REVIEWS

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Geometry: Euclid and Beyond. By Robin Hartshorne. Springer, New York, 2000, xii + 526 pp., ISBN 0-387-48650-2, $49.95.

Reviewed by Victor Pambuccian

The twentieth century has been not only the century of "Euclid must go!" but also the one in which the axiomatics of the many geometries that the nineteenth century had brought to the fore was carried out with remarkable success. These axiomatic accomplishments have been largely forgotten. Few would associate the axiomatic foundations of geometry with the names of Emil Artin, Karl Menger, Kurt Reidemeister, Max Dehn, Alfred Tarski, Reinhold Baer, Wilhelm Klingenberg, and Bartel Leenert van der Waerden, who are better known for their contributions to other fields that later became mainstream.

In the minds of most mathematicians the axiomatics of geometry is synonymous with Hilbert's *Grundlagen der Geometrie* of 1899. That this book only legitimized, by the stature of its author, an enterprise that already had a history going back to Pasch, Peano, and Pieri, and that an impressive amount of work going far beyond the original aims of the *Grundlagen* has been accomplished since, goes unnoticed.

Between 1900 and 1930 the axiomatics of geometry started attracting the attention not only of German and Italian but also of American mathematicians, foremost among them E. H. Moore, E. V. Huntington, and O. Veblen (see Scanlan [**8**]). Later, Karl Menger and Alfred Tarski imported their interest in foundational problems in geometry to the US, but the revival of interest they brought about was short-lived and confined to their immediate environment. Menger's main contribution was the realization that hyperbolic geometry can be axiomatized in terms of point-line incidence alone, whereas Tarski's, which is too vast to be summarized here (see Szczerba [**10**]), consisted of a remarkably simple axiom system for Euclidean geometry, together with a proof of its completeness and decidability.

For the absence of interest in the axiomatic enterprise in geometry, "working" geometers may argue along the following lines: Euclidean and hyperbolic geometry are two special cases of Riemannian geometry, a fact that renders unjustifiable the antiquated and time-consuming axiomatic development of these special cases. Also, since the foundations of mathematics are nowadays thought to encompass the foundations of arithmetic and set theory, but not of geometry, logicians aren't interested in the subject either.

The argument relegating Euclidean and hyperbolic geometry to footnotes of Riemannian geometry would be valid only if one were conceiving them as the "standard" geometries over the real numbers. However, this is not the case, for the elementary axiomatizations lead to geometries over arbitrary ordered fields, a class that may be narrowed by additional axioms to Pythagorean (sums of squares are squares), Euclidean (all positives are squares), or real-closed ordered fields. Significantly weaker theories have been considered as well (see Pambuccian [**7**]). These geometries, too, differ in important respects from their standard relatives and, with the exception of geometries over real-closed fields, do not fall under the jurisdiction of algebraic geometry.

What's more, unlike the version over real-closed fields, the Euclidean and hyperbolic geometries of ruler and gauge, as well as those of ruler and compass, are undecidable, as shown by M. Ziegler [**11**]. To put it simply, there is no way to "bulldoze one's way through a proof via analytic geometry." (See Davis [**1**, p. 207], where it is claimed that the belief in this possibility contributed to the decline of triangle geometry.)

The real reason for the neglect of these elementary geometries lies in a certain pragmatic form of mathematical materialism, for which geometries *ought* to be considered over the real numbers, since these are the ones modeling physical space. According to this view, the weak relatives are irrelevant theories, born out of artificial restrictions imposed by Greek mathematicians, similar to the restrictions of the constructivists. That it is due to these restrictions and to the desire of solving equations by radicals (which has no pragmatic justification either, for approximate solutions are as good for any imaginable application) that Galois theory came into being may be a neat historical accident, but it doesn't lend contemporary value to these theories, which are considered chiefly of pedagogical interest.

