the $\equiv$-relation, and all occurring relation and function symbols with the L_2-formulas that correspond to them by (iii);

(v) these conditions must also hold with 1 and 2 interchanged (k thus becomes k_2).

Constructive axiomatizations

Constructive axiomatizations of geometry were introduced in [53], and the constructive theme was continued with axiomatizations in infinitary logic by Engeler (1968) and Seeland (1978). In the finitary case, they can be characterized as being formulated in first-order languages without predicate symbols, and consisting entirely of universal axioms (a purely existential axiom eliminates the need for individual constants in the language, and will be allowed in constructive axiomatizations). These languages contain only function symbols and individual constants as primitive notions, and the axioms contain, with the possible exception of a single purely existential axiom (in case the language contains no individual constants), *no existential quantifiers*.

Such universal axiomatizations in languages without relation symbols capture the essentially constructive nature of geometry, that was the trademark of Greek geometry. For Proclus, who relates a view held by Geminus, “a postulate prescribes that we construct or provide some simple or easily grasped object for the exhibition of a character, while an axiom asserts some inherent attribute that is known at once to one’s auditors”. And “just as a problem differs from a theorem, so a postulate differs from an axiom, even though both of them are undemonstrated; the one is assumed because it is easy to construct, the other accepted because it is easy to know.” That is, *postulates* ask for the production, the $\pi o\acute{\iota}\eta\sigma\iota\varsigma$ of something not yet given, of a $\tau\iota$, whereas *axioms* refer to the $\gamma\nu\tilde{\omega}\sigma\iota\varsigma$ of a given, to insight into the validity of certain relationships that hold between given notions. In traditional axiomatizations, that contain relation symbols, and where axioms are not universal statements, such as Tarski’s, this ancient distinction no longer exists. The constructive axiomatics preserves this ancient distinction, as the ancient postulates are the primitive notions of the language, namely the individual constants and the geometric operation symbols themselves, whereas what Geminus would refer to as “axioms” are precisely the axioms of the constructive axiom system.

In a certain sense, one may think of a constructive axiomatization as one in which all the existence claims have been replaced by the existence of certain operation symbols, and where there is no need for the usual predicate symbols since they may be defined in a quantifier-free manner in terms of the operations of the constructive language.

A constructive axiom system for plane hyperbolic geometry was provided by Pambuccian (2004). It is expressed in the language $\mathcal{L}_{con} := \mathcal{L}(T', C_1, C_2, K_1, K_2, P, A', H_1, H_2)$, with *points* as variables, which contains only ternary operation symbols having the following intended interpretations: $T'(abc)$ is the point d on the ray opposite to ray $\overrightarrow{ac}$ with $ad \equiv ab$, provided that $a \neq c$ or $a = b$, and arbitrary otherwise; $C_i(abc)$, for $i = 1, 2$, stand for the two points d for which $da \equiv db$ and $da \equiv ac$, provided that $a \neq b$ and b lies between a and c, arbitrary points otherwise; $K_i(abc)$, for $i = 1, 2$, stand for the two points d for which $ad \equiv ab$ and $bd \equiv bc$, provided that (i) c lies between b and a, and is different from b or (ii) c lies strictly between

a and the reflection of b in a, two arbitrary points, otherwise; $P(abc)$ stands for the point d on the side ac or bc of triangle abc, for which $da \equiv db$ and $B(adc) \vee B(bdc)$, provided that a, b, c are three non-collinear points, an arbitrary point, otherwise; $A'(abc)$ stands for the point d on the ray $\overrightarrow{ac}$ for which $dd' \equiv ab$, where d' is the reflection of d in the line ab, provided that a, b, c are three non-collinear points, arbitrary, otherwise; $H_i(abc)$, for $i = 1, 2$, stand for the two points d for which $db \perp ba$ and $ad \perp dc$, provided that a, b, c are three different points with $B(abc)$, arbitrary, otherwise (if one does not like the fact that some operations may take "arbitrary" values for some arguments — which means that there is no axiom fixing the value of that operation for certain arguments, for which the operation is geometrically meaningless — one may add axioms stipulating a particular value in cases with no geometric significance (such as $T'(aba) = a$)). Its axioms are all universal, with one exception, a purely existential axiom, which states that there exist two different points. All the operations used are absolute, and by replacing $\neg\mathbf{R}$ (expressed in $\mathcal{L}_{con}$) with $\mathbf{R}$ we obtain an axiom system for plane Euclidean geometry.

That axiom system is the simplest possible axiom system for plane hyperbolic geometry among all axiom systems expressed in languages with only one sort of variables, to be interpreted as points, and without individual constants. The simplicity it displays is twofold. If, from the many possible ways to look at simplicity we choose the syntactic criterion which declares that axiom system to be simplest for which the maximum number of variables which occur in any of its axioms, written in prenex form, is minimal, then the axiom system referred to above is the simplest possible, regardless of language. Each axiom is a prenex statement containing no more than 4 variables. By a theorem of Scott [72] for axiom systems for Euclidean geometry which is valid in the hyperbolic case as well, there is no axiom system with individuals to be interpreted as points for plane hyperbolic geometry, consisting of at most 3-variable sentences, since all the at most 3-variable sentences which hold in plane hyperbolic geometry hold in all higher-dimensional hyperbolic geometries as well. The quantifier-complexity of its axioms is the simplest possible, as it consists of universal and existential axioms, so there are no quantifier-alternations at all in any of its axioms. It is also simplest among all constructive axiomatizations in that it uses only ternary operations, and one cannot axiomatize plane hyperbolic geometry by means of universal and existential axioms solely in terms of binary operations.

Pambuccian (2001) has also shown that plane hyperbolic geometry can be axiomatized by universal axioms in a two-sorted first-order language $\mathcal{L}$, with variables for both *points* and *lines*, to be denoted by upper-case and lower-case Latin alphabet letters respectively, three individual constants A_0, A_1, A_2, standing for three non-collinear points, and the binary operation symbols φ, ι, π_1, π_2 as primitive notions, where $\varphi(A, B) = l$, $\iota(g, h) = P$, $\pi_1(P, l) = g_1$, $\pi_2(P, l) = g_2$ may be read as: 'l is the line joining A and B' (provided that $A \neq B$, an arbitrary line, otherwise), 'P is the point of intersection of g and h' (provided that g and h are distinct and have a point of intersection, an arbitrary point, otherwise), 'g_1 and g_2 are the two limiting parallel lines from P to l' (provided that P is not on l, arbitrary lines, otherwise).

