



Another Example of an Exotic Function

Author(s): Victor Pambuccian

Source: *The American Mathematical Monthly*, Vol. 96, No. 10 (Dec., 1989), pp. 913-914

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2324587>

Accessed: 16/11/2009 17:45

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=maa>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Mathematical Association of America is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly*.

One can easily find some other consequences. We mention, for instance, the following well-known fact which contains the fundamental theorem of algebra: Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function such that there exists a positive integer m and a complex number $c \neq 0$ with $\lim_{z \rightarrow \infty} z^{-m}f(z) = c$. Then f has a root.

REFERENCES

1. J. B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1978.
2. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

Another Example of an Exotic Function

VICTOR PAMBUCCIAN

Mountain House School, 12 Lake Placid Club Drive, Lake Placid, NY 12946

The purpose of this note is to exhibit a function, which is zero almost everywhere (i.e., a function that differs from zero on a *meager* set; here “meager” stands for “of Lebesgue measure zero and of first category”) and takes every real value in any given interval. The construction is different from those in [3, Ch. 8, ex. 27], [1, Ch. I, ex. 1.2 and Th. 3.4], [4], and [5].

Let G be any meager additive subgroup of \mathbb{R} of cardinality 2^{\aleph_0} . Then $\mathbb{Q} \cdot G = \{qg | q \in \mathbb{Q}, g \in G\}$ is meager too. If $a \in \mathbb{R} \setminus \mathbb{Q} \cdot G$ and $H = a^{-1}G$, then $H \cap \mathbb{Q} = \{0\}$, $|H| = 2^{\aleph_0}$ and H is meager. Hence every x in $A = H + \mathbb{Q}$ can be uniquely written as $x = h + q$, with h in H , q in \mathbb{Q} . Let $\phi: H \rightarrow \mathbb{R}$ be any one-to-one function and define ψ by

$$\psi(x) = \begin{cases} \phi(h), & \text{if } x \in A, x = h + q \\ 0, & \text{otherwise.} \end{cases}$$

Then ψ has all the required properties. The construction depends on the existence of a meager additive subgroup of \mathbb{R} of cardinality 2^{\aleph_0} . The additive subgroup of \mathbb{R} generated by the set

$$\left\{ x \in \mathbb{R} \mid x = \sum_{i=1}^{\infty} \frac{a_i}{(2i)!}, \quad 0 \leq a_i < 2i \right\}$$

is such an example (for details concerning its measurability see [2, p. 191]).

Any example of such an exotic function can be used to show the ubiquity of everywhere discontinuous functions that have the Darboux property, in the following sense (see [1, Th. 3.4] and [5]; the result was first proved in [4]): For any function $f: \mathbb{R} \rightarrow \mathbb{R}$, there is a function $g: \mathbb{R} \rightarrow \mathbb{R}$,

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{R} \setminus A \\ \psi(x), & \text{if } x \in A \end{cases}$$

that differs from f on a set of measure zero and of first category and has the intermediate value property, although it is nowhere continuous.

REFERENCES

1. A. M. Bruckner, *Differentiability of real functions*, Lecture Notes in Mathematics, Vol. 659, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

2. P. Erdős, K. Kunen, and R. Daniel Mauldin, Some additive properties of sets of real numbers, *Fund. Math.*, 113 (1981) 187–199.
3. B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis*, Holden-Day, San Francisco, 1964.
4. S. Marcus, Sur une propriété appartenant à toutes les fonctions réelles d'une variable réelle, *Indian J. Math.*, 9 (1967) 457–460.
5. Solution to Problem 6505, this MONTHLY, 94 (1987) 560.