It is in general ignored by almost everyone outside the negligibly tiny community of logicians that there are serious, unsurmountable foundational problems regarding the real numbers, which are highly sensitive to the particular set-theoretical axiom system used. And what if there are problems? After all, are the reals not dense in mathematics, with the exception of pure algebra? That the honorable structure of the field of real numbers cannot be described in the language of ordered fields in first-order logic is a problem that need not bother the "working" geometer and is best relegated to philosophical discussions to be pursued after retirement. The related argument that geometry ought not to depend on set theory, that spatial intuition is of a more immediate nature than that of infinite sets, would encounter a similar fate.

So, why would one care about the axiomatization of fragments of these standard geometries? Why wonder which theorems one could prove using fewer assumptions, when the "real world" satisfies so much more than those pitiful "assumptions"? Is this type of undertaking not too close to that of the Greeks, who first conceived of mathematics as being the art of conditional statements? Shouldn't one, in multicultural earnest, make room for the mathematics of those the Greeks called "barbarians" and study geometry as the science of the real world, not some kind of statement dependency game?

In particular, if one addresses an audience of future secondary education teachers, should one still teach Euclid and the modern renaissance of the axiomatic method? After all, what high school geometry text in use today in public schools in the USA asks students to do more than find the side or a median of a triangle knowing a few other data or, for the advanced learner, find the area of a cyclic quadrilateral given the sides? Why ask of them what most professional mathematicians were never taught? Isn't axiomatics best relegated to the (modern) algebra courses, where it is indispensable?

I have been faced with this dilemma for the past several years, during which I have taught an "Introduction to Geometry" class attended primarily by secondary education majors. For six years I used Greenberg [**4**], a logically sound axiomatic development of Euclidean and hyperbolic geometry (no "ruler axiom"), as a text, supplemented with lecture notes on Euclidean geometry and problem sets containing problems of the mathematical competition type, as well as problems of an axiomatic nature for homework. Invariably, the problems with an axiomatic or model-construction flavor were ignored by the students, who showed very little interest in the axiomatic enterprise. Recently I have decided to capitulate and adopt Isaacs [**6**], which shuns both axiomatics and hyperbolic geometry in favor of actual problem solving and construction problems in standard Euclidean geometry.

There are two main objectives I have in mind when teaching the class. One is to use the opportunity to convey an understanding of the conditional nature of mathematical statements, on which the modern understanding of mathematics is based. The second is to have students understand the solutions of—and ideally be capable of solving by themselves—the geometry problems that regularly appear in mathematical competitions around the world (say, in the St. Petersburg Olympiads for seventh or eighth graders [**2**]) or the problems appearing as "review problems from sixth grade geometry" in the geometry textbook for seventh graders in Rumania (e.g., given four points, construct the square whose sides pass through those points, making clear when the construction is possible and when not). Since the first aim does not appear to be either popular or achievable with the audiences I am addressing, I have decided to concentrate on the second aim.

The book under review is for those who can pursue both aims. It should be used in conjunction with Books I–IV of Euclid's *Elements*.

Its first chapter deals with Euclidean geometry, both from Euclid's and from more recent times, including constructions. It thus addresses the second aim discussed above in the rather short space of sixty-three pages. The second chapter presents a development of absolute and Euclidean geometry based on Hilbert's axioms. It also introduces in an informal way the notions of "models of independence" and "consistency."

On page 70 it is implied, as in all textbooks on the subject, that, given Gödel's theorem on the unprovability of the consistency of any "reasonably rich" set of axioms, one has to settle for something less than a proof of the full consistency of geometry, that consistency relative to another (perhaps intuitively more convincing) theory is all one can reasonably expect. This is an unnecessarily modest statement for geometries axiomatized in first-order logic, because their axiom systems are not rich enough to interpret arithmetic, so one cannot express their consistency in their own language and Gödel's theorem does not apply. Moreover, the geometries axiomatized in the book have consistent and decidable extensions, namely, Euclidean or hyperbolic geometry over real-closed fields. The decidability of these theories follows from Tarski's quantifier-elimination procedure for real-closed fields. Their consistency can be proved by showing (i) that the quantifier-free formulas that Tarski's decision procedure T (or any other quantifier-elimination procedure, for that matter) associates to the axioms turn out to be tautologies, and (ii) that in a fixed deductive system for first-order logic that works entirely with sentences, all the logical rules preserve truth (i.e., if the quantifier-free formulas that T associates to the premises are tautologies, then the quantifier-free formula that T associates to the conclusion is a tautology as well).[1]