Another constructive axiomatization with *points* as variables, containing three individual constants a_0, a_1, a_2, standing for three non-collinear points, with $\Pi(a_0a_1) = \pi/3$ ($\Pi(xy)$ stands for the Lobachevsky function associating the angle of parallelism to the segment xy), one quaternary operation symbol $\tilde{\iota}$, with $\tilde{\iota}(abcd) = p$ to be interpreted as 'p is the point of intersection of lines $\overline{ab}$ and $\overline{cd}$, provided that lines $\overline{ab}$ and $\overline{cd}$ are distinct and have a point of intersection, an arbitrary point, otherwise', and two ternary operation symbols, $\varepsilon_1(abc)$ and $\varepsilon_2(abc)$, with $\varepsilon_i(abc) = d_i$ (for $i = 1, 2$) to be interpreted as 'd_1 and d_2 are two distinct points on line $\overline{ac}$ such that $ad_1 \equiv ad_2 \equiv ab$, provided that $a \neq c$, an arbitrary point, otherwise', was provided by Klawitter (2003).

Constructive axiomatizations also serve as a means to show that certain elementary hyperbolic geometries are in a precise sense "naturally occurring". If we were to explore the land of plane hyperbolic geometry in one of its models of the real numbers, and all the notes we can take of all the wonders we see have to be written down without the use of quantifiers, being allowed to use only the three constants A_0, A_1, A_2, which are marked in the model we are visiting and represent three generic fixed non-collinear points, as well as the joining, intersection, and hyperbolic parallels operations φ, ι, π_1, π_2, then all the notes we take will be theorems of plane elementary hyperbolic geometry as conceived by Hilbert. If we are as thorough as possible in our observations, then we would have written down an axiom system for that geometry. This may not be so surprising, given that we have the operations π_1 and π_2 as part of our language, so it may be said that we have built into our language the element of surprise we claim to have obtained.

However, even if the language had been perfectly neutral vis-à-vis $\mathbf{HPA}$, such as $\mathcal{L}_{con}$, to which we add three individual constants a_0, a_1, a_2, standing for three non-collinear points, the story we could possibly tell of our visit to the land of plane hyperbolic geometry over the reals is that of plane elementary hyperbolic geometry as conceived by Hilbert. Since all the operations in $\mathcal{L}_{con}$ are absolute, in the sense of having a perfectly meaningful interpretation should we have landed in the Euclidean kingdom, the proof that Hilbert's elementary hyperbolic geometry *is* a most natural fragment of full plane hyperbolic geometry over the reals no longer suffers from the shortcoming of the previous one.

Another constructive axiomatization with a similar property of being formulated in a language containing only absolute operations will be referred to in the survey of H-planes.

Other languages with simple axiom systems

The simplest language in which n-dimensional hyperbolic (as well as Euclidean) geometry, i. e. of a theory synonymous with $\mathcal{H}_n$, can be axiomatized, with individuals to be interpreted as *points*, is one containing a single ternary relation such as Pieri's P ($P(abc)$ standing for $ab \equiv ac$) or the perpendicularity predicate $\perp$ (cf. [71]). That no finite set of binary relations with points as variables can axiomatize hyperbolic (or Euclidean) geometry was shown by Robinson (1959) (cf. [71]).

There are several results related to the axiomatizability of hyperbolic geometry in languages in which the individual variables have interpretations other than points or points and lines. In the two-dimensional case, Prażmowski (1986) has shown that hyperbolic geometry can be axiomatized with individual variables to be interpreted as *equidistant lines*, or *circles*, or *horocycles*, in languages containing a single ternary relation, or several binary relations. Prażmowski (1984, 1986) also showed that *horocycles* or *equidistant lines* or *circles* and *points*, with point-horocycle or point-equidistant line, or point-circle incidence can serve as a language in which to axiomatize plane hyperbolic geometry.

Unlike the Euclidean 2-dimensional case, for $n = 2$ and for $n \geq 4$, n-dimensional hyperbolic geometry over Euclidean ordered fields can be (as shown in [62], using a result from [69]) axiomatized by means of $\forall\exists$-axioms with *lines* as individual variables by using only the binary relation of line orthogonality (with intersection) as primitive notion. Just like Euclidean three-space, hyperbolic three-space cannot be axiomatized with line perpendicularity alone (as shown by List [51]), but it can be axiomatized — as noticed by Pambuccian (2000) — with *planes* as individual variables and plane-perpendicularity as the only primitive notion. More remarkable, as noticed in [61], both n-dimensional hyperbolic and Euclidean geometry (coordinatized by Euclidean fields) can be axiomatized with *spheres* as individuals and the single binary predicate of sphere tangency for all $n \geq 2$.

Generalized hyperbolic geometries

Order based generalizations of hyperbolic geometry. A generalization of the ordered structure of hyperbolic planes was provided by Prażmowski [68, §2.1] under the name *quasihyperbolic plane*, in a language with points, lines, point-line incidence, and a quaternary relation among points, with $ab \upharpoonleft\!\upharpoonright cd$ to be interpreted as 'ray $\vec{ab}$ is parallel to $\vec{cd}$ or $a = b$ or $c = d$'. Another (dimension-free) generalization was proposed by Karzel and Konrad [38] (cf. also [36], [45]). It is not known how the two generalizations, the quasihyperbolic and that of [38], are related.

Plane geometries. Klingenberg [43] introduced the most important fragment of hyperbolic geometry, the generalized hyperbolic geometry over arbitrary ordered fields. Its axiom system, consists of the axioms for metric planes, i. e. A2.13-A2.22, and the two axioms $(\exists ab)\, U(ab)$ and $\bigwedge_{i=1}^{3}(pq|a_i \wedge U(a_i g)) \rightarrow \bigvee_{i=1}^{3} a_i = a_{i+1}$ (addition in the indices being mod 3), with $U(ab)$, to be read 'the lines a and b have neither a point nor a perpendicular in common', being defined by $U(ab) :\Leftrightarrow (\forall xy)\, \neg(xy|a \wedge xy|b) \wedge \neg(x|a \wedge x|b)$ (a different axiom system, based on that of semi-absolute planes can be found in [8]). The axiom system could also have been expressed by means of universal axioms in the bi-sorted first-order language

