One can easily find some other consequences. We mention, for instance, the following well-known fact which contains the fundamental theorem of algebra: Let $f\colon \mathbb{C} \to \mathbb{C}$ be a continuous function such that there exists a positive integer m and a complex number $c \neq 0$ with $\lim_{z\to\infty} z^{-m}f(z) = c$. Then f has a root.

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Another Example of an Exotic Function

VICTOR PAMBUCCIAN
Mountain House School, 12 Lake Placid Club Drive, Lake Placid, NY 12946

The purpose of this note is to exhibit a function, which is zero almost everywhere (i.e., a function that differs from zero on a *meager* set; here "meager" stands for "of Lebesgue measure zero and of first category") and takes every real value in any given interval. The construction is different from those in [**3**, Ch. 8, ex. 27], [**1**, Ch. I, ex. 1.2 and Th. 3.4], [**4**], and [**5**].

Let G be any meager additive subgroup of $\mathbb{R}$ of cardinality $2^{\aleph_0}$. Then $\mathbb{Q} \cdot G = \{qg | q \in \mathbb{Q},\ g \in G\}$ is meager too. If $a \in \mathbb{R} \setminus \mathbb{Q} \cdot G$ and $H = a^{-1}G$, then $H \cap \mathbb{Q} = \{0\}$, $|H| = 2^{\aleph_0}$ and H is meager. Hence every x in $A = H + \mathbb{Q}$ can be uniquely written as $x = h + q$, with h in H, q in $\mathbb{Q}$. Let $\phi\colon H \to \mathbb{R}$ be any one-to-one function and define ψ by

$$\psi(x) = \begin{cases} \phi(h), & \text{if } x \in A,\ x = h + q \\ 0, & \text{otherwise.} \end{cases}$$

Then ψ has all the required properties. The construction depends on the existence of a meager additive subgroup of $\mathbb{R}$ of cardinality $2^{\aleph_0}$. The additive subgroup of $\mathbb{R}$ generated by the set

$$\left\{ x \in \mathbb{R} | x = \sum_{i=1}^{\infty} \frac{a_i}{(2i)!}, \qquad 0 \leq a_i < 2i \right\}$$

is such an example (for details concerning its measurability see [**2**, p. 191]).

Any example of such an exotic function can be used to show the ubiquity of everywhere discontinuous functions that have the Darboux property, in the following sense (see [**1**, Th. 3.4] and [**5**]; the result was first proved in [**4**]): For any function $f\colon \mathbb{R} \to \mathbb{R}$, there is a function $g\colon \mathbb{R} \to \mathbb{R}$,

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{R} \setminus A \\ \psi(x), & \text{if } x \in A \end{cases}$$

that differs from f on a set of measure zero and of first category and has the intermediate value property, although it is nowhere continuous.

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