This is surely not a proof of decidability inside the geometric theory itself, but it is the closest one could possibly get to proving consistency without relying on the consistency of some set theory by way of models.

Chapters 3 and 4 are devoted to the introduction of coordinates in Euclidean geometry, leading to representation theorems for both ruler and gauge and ruler and compass geometries (all models of the respective axiom systems are isomorphic to Cartesian planes over Pythagorean or Euclidean ordered fields). For the first time in print one finds an exact statement of the concept of "free mobility" together with a precise statement and proof of its equivalence to the side-angle-side axiom. Chap-

[1]This approach was suggested to me by Andreas Blass on 21 September 1999. Such a consistency proof was carried out in detail in [**5**].

ter 5 is devoted to the Euclidean (Hilbertian) theory of area in Euclidean geometry, covering the relationship between "equal content" (Hilbert's *Ergänzungsgleichheit*) and "equidecomposability" (Hilbert's *Zerlegungsgleichheit*), as well as—a premiere in geometry textbooks—Dehn's solution to Hilbert's Third Problem! In Chapter 6 the author mentions the impossibility of squaring the circle, proves all classical impossibilities of construction that are reducible to field extension theory, and gives the characterization of regular polygons constructible with compass and marked ruler.

Chapter 7 is the most extensive and elementary treatment of non-Euclidean geometry to be found in any textbook. From its many remarkable features I will mention two: first, the treatment of area in hyperbolic geometry, including an elementary proof of the Bolyai-Gerwien theorem that (in accordance with the guiding ethos of the whole book) does not use any type of continuity consideration; second, the detailed introduction of coordinates in the Hilbertian end-calculus fashion, including (again for the first time in a textbook) the introduction of a multiplicative distance function, which allows the definition of trigonometric functions in the absence of continuity (usually some e^x unexpectedly pops up). That this can be done was considered possible by Hilbert, who nevertheless did not provide any hint regarding its implementation. Such an end-calculus, continuity-free hyperbolic trigonometry was first carried out by J. H. C. Gerretsen [**3**], and considerably simplified by P. Szász [**9**]. The reviewer has reasons to believe that the author reinvented it. A novelty is to be found among the exercises on page 370 as well: a non-Euclidean analogue of Euclid's III.36, the theorem leading to the definition of the the power of a point with respect to a circle. Another special feature is the extensive discussion of non-Archimedean Hilbert planes (models of absolute geometry), including references to the work of F. Bachmann and to Pejas's classification of Hilbert planes. The last chapter deals with regular polyhedra and proves not only the classification of regular convex polyhedra, but also that of the Archimedean semiregular convex solids.

The author's style is conversational, and the typical page contains more English than mathematical symbols. Every effort is made to tell the story in unabbreviated and jargon-free language; even the equality sign is sometimes replaced with "is equal to." Motivated students with the necessary maturity will find an ample set of problems of varying levels of complexity, most of which considerably deepen the understanding of the section they follow.

The level of mathematical maturity, dedication, vocation, and inspiration this book demands of its audience will reduce its suitability as a textbook for a geometry class taken primarily by secondary education majors. However, this book is a gem as a textbook for classes designed for mathematics majors. If one thinks that the raison d'être of undergraduate classes is to provide a stepping stone for their higher level graduate classes, then there is not much reason to teach such a class to mathematics majors. However, by the same token, many of those who take an undergraduate number theory class will never use it in graduate school or in their careers. It is, with Jacobi, "pour l'honneur de l'esprit humain" that they take it, and the same reason ought to apply to a course based on this book.