$\mathsf{L}(A_0, A_1, A_2, A_3, \varphi, F, \pi, \iota, \zeta)$, with points and lines as individual variables, where the A_i are point constants such that $\varphi(A_0, A_1)$ and $\varphi(A_2, A_3)$ have neither a common point nor a common perpendicular, ζ is a binary operation with lines as arguments and a line as value, with $\zeta(g, h)$ to be interpreted as 'the common perpendicular to g and h, provided that $g \neq h$ and that the common perpendicular exists, an arbitrary line otherwise'; F is a ternary operation, $F(abc)$ standing for the footpoint of the perpendicular from c to the line ab, provided that $a \neq b$, an arbitrary point, otherwise; and π a ternary operation symbol, $\pi(abc)$ being interpreted as the fourth reflection point whenever a, b, c are collinear points with $a \neq b$ and $b \neq c$, and arbitrary otherwise. By *fourth reflection point* we mean the following: if we designate by σ_x the mapping defined by $\sigma_x(y) = \sigma(xy)$, i. e. the reflection of y in the point x, then, if a, b, c are three collinear points, by [4, §3,9, Satz 24b], the composition $\sigma_c\sigma_b\sigma_a$, is the reflection in a point, which lies on the same line as a, b, c. That point is designated by $\pi(abc)$. If we denote by Ξ the axiom system expressed in this language, then we can prove that Ξ is the axiom system for the universal $\mathsf{L}(A_0, A_1, A_2, A_3, \varphi, F, \pi, \iota, \zeta)$-theory of the standard Kleinian model of the hyperbolic plane over the real numbers. In other words, that if we are allowed to express ourselves only by using the above operations, and none of our sentences is allowed to have existential quantifiers, then all we could say about the phenomena taking place in the hyperbolic plane over the reals in which there are four fixed points A_0, A_1, A_2, A_3, such that the lines $\varphi(A_0, A_1)$ and $\varphi(A_2, A_3)$ are hyperbolically parallel, is precisely the theory of Klingenberg's hyperbolic planes. In this sense, Klingenberg's hyperbolic geometry is a naturally occurring fragment of full hyperbolic geometry. As shown in [43] and [4], all models of Klingenberg's axiom system are isomorphic to the generalized Kleinian models over ordered fields K. Their point-set consists of the points of a hyperbolic projective-metric plane over K that lie inside the absolute, the lines being all the lines of the hyperbolic projective-metric plane that pass through points that are interior to the absolute, and the operations have the intended interpretation. The difference between generalized Kleinian models and Kleinian models over Euclidean ordered fields is that in the former neither midpoints of segments nor hyperbolic (limiting) parallels from a point to a line (in other words intersection points of lines with the absolute) need to exist. In fact, if we add to the $\mathcal{L}(\equiv)$ axiom system for Klingenberg's generalized hyperbolic planes an axiom stating the existence of the midpoint of every segment, we obtain an axiom system for $\mathcal{H}_2$.

A hyperbolic projective polarity ω defined in a Pappian projective plane induces a notion of perpendicularity: two lines are perpendicular if each passes through the pole of the other. If the projective plane is orderable, then one obtains a hyperbolic geometry by the process described earlier. If the original projective plane is not orderable, then one cannot define hyperbolic geometry in this way, but the whole projective plane, with the exception of those lines which pass through their poles, with its perpendicularity relation defined by ω may be considered as a geometry of hyperbolic-type, whose models are called *hyperbolic projective-metric* planes. They were axiomatized by Lingenberg, who also provided an axiomatization for these planes over quadratically closed fields (see [50] and the literature cited therein). As shown in [63], they can be axiomatized in terms of *lines* and *orthogonality*.

Treffgeradenebenen have been introduced by Bachmann [4, §18,6], as models where lines are all the *Treffgeraden*, i. e. lines in $\mathfrak{A}(K,-1)$, with K a Pythagorean field, which intersect the unit circle (the set of all points (x,y) with $x^2+y^2=1$) in two points, and where points are all points of $\mathfrak{A}(K,-1)$ for which all lines of $\mathfrak{A}(K,-1)$ which pass through them are *Treffgeraden*. For a constructive axiom system see [59].

Further generalizations of hyperbolic planes, with a poorly understood class of models, have been put forward by Baer [9] (and formally axiomatized by Pambuccian (2001)) and Artzy [3]. All models of the former must have infinitely many points and lines, whereas the models of the latter may be finite as well.

Higher-dimensional geometries. Axiom systems for both three-dimensional hyperbolic geometry over Euclidean ordered fields, and for more general hyperbolic geometries in terms of planes and reflections in planes, obtained from that of Ahrens by adding certain axioms to it, were presented by Scherf (1961). Hübner [35] described algebraically the models of Kinder's (1965) axiom system for n-dimensional absolute metric geometry to which certain additional axioms, in particular axioms of hyperbolic type, have been added, thus generalizing Scherf's work to the n-dimensional case with $n \geq 2$. Kroll and Sörensen [48] have axiomatized a dimension-free hyperbolic geometry, all of whose planes are generalized hyperbolic planes in the sense of Klingenberg.

2. Absolute Geometry

The concept of *absolute* geometry was introduced by Bolyai in §15 of his *Appenidx*, its theorems being those that do not depend upon the assumptions of the existence of no more than one or of several parallels. It turned out that this concept has even farther-reaching consequences than that of hyperbolic geometry, for it provides, for an era which knows that geometry cannot be equated with Euclidean geometry, a framework for a definition of what one means by geometry. In a first approximation one would think that the body of theorems common to Euclidean and hyperbolic geometry would form *geometry per se*, and it is this geometry that was first thoroughly studied. One can also conceive of any body of theorems common to Euclidean and hyperbolic geometry as forming an absolute geometry, and it is this more liberal view that we take in this section.