REFERENCES

1. P. J. Davis, The rise, fall, and possible transfiguration of triangle geometry: a mini-history, this MONTHLY **102** (1995) 204–214.
2. V. Fomin and A. Kirichenko, *Leningrad Mathematical Olympiads*, MathPro Press, Westford, MA, 1994.
3. J. C. H. Gerretsen, Die Begründung der Trigonometrie in der hyperbolischen Ebene. I–III, *Nederl. Akad. Wetensch. Proc.* **45** (1942) 360–366, 479–483, 559–566.
4. M. J. Greenberg, *Euclidean and Non-Euclidean Geometries*, 3rd ed., W. H. Freeman, New York, 1993.

5. H. Friedman, A consistency proof for elementary algebra and geometry, 1999, http://www.math.ohio-state.edu/foundations/manuscripts.html
6. I. M. Isaacs, *Geometry for College Students*, Brooks/Cole, Pacific Grove, CA, 2001.
7. V. Pambuccian, Fragments of Euclidean and hyperbolic geometry, *Sci. Math. Jpn.* **53** (2001) 361–400.
8. M. Scanlan, Who were the American postulate theorists?, *J. Symbolic Logic* **56** (1991) 981–1002.
9. P. Szász, Einfache Herstellung der hyperbolischen Trigonometrie in der Ebene auf Grund der Hilbertschen Endenrechnung, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **5** (1962) 79–85.
10. L. W. Szczerba, Tarski and geometry, *J. Symbolic Logic* **51** (1986) 907–912.
11. M. Ziegler, Einige unentscheidbare Körpertheorien, *Enseign. Math.* **28** (1982) 269–280.

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Rockefeller and the Internationalization of Mathematics Between the Two World Wars: Documents and Studies for the Social History of Mathematics in the 20th Century. By Reinhard Siegmund-Schultze. Birkhäuser, Basel, 2001, xiii + 341 pp., ISBN 4-7643-6468-8, $94.95.

Reviewed by **Della D. Fenster**

Mark McGwire announced his retirement from baseball in November 2001. He stepped aside for someone he felt would better serve the team. In a single moment, baseball—and all those loyal to it—had a hero comparable to those of half a century earlier. Instantly, it seemed, baseball assumed the stature it had when, day after day, Lou Gehrig returned to the ball park knowing his career would come to an end well ahead of schedule. A decade later, Jackie Robinson courageously stepped into the batter's box on a daily basis. One particularly contentious afternoon, Pee Wee Reese, a man from a little known town perched on the falls of the Ohio River, strolled over to first base and placed his white arm around Jackie Robinson's black shoulders. These were seemingly ordinary moments for Gehrig, Robinson, and Reese on otherwise ordinary days that added up to something much larger than the sum of their parts. Apparently, the same calculus applied to Mark McGwire's career. As he described it, "I am walking away from the game that has provided me opportunities, experiences, memories, and friendships to fill ten lifetimes."

Reinhard Siegmund-Schultze develops a similar theme in his *Rockefeller and the Internationalization of Mathematics Between the Two World Wars*, which focuses on how the daily decisions regarding the allocation of funds by the Rockefeller philanthropies came together to affect the development of mathematics. Specifically, Siegmund-Schultze combines attention to detail with a fresh and seemingly inexhaustible supply of primary source material to begin to answer a question that Robert Kohler deemed "unanswerable" in his classic book on foundation patronage [**4**, p. 2]: how such patronage has influenced a particular scientific discipline, in this case, mathematics. As Siegmund-Schultze describes it, his aim is to provide insight into "the overall process of 'internationalization' of science and mathematics in the first part of the 20th century" (p. 1) by examining "the place and the function of American philanthropists, especially of the Rockefeller philanthropy, in this process ..." (p. 10). But more than that, his overall approach unveils the decidedly personal aspect of this history, and his emphasis on personal letters and diary entries ("logs") reveals how the