Order-based absolute geometries

Ordered Geometry. A most natural choice for a geometry to be called *absolute* would be one based on the groups I and II of Hilbert's axioms, i. e. a geometry of incidence and order, with no mention of parallels. We could call this geometry with Coxeter [14], who follows Artin [2], *ordered geometry*, or the geometry of *convexity* *spaces*, which were axiomatized by Bryant [11] and shown to be equivalent to ordered geometry by Precup (1980). Its axioms, expressed in $\mathcal{L}(B)$, with L defined by (1), are: A1.8, A1.6, and

A 2.1. $B(aab)$,

A 2.2. $L(abc) \rightarrow L(cba) \wedge L(bac)$,

A 2.3. $a \neq b \wedge L(abc) \wedge L(abd) \rightarrow L(acd)$,

A 2.4. $B(abc) \rightarrow B(cba)$,

A 2.5. $B(abc) \wedge B(acd) \rightarrow B(bcd)$,

A 2.6. $(\forall ab)(\exists c)\, B(cab) \wedge c \neq a$,

A 2.7. $(\forall abcxy)(\exists z)\, \neg L(abc) \wedge B(bxc) \wedge x \neq b \wedge x \neq c \wedge B(ayx) \wedge y \neq a$
$\wedge y \neq x \rightarrow L(cyz) \wedge B(azb) \wedge z \neq a \wedge z \neq b.$

In the presence of Pasch's axiom A2.7, one can define in a first-order manner the concept of dimension in ordered geometry. Pasch [64] has shown that models of 3-dimensional ordered geometries can be embedded in ordered projective spaces of the same dimension if they satisfy additional conditions. Kahn (1980) has shown that the Desargues theorem need not be postulated in at least 3-dimensional spaces, provided that a condition which ordered spaces do satisfy, holds. Sörensen (1986), Kreuzer (1989), [47] and Frank (1988) offered shorter or more easy to follow presentations of these results, which rely on minimal sets of assumptions. The extent to which convex geometry can be developed within a very weak ordered geometry, which may also be formulated in $\mathcal{L}(B)$ can be read from [13].

Grochowska-Prażmowska [25] has provided an axiom system for an ordered geometry based on the quaternary relation of *oriented parallelism* $\upharpoonleft\!\upharpoonright$. That axiom system is equivalent to that of ordered geometry to which an axiom on ray parallels has been added. That axiom states the existence, for every triple a, b, c of non-collinear points, of a point d, different from c, such that the rays $\overrightarrow{ab}$ and $\overrightarrow{cd}$ are parallel, ray parallelism being defined in the Menger-Skala manner presented earlier. It has both affine ordered planes and hyperbolic planes as models.

Weak general affine geometry. Szczerba [84] has shown that if one adds to A1.6, A1.8, A2.6, and the projective form of the Desargues axiom, the following axioms, of which A2.11 is an upper dimension axiom, A2.10 the outer form of the Pasch axiom, and A2.12 an axiom stating that there is a line in the projective closure of the plane which lies outside the plane,

A 2.8. $B(xyz) \wedge B(yzu) \wedge y \neq z \rightarrow B(xyu)$,

A 2.9. $B(xyz) \wedge B(xyu) \wedge x \neq y \rightarrow B(yzu) \vee B(yuz)$, A 2.10. $(\forall txyzu)(\exists v)\, B(xtu) \wedge B(yuz) \rightarrow B(xvy) \wedge B(ztv)$,

A 2.11. $(\forall xyzt)(\exists u)\, (L(yuz) \wedge L(xtu)) \vee (L(xuy) \wedge L(ztu)) \vee (L(xuz) \wedge L(yut))$,

A 2.12. $(\exists abcd)(\forall xyzuvs)\, L(axb) \wedge L(byc) \wedge L(czd) \wedge L(dua) \wedge L(avc)$
$\wedge L(xvz) \wedge L(yvu) \wedge L(xus) \wedge L(yzs) \rightarrow x = u \vee y = z$,

one obtains an axiom system Ψ for a general affine geometry, all of whose models are isomorphic to open, convex sets in affine betweenness planes over ordered skew fields, an affine betweenness plane $\mathfrak{B}(F)$ over the ordered skew field F being the structure $\langle F \times F, \mathbf{B}_F \rangle$, with $\mathbf{B}_F(\mathbf{abc})$ if and only if $\mathbf{b} = \mathbf{a}(1-\xi) + \mathbf{c}\xi$ for some $\xi \in F$. If one adds a projective form of Pappus' axiom to Ψ, an extension giving rise to an axiom system Ψ_a, then the skew fields in the representation theorem for Ψ become commutative. This is another naturally occurring theory. Pappian general affine geometry is the $\mathsf{L}(B)$-theory common to ordered affine Pappian geometry and to Klingenberg's generalized Kleinian models, i.e. it contains precisely those $\mathsf{L}(B)$-sentences which are true in both of these geometries. Szczerba [84] also proves a representation theorem for the weaker theory axiomatized by $\Psi \setminus \{\text{A2.12}\}$.

Plane metric ordered geometries

H-planes. Following Bolyai, a vast literature on absolute geometries has come into being. The main aim of the authors of this literature is that of establishing systems of axioms that are on the one hand *weak* enough to be common to various geometries, among which the Euclidean and the hyperbolic, and on the other hand, *strong* enough to allow the proof of a substantial part of the theorems of elementary (Euclidean) geometry, and to allow an algebraic description as subgeometries of some projective geometry with a metric defined in the manner of Cayley [12].

The first three groups of Hilbert's [31] axioms provided the first elementary axiomatization of an absolute geometry, which we have already encountered, expressed in $\mathcal{L}(B, \equiv)$ as $\mathcal{A}_3$ (which is $\mathcal{A}$ together with an axiom fixing the dimension to 3).

Of great importance, since it facilitates absolute proofs of theorems, was Pejas's [66] algebraic description of all models of $\mathcal{A}_2$ (also referred to in [29] as Hilbert planes, or H-planes). It reads:

Let K be a field of characteristic $\neq 2$, and k an element of K. By the *affine-metric plane* $\mathfrak{A}(K, k)$ (cf. [29, p.215]) over the field K with the *metric constant* k we mean the projective plane $\mathfrak{P}(K)$ over the field K from which the line $[0, 0, 1]$, as well as all the points on it have been removed (and we write $\mathfrak{A}(K)$ for the remaining point-set), for whose points of the form $(x, y, 1)$ we shall write (x, y) (which is incident with a line $[u, v, w]$ if and only if $xu + yv + w = 0$), together with a notion of orthogonality, the lines $[u, v, w]$ and $[u', v', w']$ being orthogonal if and only if $uu' + vv' + kww' = 0$. If K is an ordered field, then one can order $\mathfrak{A}(K)$ in the usual way.

The algebraic characterization of the H-planes consists in specifying a point-set E of an affine-metric plane $\mathfrak{A}(K, k)$, which is the universe of the H-plane. Since E

will always lie in $\mathfrak{A}(K)$, the H-plane will inherit the order relation B from $\mathfrak{A}(K)$. The congruence of two segments $\mathbf{ab}$ and $\mathbf{cd}$ will be given by the usual Euclidean formula $(a_1 - b_1)^2 + (a_2 - b_2)^2 = (c_1 - d_1)^2 + (c_2 - d_2)^2$ if $E \subset \mathfrak{A}(K, 0)$, and by

$$\frac{F(\mathbf{a}, \mathbf{b})^2}{Q(\mathbf{a})Q(\mathbf{b})} = \frac{F(\mathbf{c}, \mathbf{d})^2}{Q(\mathbf{c})Q(\mathbf{d})}, \tag{3}$$

if $E \subset \mathfrak{A}(K, k)$ with $k \neq 0$, where $F(\mathbf{x}, \mathbf{y}) = k(x_1y_1 + x_2y_2) + 1$, $Q(\mathbf{x}) = F(\mathbf{x}, \mathbf{x})$, and $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$.

Let now K be an ordered Pythagorean field (i. e. the sum of any two squares of elements of K is the square of an element of K), R the ring of *finite* elements, i. e. $R = \{x \in K \,:\, (\exists n \in \mathbf{N})\, |x| < n\}$ and P the ideal of *infinitely small* elements of K, i. e. $P = \{0\} \cup \{x \in K \,:\, x^{-1} \notin R\}$. All H-planes are isomorphic to a plane of one of the following three types:

Type I. *$E = \{(a, b) \mid a, b \in M\} \subset \mathfrak{A}(K, 0)$, where M is an R-module $\neq (0)$;*

Type II. *$E = \{(a, b) \mid a, b \in M\} \subset \mathfrak{A}(K, k)$ with $k \neq 0$, where M is an R-module $\neq (0)$ included in $\{a \in K \mid ka^2 \in P\}$, that satisfies the condition*

$$a \in M \Rightarrow ka^2 + 1 \in K^2;$$

Type III. *$E = \{\mathbf{x} \mid Q(\mathbf{x}) > 0, Q(\mathbf{x}) \notin J\} \subset \mathfrak{A}(K, k)$ with $k < 0$, where $J \subseteq P$ is a prime ideal of R that satisfies the condition*

$$ka^2 + 1 > 0, ka^2 + 1 \notin J \Rightarrow ka^2 + 1 \in K^2,$$

with K satisfying

$$\{a \in K \mid ka^2 \in R \setminus P\} \neq \emptyset.$$

In planes of type I there exist rectangles, so their metric is Euclidean, and we may think of them as ‘finite’ neighborhoods of the origin inside a Cartesian plane. Those of type II can be thought of as infinitesimally small neighborhoods of the origin in a non-Archimedean ordered affine-metric plane. There is no rectangle in them, and their metric may be of hyperbolic type (should $k < 0$) or of elliptic type (should $k > 0$) — in the latter the sum of the angles of a triangle can exceed two right angles only by an infinitesimal amount. Planes of type III are generalizations of the Klein inner-disc model of hyperbolic geometry. A certain infinitesimal collar around the boundary may be deleted from the inside of a disc, and the metric constant k, although negative, may not be normalizable to -1, as the coordinate field is only Pythagorean and not necessarily Euclidean. In case K is a Euclidean field (every positive element has a square root) and $J = (0)$, we can normalize the metric constant k to -1 and we have Klein’s inner-disc model of plane hyperbolic geometry with K as coordinate field.

The axioms for absolute geometry, in particular the *five-segment axiom* A1.5, have been the subject of intensive research. Significant simplifications, which allow the formulation of all plane axioms as prenex sentences with at most six variables, have been achieved by Rigby (1968, 1975), building up on the results obtained by Mollerup (1904), R. L. Moore (1908) Dorroh (1928,1930), Piesyk (1965), Forder (1947), Szász (1961). As shown by Pambuccian (1997), it is not possible to axiomatize $\mathcal{A}_2$ in $\mathcal{L}(B, \equiv)$ by means of prenex axioms containing at most 4 variables, raising the question whether it is possible to do so with at most 5 variables. The axiomatization of Hilbert planes can also be achieved by completely separating the axioms of order from those of congruence and collinearity, i. e. if one expresses the axiom system in $\mathcal{L}(L, B, \equiv)$, then the symbol B does not occur in any axiom in which the symbol $\equiv$ occurs. This was shown in polished form by Sörensen [78]. A constructive axiomatization for a theory $\mathcal{CA}_2$ synonymous with $\mathcal{A}_2$ was provided in [58]. It is expressed in $\mathsf{L}(a_0, a_1, a_2, T', J')$, where the a_i stand for three non-collinear points, T' being the segment transport operation encountered earlier, and J' is a quaternary segment-intersection predicate, $J'(abcd)$ being interpreted as the point of intersection of the segments ab and cd, provided that a and b are two distinct points that lie on different sides of the line cd, and c and d are two distinct points that lie on different sides of the line ab, and arbitrary otherwise. This shows the remarkable fact that plane absolute geometry is a theory of two geometric instruments: segment-transporter and segment-intersector. If we enlarge the language by adding a ternary operation A — with $A(abc)$ representing the point on the ray $\vec{ac}$, whose distance from the line ab is congruent to the segment ab, provided that a, b, c are three non-collinear points, and an arbitrary point otherwise — we can express constructively a strengthened version of Aristotle's axiom, to be denoted by **Ars**, as: $\neg L(abc) \rightarrow (B(aA(abc)c) \vee B(acA(abc))) \wedge ab \equiv A(abc)F(abA(abc))$. If we add **Ars** and **R** to the axiom system for $\mathcal{CA}_2$, we obtain a constructive axiom system for plane Euclidean geometry over Pythagorean ordered fields, whereas if we add **Ars** and **HM** to the axiom system for $\mathcal{CA}_2$ we obtain a constructive axiom system for plane hyperbolic geometry over Euclidean ordered fields, i. e. a theory synonymous with $\mathcal{H}_2$. Thus both Euclidean and hyperbolic plane geometry may be axiomatized in the same language $\mathsf{L}(a_0, a_1, a_2, T', J', A)$, the only difference consisting in the axiom specifying the metric, no specifically Euclidean or specifically hyperbolic operation symbol being needed to constructively axiomatize the two geometries.

An interesting constructive axiomatization of H-planes over Euclidean ordered fields can be obtained by translating into constructive axiomatizability results the theorems of Strommer (1977) (proved independently by Katzarova (1981) as well), or their generalization in [15], where it is shown that Steiner's theorem on constructions with the ruler, given a circle and its centre C, can be generalized to the absolute setting by having a few additional fixed points in the plane (such as two points P and Q together with the midpoint M of the segment PQ, provided that $CP \not\equiv CQ$).

A very interesting, but never cited, absolute geometry weaker than that of H-planes, is the one considered by Smid [74], who also characterized it algebraically by showing that it can be embedded in a projective metric plane. That geometry is obtained from Hilbert's axiom system of H-planes by replacing the axiom of segment transport with the weaker version which asks, given two point-pairs a, b and c, d, that one of the following two statements hold: (i) the segment ab can be transported on ray $\overrightarrow{cd}$ or (ii) the segment cd can be transported on ray $\overrightarrow{ab}$. It holds in all open convex subsets of H-planes.

Metric planes

Schmidt-Bachmann planes. The geometry of metric planes can be thought of as the metric geometry common to the three classical plane geometries (Euclidean, hyperbolic, and elliptic). Neither order nor free mobility is assumed.

It originates in the observation that a significant amount of geometric theorems can be proved with the help of the three-reflection theorem, which proved for the first time its usefulness in [32]. The axiomatization of metric planes grew out of the work of Hessenberg, Hjelmslev, A. Schmidt, and Bachmann, whose life and students' work have been devoted to their study. Metric planes have never been presented as models of an axiom system in the logical sense of the word, but as a description of a subset satisfying certain conditions inside a group. Given that most mathematicians were familiar with groups, but not with logic — which is the area in which Bachmann had worked before embarking on the reflection-geometric journey — and that the group-theoretical presentation flows smoothly and gracefully, which cannot be said of the formal-logical one, it is perfectly reasonable to present it the way Bachmann did, when writing a book on the subject. Since we are interested here only in the axiom systems themselves, and not in the development of a theory based on them, it is natural to present these structures as axiomatized in first-order logic.

There are two main problems for these purely metric plane geometries: (i) that of their embeddability in a Pappian projective plane, where line-perpendicularity and line-reflection are represented in the usual manner by means of a quadratic form, and (ii) that of characterizing algebraically those subsets of lines in the projective plane in which the metric plane has been embedded, which are the lines of the metric plane. While the first problem has been successfully solved whenever it had a solution, there are only partial results concerning the second one, complete representation theorems being known only for metric planes satisfying additional requirements (such as free mobility or orderability).

In its most polished form, to be found in [4], the axiom system can be understood as being expressed with one sort of variables for *lines*, and a binary operation ϱ, with $\varrho(a, b)$ to be interpreted as 'the reflection of line b in line a'. To improve the readability of the axioms, we shall use the following abbreviations ($p|q$ may be read 'p is orthogonal to q' (i. e., given A2.20, $\varrho(p, q) = q$) — and we may think of the pair (p, q) with $p|q$ as a 'point', namely the intersection point of p and q — and $pq|a$ may be read 'a passes through the intersection point of p and q, two orthogonal lines' or 'the point pq lies on a'):

$$g_1 \dots g_n = h_1 \dots h_m :\Leftrightarrow (\forall x)\, \varrho(g_1, \dots \varrho(g_n, x) \dots) = \varrho(h_1, \dots \varrho(h_m, x) \dots),$$
$$(a_1 \dots a_n)^2 = 1 :\Leftrightarrow (\forall x)\, \varrho(a_1, \dots \varrho(a_n, \varrho(a_1, \dots \varrho(a_n, x) \dots) = x,$$
$$a|b :\Leftrightarrow a \neq b \wedge (ab)^2 = 1, J(abc) :\Leftrightarrow abc \neq 1 \wedge (abc)^2 = 1,$$
$$pq|a :\Leftrightarrow p|q \wedge J(pqa).$$

A 2.13. $a^2 = 1,$

A 2.14. $(\forall ab)(\exists c)\, aba = c,$

A 2.15. $(\forall abcd)(\exists g)\, a|b \wedge c|d \rightarrow J(abg) \wedge J(cdg),$

A 2.16. $ab|g \wedge cd|g \wedge ab|h \wedge cd|h \rightarrow (g = h \vee ab = cd),$

A 2.17. $(\forall a_1a_2a_3pq)(\exists d)\, \bigwedge_{i=1}^{3} pq|a_i \rightarrow a_1a_2a_3 = d,$

A 2.18. $(\forall a_1a_2a_3g)(\exists d)\, \bigwedge_{i=1}^{3} a_i|g \rightarrow a_1a_2a_3 = d,$

A 2.19. $(\exists ghj)\, g|h \wedge \neg j|g \wedge \neg j|h \wedge \neg J(jgh),$

A 2.20. $a|b \rightarrow \varrho(a, b) = b,$

A 2.21. $a \neq b \wedge \varrho(a, b) = b \rightarrow a|b,$

A 2.22. $\varrho(a, b) = a \rightarrow a = b.$

A2.13 states that reflections in lines are involutions; A2.14 that for all line-reflections a and b, aba is a line-reflection as well; A2.15 that for any two points (a, b) and (c, d) there is a line g joining them; A2.16 states that the joining line of two different points is unique; A2.17 and A2.18 that the composition of three reflections in lines with a point or with a common perpendicular in common is a reflection in a line; A2.19 that there are three lines forming a right triangle; A2.20-A2.22 ensure that ϱ has the desired interpretation.

The Euclideanity of a Euclidean plane may be considered as being determined by its affine structure (i. e. by the fact that an Euclidean plane is an affine plane), or as being determined by its Euclidean metric, i. e. by the fact that there are rectangles in that plane. On the basis of orthogonality, one may define in the usual manner a notion of parallelism, and ask whether having a Euclidean metric implies the affine structure, i. e. the intersection of non-parallel lines). It was shown by Dehn [16] that the latter is not the case, i. e. that there are planes with a Euclidean metric, to be called *metric-Euclidean planes*, that are not Euclidean planes (i. e. where the parallel axiom does not hold). Such planes, which are precisely the planes of type I in Pejas's classification for which $E \neq \mathfrak{A}(K, 0)$, must be non-Archimedean.

Metric-Euclidean planes were introduced by Bachmann (1948) (see also [4]), as metric planes in which the rectangle axiom, i. e. $(\exists abcd)\, a \neq b \wedge c \neq d \wedge a|c \wedge a|d \wedge$

$b|c \wedge b|d$, holds. The point-set of a metric-Euclidean plane of characteristic $\neq 2$ is a subset of the Gaussian plane over (L, K), where L is a quadratic extension of K (a generalization of the standard complex numbers plane), which contains 0, 1, is closed under translations and rotations around 0, and contains the midpoints of any point-pair consisting of an arbitrary point and its image under a rotation around 0.

Non-elliptic metric planes (i. e. those satisfying the axiom $(\forall abc)\, abc \neq 1$, which will be referred to as $\neg\mathbf{P}$) can also be axiomatized in $\mathcal{L}(\equiv)$, as proposed in [77] (see [59] for a formalization of that axiom system).

By an *ordinary metric-projective* plane $\mathfrak{P}(K, \mathfrak{f})$ over a field K of characteristic $\neq 2$, with $\mathfrak{f}$ a symmetric bilinear form, which may be chosen to be defined by $\mathfrak{f}(\mathbf{x}, \mathbf{y}) = \alpha x_1 y_1 + \beta x_2 y_2 + \gamma x_3 y_3$, with $\alpha\beta\gamma \neq 0$, for $\mathbf{x}, \mathbf{y} \in K^3$ (where $\mathbf{u}$ always denotes the triple (u_1, u_2, u_3), line or point, according to context), we understand a set of points and lines, the former to be denoted by (x, y, z) the latter by $[u, v, w]$ (determined up to multiplication by a non-zero scalar, not all coordinates being allowed to be 0), endowed with a notion of incidence, point (x, y, z) being incident with line $[u, v, w]$ if and only if $xu + yv + zw = 0$, an orthogonality of lines defined by $\mathfrak{f}$, under which lines $\mathbf{g}$ and $\mathbf{g}'$ are orthogonal if and only if $\mathfrak{f}(\mathbf{g}, \mathbf{g}') = 0$, and a segment congruence relation defined by (3) for points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ for which $Q(\mathbf{a}), Q(\mathbf{b}), Q(\mathbf{c}), Q(\mathbf{d})$ are all $\neq 0$, with $F(\mathbf{x}, \mathbf{y}) = \beta\gamma x_1 y_1 + \alpha\gamma x_2 y_2 + \alpha\beta x_3 y_3$. An ordinary projective metric plane is called *hyperbolic* if $F(\mathbf{a}, \mathbf{a}) = 0$ has non-zero ($\mathbf{a} \neq \mathbf{0}$) solutions, in which case the set of solutions forms a conic section, the *absolute* of that projective-metric plane.

The algebraic characterization of non-elliptic metric planes is given by

Representation Theorem 2. *Every model of a non-elliptic metric plane is either a metric-Euclidean plane, or else it can be represented as an embedded subplane (i. e. containing with every point all the lines of the projective-metric plane that are incident with it) that contains the point* $(0, 0, 1)$ *of a projective-metric plane* $\mathfrak{P}(K, \mathfrak{f})$ *over a field* K *of characteristic* $\neq 2$, *in which no point lies on the line* $[0, 0, 1]$, *from which it inherits the collinearity and segment congruence relations.*

The proof of this most important representation theorem follows, to some extent, in the non-elliptic (and most difficult) case, the pattern of the proof of Representation Theorem 1. One defines, for any two lines a and b, the *pencil* of lines defined by a and b to be the set $G(ab) := \{c \mid (\exists d)\, abc = d\}$). One can extend the set of lines and points of the metric plane to an *ideal plane* by letting the set of all line pencils be the set of points (pencils $G(ab)$ which contain two orthogonal lines m and n (in other words, for which there is a point mn on both a and b) can be thought of as representing points of the metric plane (the point mn in our example), whereas those which do not contain two orthogonal lines are *ideal points*, i. e. points that have been added to the metric plane). In analogy to the hyperbolic case, among the ideal points $G(ab)$, one may think of those for which there is a line g with $a|g$ and $b|g$ (i. e. a common perpendicular to a and b), as representing what used to be the exterior points, and of those for which there is neither a point on both a and b nor a common perpendicular (should such pencils exist), as representing what used to be absolute points. Given that two distinct pencils have at most one common line, one can extend the set of lines with *ideal lines* to make sure that any two points are joined by a line. That this is possible and that the ideal plane turns out to be a Pappian projective plane is due to several crucial ideas and theorems due to Hjelmslev, such as the concept of semi-rotation, and his *Lotensatz* (cf. [4, §3,6; §6], [29, §42] for full details of the proof of this representation theorem).

Non-elliptic metric planes have also been constructively axiomatized by Pambuccian [60] in the language $\mathcal{L}(a_0, a_1, a_2, F, \pi)$ with individual variables to be interpreted as *points*, with a_0, a_1, a_2 standing for three non-collinear points, and with F and π as described earlier. A representation theorem for ordered metric planes with non-Euclidean metric (i. e. in which there are no rectangles) has been provided by Pejas [67]. Metric planes with various kinds of degrees of free mobility (all of which are expressed inside $\mathcal{L}(\varrho)$) have been studied by Diller [17], who provides an algebraic description for their models.

Among generalizations of metric planes, which can be expressed in the same language $\mathcal{L}(\varrho)$, one that would encompass Minkowski planes as well was proposed by Wolff (1967) (cf. [4]). It arises from the axiom system for metric planes by weakening A2.15 and changing A2.19. In it two "points" do not always have a joining line.

Sperner planes and their generalizations. Asking for the weakest axiom system in terms of line-reflections from which one could prove the projective form of Desargues theorem — expressed in a particular manner in the language of line-reflections — Sperner [79] provided an axiom system for a large class of planes, to be referred to as *Sperner planes*, and opened up a vast area of research into axiom system weaker than the Schmidt-Bachmann one. Unlike the axiom system for metric planes, Sperner's axiom system also allows geometries over fields of characteristic 2. It follows from [19] that Sperner planes are embeddable in Pappian projective planes, and there is a quadratic form $\mathbf{Q}$ such that line-reflection has the usual algebraic expression relative to $\mathbf{Q}$.

Building upon the work of Sperner [79], Lingenberg (1959-1965) has introduced in a long series of papers a class of planes considerably larger than that of metric planes, and later presented his results in book form in [50].

Hjelmslev and semiabsolute planes. Hjelmslev [34] considered geometries in which the uniqueness of the line joining two points, A2.16 in Bachmann's axiom system for metric planes, is no longer required, being replaced by that of the uniqueness of the perpendicular raised from a point of a line to that line. Such geometries were axiomatized in $\mathcal{L}(\varrho)$ by Bachmann [7].

More general structures, called pre-Hjelmslev planes, were axiomatized in [44]. Representation theorems for ordered Hjelmslev planes with or without free mobility, similar to those of Pejas, have been provided by Kunze (1981).

Semiabsolute planes, which were defined and studied in [8] as models of A2.13-A2.16, A2.18-A2.22, and $(\forall abg)(\exists c)\, ab|g \rightarrow abg = c$, cannot, in general, be embedded in projective planes.

Three- and higher-dimensional absolute geometries

The first three-dimensional generalization of Bachmann's plane absolute geometry was provided by Ahrens [1]. A weakening of Ahrens's axiom system in the non-elliptic case, which may be understood as a three-dimensional variant of Lingenberg's planes (axiomatized by **S, EB***, and **D**) was provided by Nolte [55]. Nolte's axiom system can be formulated in a one-sorted language $\mathcal{L}(\varrho)$, with *planes* as individual variables and the binary operation ϱ, with $\varrho(g, h)$ to be interpreted as the reflection of the plane h in the plane g.

Dimension-free ordered spaces with congruence and free mobility with all subplanes H-planes, have been considered in Karzel and König [37], where a representation theorem for the models of these spaces, amounting to their embeddability in a Pappian affine plane, is proved. A like-minded, more general geometry was studied in [45] (see also [36]).

An n-dimensional generalization of Bachmann's [4] axiom system for metric planes, with *hyperplanes* as individual variables, and a binary operation ϱ_n with $\varrho_n(ab)$ to be interpreted as "the reflection of b in a", which is equivalent to that of [1] for $n = 3$, was proposed by Kinder (1965). A first algebraic description of Kinder's axiom system was provided in [35]. An algebraic description of some classes of the *ordered* n-dimensional absolute geometries axiomatized by Kinder, generalizing the results of [67], was provided in great detail in [28], after one for the ordered ones with free mobility had been provided by Klopsch (1985). The situation in higher dimensions is significantly more complex than in the two-dimensional case. Just as Kinder's axiom system generalizes Ahrens's axiom system to finite dimensions, [56] generalizes the axiom system from [55] to finite dimensions. A like-minded, but less researched, axiom system was introduced by Lenz (1974).

A dimension-free absolute metric geometry based on incidence and orthogonality was first proposed by Lenz (1962). It has been weakened to admit elliptic models as well in [75], [76]. The axiom system from [75] can be formalized in $\mathcal{L}(L, \kappa, \perp)$, with individuals to be interpreted as *points*, L a ternary relation standing for collinearity, κ a quaternary relation standing for coplanarity, and $\perp$ a ternary relation standing for orthogonality, with $\perp(abc)$ to be interpreted as ab is perpendicular to ac.

An axiom system for these spaces, but excluding the elliptic case, formulated in a language containing two sorts of variables, for points and hyperplanes, the binary relation of point-hyperplane incidence, and the binary notion of hyperplane orthogonality was presented in [76]. A like-minded axiom system, for dimension-free absolute metric geometry, with points and lines as individual variables, was presented in [20], simplified in [27], and reformulated in a different language by J. T. Smith (1985). If one adds to that axiom system axioms implying that the space has finite dimension $n \geq 3$, then the axiom system is equivalent to that proposed by Kinder (1965), and all finite-dimensional models can be embedded in a projective geometry over a field of characteristic different from 2 of elliptic, hyperbolic or Euclidean type, the orthogonality being given by a symmetric bilinear form. A weaker version of the above geometry was considered by J. T. Smith (1974). The most general axiom system for dimension-free absolute metric geometries, with points and lines as variables, was provided in [70], the last paper on this subject.

Axiomatizations of a large class of dimension-free absolute geometries with *points* as variables and the binary operation of point-reflection were proposed in [39] and [22]. In an earlier paper, Karzel (1971) had shown that axiom systems for line-reflections (formulated in $\mathcal{L}(\varrho)$) satisfying Sperner-Lingenberg-type axioms, can be interpreted not only as axiomatizing plane geometries, but also as axiomatizing geometries of dimension ≥ 3.

A very general absolute geometry, axiomatized in $\mathcal{L}(B, \equiv)$, whose betweenness and equidistance can be represented by a special class of generalized metrics, with values in ordered Abelian groups, was proposed by Moszyńska [54].

Reverse geometry

One source of motivations for research in absolute geometries has been mentioned already. It is the quest for a theory common to Euclidean and hyperbolic geometry, which would be rich enough to allow its models to be embedded in projective spaces coordinatized by commutative fields (Pappian projective spaces), and, in case a notion of orthogonality is definable inside that theory, that it may be represented in the usual manner by means of a quadratic form.

A second source can be discerned in questions of what one might call *reverse geometry*: Which axioms are needed to prove a particular theorem? The programme of reverse geometry was stated by Hilbert [30, p. 50], with his characteristic eloquence (cf. also [65] for the history of the regressive method):

> Unter der axiomatischen Erforschung einer mathematischen Wahrheit verstehe ich eine Untersuchung, welche nicht dahin zielt, im Zusammenhange mit jener Wahrheit neue oder allgemeinere Sätze zu entdecken, sondern die vielmehr die Stellung jenes Satzes innerhalb des Systems der bekannten Wahrheiten und ihren logischen Zusammenhang in der Weise klarzulegen sucht, daß sich sicher angeben läßt, welche Voraussetzungen zur Begründung jener Wahrheit notwendig und hinreichend sind.

An early example of an explicitly reverse geometric undertaking can be found in Barbilian [10], where it is analyzed on which of Hilbert's axioms the truth of a theorem due to Pompeiu, on the distances from a point to the vertices of an equilateral triangle, depends.

Much more common are investigations on the possibility of proving inside $\mathcal{A}$ theorems first established in Euclidean geometry, among the latest being that of Pambuccian (1998) on the existence of equilateral triangles over any given segment, or of the existence of even one equilateral triangle in H-planes.

Bachmann [6], following an earlier investigation by Toepken (1941), embarked on a reverse study of the altitude theorem, which states that the three altitudes of a triangle are concurrent. The axiom system for metric planes itself can be seen as an answer to the question: Which axioms are needed to prove all four important line concurrency theorems in triangles (concurrency of altitudes, angle bisectors, medians, and perpendicular bisectors)? All of these theorems figure prominently in both Hjelmslev [33], [34] and Bachmann [4, §4].

Explicitely reverse are also the proof in ordered geometry of the Sylvester-Gallai theorem in [14], as well as those of Pambuccian (2001, 2003).

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